Variational properties of a nonlinear elliptic equation and rigidity

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Abstract. We consider in this paper elliptic equations which are perturbations of Laplace's equation by a compactly supported potential. We show that in dimension greater than three for a wide class of potentials all the solutions are globally minimising. However, in dimension two the situation is different. We show that for radially symmetric potentials there always exist solutions which are not locally minimal unless the potential vanishes identically. We discuss the relations of this with the so-called Hopf rigidity phenomenon.

1 Introduction and main results

In this paper we discuss variational properties of classical solutions of the nonlinear equation

$$\Delta u = -V'_u\left(u, x_1, \ldots, x_n\right) \tag{1.1}$$

which is the Euler-Lagrange equation of the functional

$$I(u) = \int \frac{1}{2}\left(\nabla u\right)^2 - V\left(u, x_1, \ldots, x_n\right) dx_1 \ldots dx_n \tag{1.2}$$

The main question which is addressed here is the following: under what conditions on the potential V are all classical solutions of (1.1) globally minimising for the functional (1.2)? By a global minimiser we mean a smooth function on a domain of $\mathbb{R}^n$ minimising the integral (1.2) over all bounded subdomains with smooth boundaries with respect to smooth functions with the same boundary values.

The motivation for this question comes from variational problems of classical mechanics. In geodesic problems it is well known that all geodesics are globally minimising on manifolds of negative sectional curvature. However, it was shown first by E Hopf and L Green that the situation is completely different for Riemannian (two) tori or for Riemannian planes which are flat outside a compact set. They proved ([12], [9]) that for these manifolds there always exists geodesics with conjugate points and therefore non-minimal, unless the metric is flat. We refer the reader to [4], [6] for higher dimensional generalisations of E Hopf and Green's theorems and to [8], [7], [13], [10] for very important previous developments.

It was observed first in [2] that the Hopf phenomenon is not entirely Riemannian. In [3] we have shown that a similar type of rigidity holds true for Newton's equations with periodic or compactly supported potentials.

For the equation (1.1) with periodic potentials it was shown in [15], [16] that far-going generalisations of Aubry-Mather and KAM theories apply. Using these theories one can construct families of minimal solutions which form laminations or sometimes even foliations of the configuration torus. It is an interesting open question, however, if it happens that for all "slope" vectors these laminations are genuine foliations. This question is tightly related to the one we are addressing in this paper, since all the leaves of a foliation are globally minimal.

We shall assume throughout this paper that the potential V is compactly supported. The main reason for this is to make the analogy with the above mentioned situations of Hopf rigidity and to exclude, in particular, the case with $V''_{uu} \leq 0$ everywhere which is analogous to non-positive curvature. Nevertheless, several of our results apply in more generality, as we shall indicate.

We will show that in case of dimensions greater than 2 there are many potentials such that all solutions of (1.1) are minimising (Theorem 1). In dimension 2, however, at least for radially symmetric potentials there always exist non-minimal solutions of (1.1) unless V vanishes identically (Theorem 2). We state theorems 1 and 2 here.

Theorem 1 *For $n \geq 3$ let V be a compactly supported potential on $\mathbb{R}^{n+1}$. Assume that $V''_{uu}(u,x) \leq U(x)$ for some function U such that either*

(A) $U(x) \leq \left(\frac{n-2}{2}\right)^2 \frac{1}{||x-x_0||^2}$ *for some point $x_0 \in \mathbb{R}^n$, or*

(B) $||U||_{n/2} \leq \frac{n(n-2)}{4}|\mathbb{S}^n|$ *where $|\mathbb{S}^n|$ is the volume of the unit n-sphere.*

Then any solution of (1.1) is globally minimising.

Theorem 2 *Let $V(u, x_1, x_2)$ be a radially symmetric compactly supported potential ($n = 2$). There always exist radial non-minimal solutions of (1.1) unless V vanishes identically.*

Our approach to the proof of the theorem 2 is based on the reduction to a Newton equation with the potential supported in the semi-strip. It turns out that a technique analogous to Hopf's can be applied for such a shape of support.

In Section 3 we will consider the case of radially symmetric potentials for $n \geq 3$. We illustrate the minimality property of radial solutions of (1.1) organising them in foliations (Theorem 3).

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2 Proof of Theorem 1

Let u be a solution of (1.1) in a domain Ω and v be any function with the same boundary values. Set $\xi = v - u$ and write

$$I(v) - I(u) = \int_\Omega \nabla u \nabla \xi + \frac{(\nabla \xi)^2}{2} - (V(u+\xi, x) - V(u, x))\, d^n x \tag{2.1}$$

Integrating the first term by parts and using the equation (1.1) we have

$$\int_\Omega \nabla u \nabla \xi dx = \int_\Omega V'_u(u, x) \xi \, d^n x$$

By the assumption on V we have

$$V(u+\xi, x) - V(u, x) - V'_u(u, x)\xi \leq \frac{1}{2} U(x) \xi^2$$

Substituting this in (2.1) we obtain

$$I(v) - I(u) \geq \int_\Omega \frac{(\nabla \xi)^2}{2} - \frac{1}{2} U(x) \xi^2 \, d^n x \tag{2.2}$$

We show that under hypothesis (A) or (B) the last integral is always positive unless $\xi \equiv 0$. Indeed, for case (A) introduce spherical coordinates centred at x_0, $r = ||x - x_0||$. The required assertion follows from the following

Lemma 1 *For any function $\xi(r)$ defined on $[r_1, r_2]$ $(0 \leq r_1 < r_2 \leq \infty)$ with $\xi(r_2) = 0$ it follows that*

$$\int_{r_1}^{r_2} r^{n-1} \left((\xi')^2 - \left(\frac{n-2}{2} \right)^2 \frac{\xi^2}{r^2} \right) dr \geq 0$$

with equality for $\xi \equiv 0$ only.

Proof of Lemma 1

Introduce the new function $\varphi(r) = \xi(r) \cdot r^{n/2-1}$ and substitute into the integral. We have

$$\int_{r_1}^{r_2} r^{n-1} \left[\left(r^{1-\frac{n}{2}} \varphi' + \left(\frac{2-n}{2} \right) r^{-\frac{n}{2}} \varphi \right)^2 - \frac{(n-2)^2}{4} r^{-n} \varphi^2 \right] dr$$

$$= \int_{r_1}^{r_2} \left(r \cdot (\varphi')^2 + (2-n) \varphi \varphi' \right) dr$$

$$= \frac{n-2}{2} \varphi^2 (r_1) + \int_{r_1}^{r_2} r (\varphi')^2 \, dr$$

Since $n \geq 3$ we see that the last expression is always non-negative and equals zero only for $\xi \equiv 0$. This completes the proof of Lemma 1 and of the theorem in case (A).

Alternatively, under hypothesis (B), Sobolev's inequality (e.g. §8.3 of [14]) implies that

$$\int |\nabla \xi|^2 \, d^n x \geq S_n ||\xi||^2_{\frac{2n}{n-2}},$$

where

$$S_n = \frac{n(n-2)}{4} |\mathbb{S}^n|.$$

Then Hölder's inequality applied to the second term of the integrand (as in §11.3 of [14]) yields

$$|\int U \xi^2 \, d^n x| \leq ||\xi^2||_{\frac{n}{n-2}} ||U||_{\frac{n}{2}}.$$

Thus

$$I(v) - I(u) \geq \frac{1}{2} (S_n - ||U||_{\frac{n}{2}}) ||\xi||^2_{\frac{2n}{n-2}}.$$

This completes the proof in case (B). □

Remark 1 *Lemma 1 and its proof have a precursor in [5] (Ch6 §5), and the whole of Theorem 1 fits in the general domain of "absence of bound states" described for example in [17] (§8.3).*

Remark 2 *Our proof shows in addition that any solution is a non-degenerate minimum for (1.2). In a different way one can say this as follows: for any solution u on Ω, the linearised equation*

$$\Delta \xi + V''_{uu}(u, x) \xi = 0$$

has only the trivial solution satisfying the zero boundary conditions $\xi|_{\partial\Omega} = 0$.

3 Radial potentials

Let us consider now the case of radial, compactly supported potentials $V(u, r)$ satisfying $V''_{uu}(u, r) \leq \frac{1}{4r^2}$. Radial solutions $u(r)$ of (1.1) are given by the following:

$$u'' + \frac{n-1}{r} u' + V'_u(u, r) = 0$$

It follows from Theorem 1A that for $n \geq 3$ all radial solutions are without conjugate points. Moreover, Lemma 1 gives that the points r_1 and ∞ are not conjugate for any $r_1 > 0$. This fact enables us to organise the solutions into foliations in the following way. Denote by N_A the class of all those solutions which can be written as $u(r, \alpha) = \frac{\alpha}{r^{n-2}} + A$ for some α outside the support of V.

Theorem 3 *For compactly supported radial potential V satisfying $V''_{uu}(u, r) \leq \frac{1}{4r^2}$ the set N_A is totally ordered and the graphs of the solutions define a smooth foliation of $\mathbb{R}^{n+1} - \mathbb{R}(u) \times \{x = 0\}$.*

Proof of the theorem

Given A we have to show that the function $u(r, \alpha)$ is monotone in α. Indeed the function $\xi(r) = \frac{\partial u}{\partial \alpha}$ is a solution of the linearised equation. Note that by definition $\xi(\infty) = 0$. Then the non-conjugacy property implies that $\xi > 0$ and then it is easy to complete the proof. $\square$

Remark 3 *In some cases there is another way of organising the set of all solutions into foliations. Assume we are given a radial potential with $V(u, r) \equiv 0$ for $0 < r \leq r_0$ and $r \geq R_0$ satisfying the inequality of Theorem 3. Define the set M_A of all those solutions which can be written as $u(r, \alpha) = \frac{A}{r^{n-2}} + \alpha$ when $r \leq r_0$. Then one shows that for any given A the set M_A is ordered and the graphs of the solutions smoothly foliate the space $\mathbb{R}^{n+1} - \mathbb{R}(u) \times \{x = 0\}$. In addition M_0 foliates the whole of $\mathbb{R}^{n+1}$. In order to check the order property one shows that the linearised equation has no focal points in the following sense: any solution satisfying $\dot{\xi}(r_1) = \xi(r_2) = 0$ is trivial provided $r_1 < r_2$ (this is not necessarily true for $r_1 > r_2$). Then one proceeds exactly as in the proof of theorem 3.*

4 Rigidity for the case $n = 2$

Let $u(r)$ be a radial solution of the equation (1.1) for compactly supported rotationally symmetric $V(u, r)$, $V(u, r) \equiv 0$ for $|u| \geq U$ or $r > R$. With the substitution $r = e^t$ the equation for u as a function of t can be written in the Hamiltonian form

$$\begin{aligned} \dot{u} &= p \\ \dot{p} &= -e^{2t} W'_u(u, t) \end{aligned} \tag{4.1}$$

The Hamiltonian function of (4.1) is

$$H = \frac{1}{2} p^2 + e^{2t} W(u, t)$$

with the function $W(u, t) = V(u, e^t)$. Note that the support of W is contained in the semi-strip $\Pi = \{|u| \leq U, t \leq T = lnR\}$.

We shall prove the following rigidity result which implies theorem 2.

Theorem 4 *There always exist solutions with conjugate points for (4.1) unless W vanishes identically.*

The strategy of the proof will follow the original one of E Hopf, but will take special care about the non-compactness of the situation which requires careful estimates on ω given in Lemmas. In what follows, we will assume that all solutions of (4.1) are without conjugate points.

The first step in the proof is the following very well known construction: if the solution $u(t)$ has no conjugate points, then one can easily construct a *non-vanishing* solution ξ of the linearised Jacobi equation

$$\xi'' + e^{2t} W''_{uu}(u(t), t)\xi = 0$$

Having such a $\xi(t)$ for every $u(t)$ one defines the function $\omega(p, u, t)$ by the formula

$$\omega(p, u, t) = \frac{\dot{\xi}(t)}{\xi(t)} \quad \text{when} \quad p = \dot{u}(t),\ u = u(t).$$

Then ω satisfies the Ricatti equation along the flow of (4.1).

$$\dot{\omega} + \omega^2 + e^{2t} W''_{uu}(u(t), t) = 0 \tag{4.2}$$

Here $\cdot$ stands for the derivative along the flow. It should be mentioned that by the construction of ξ and ω the function ω is a-priori only measurable (and smooth along the flow).

Denote by

$$K = \sqrt{sup_{(t,u)\epsilon\Pi} W''_{uu}(u, t)}$$

We will need the following two lemmas specifying the behaviour of the function ω at infinity:

Lemma 2 *The following statements hold true:*

$$|\omega(p, u, t)| \leq Ke^T \quad \textit{for all} \quad (p, u, t) \tag{4.3}$$

$$0 \leq \omega(p, u, t) < \frac{1}{t - T} \quad \textit{for} \quad t > T \tag{4.4}$$

There exists a constant $\tilde{K}$ such that for all (p, u, t) with $t < T$

$$-\tilde{K}e^{t/4} < \omega(p, u, t) < Ke^t \tag{4.5}$$

Lemma 3 *The function ω satisfies the inequalities*

I. For $t \leq T$

$$\textit{if} \quad u > U,\ p > 0 \quad \textit{then} \quad 0 \leq \omega < Ke^{t - \frac{u-U}{p}} \tag{4.6}$$

$$\textit{if} \quad u < -U,\ p < 0 \quad \textit{then} \quad 0 \leq \omega < Ke^{t + \frac{-U-u}{p}} \tag{4.7}$$

$$\textit{if} \quad u > U - p(T-t),\ p \leq 0 \quad \textit{or} \quad u < -U - p(T-t),\ p \geq 0 \quad \textit{then} \quad \omega \equiv 0. \tag{4.8}$$

II. For $t > T$

$$\begin{aligned}
&\textit{if} \quad u > U + p(t-T),\ p > 0 \quad \textit{then} \quad 0 \le \omega < Ke^{t-\frac{u-U}{p}} && (4.9)\\
&\textit{if} \quad u < U + p(t-T),\ p < 0 \quad \textit{then} \quad 0 \le \omega < Ke^{t+\frac{-u-U}{p}} && (4.10)\\
&\textit{if} \quad u < -U,\ p \ge 0 \quad \textit{or} \quad u > U,\ p \le 0 \quad \textit{then} \quad \omega \equiv 0. && (4.11)
\end{aligned}$$

We postpone the proof of Lemmas 2 and 3 and first finish the proof of the theorem.

Proof of the theorem

In order to achieve decay also in the p direction, introduce the Gibbs density

$$\alpha(p,u,t) = e^{-H} = e^{-1/2p^2 - e^{2t}W(u,t)}$$

We have

$$\dot{\alpha} = -e^{-H}\dot{H} = -e^{-H}H_t = -\alpha\left(e^{2t}W\right)_t. \tag{4.12}$$

Now recall that the Hamiltonian flow of (4.1) preserves the Liouville measure $d\mu = dpdu$. Multiply the Ricatti equation (4.2) by α and write it in the form

$$\dot{\widehat{\alpha\omega}} - \dot{\alpha}\omega + \alpha\omega^2 + \alpha e^{2t}W''_{uu} = 0$$

Substitute equation (4.12) to obtain

$$\dot{\widehat{\alpha\omega}} + \alpha\omega\left(e^{2t}W\right)_t + \alpha\omega^2 + \alpha e^{2t}W''_{uu}(u,t) = 0$$

Owing to the estimates of Lemmas 2 and 3 one can integrate this equation over the whole (p,u) space. Use in addition the invariance of the measure and write

$$\frac{d}{dt}\left(\int \alpha\omega d\mu\right) + \int \alpha\omega\left(e^{2t}W\right)_t d\mu + \int \alpha\omega^2 d\mu + \int \alpha e^{2t}W''_{uu} d\mu = 0 \tag{4.13}$$

Using integration by parts the last term can be replaced by $\int \alpha\left(e^{2t}W_u\right)^2 d\mu$.

Integrate now the last equation (4.13) for $-A \le t \le A$ for a large constant A and pass to the limit $A \to +\infty$. Note that by the uniform estimate of Lemma 2 the term $\int \alpha\omega d\mu|_{-A}^{A}$ vanishes in the limit. So we have

$$\int \alpha\omega\left(We^{2t}\right)_t d\mu dt + \int \alpha\omega^2 d\mu dt + \int \alpha\left(e^{2t}W\right)^2 d\mu dt = 0 \tag{4.14}$$

By the Cauchy–Schwarz inequality we can estimate the first integral of (4.14) by

$$\int \alpha\omega\left(We^{2t}\right)_t d\mu dt \ge -\left[\int \alpha\omega^2 d\mu dt \int \alpha\left(\left(We^{2t}\right)_t\right)^2 d\mu dt\right]^{1/2}$$

With the notation $x = \left(\int \alpha\omega^2 d\mu dt\right)^{1/2}$ we have the quadratic inequality

$$x^2 - x \cdot \int \alpha\left[\left(We^{2t}\right)_t\right]^2 d\mu dt + \int \left(e^{2t}W_u\right)^2 \alpha d\mu dt \le 0.$$

Then its discriminant must be non-negative:

$$4\int \alpha \left(e^{2t}W_u\right)^2 d\mu dt \le \int \alpha \left[\left(e^{2t}W\right)_t\right]^2 d\mu dt \qquad (4.15)$$

The final argument in the proof is the following rescaling trick. It is similar to one invented in [3] for periodic potentials. Consider the family of Hamiltonians for every natural number N

$$H_N = \frac{1}{N^2}H\left(Np, Nu, t\right) = \frac{1}{2}p^2 + \frac{1}{N^2}e^{2t}W\left(Nu, t\right)$$

It can be immediately checked that the property of having all solutions without conjugate points remains valid for all N. Thus the inequality (4.15) implies the following inequalities for all N:

$$\begin{aligned} 4 \ \int \ & e^{-\frac{1}{N^2}W(Nu,t)e^{2t}} \left(e^{2t}\frac{1}{N}W_u'\left(Nu,t\right)\right)^2 dudt \le \\ \int \ & e^{-\frac{1}{N^2}W(Nu,t)e^{2t}} \left[\frac{1}{N^2}\left(e^{2t}W\left(Nu,t\right)\right)_t\right]^2 dudt \end{aligned}$$

(Here the integration with respect to p has been performed on both sides.)

Change the variable in both integrals to $v = Nu$. We obtain the inequality:

$$\begin{aligned} \frac{4}{N^3} \ \int \ & e^{-\frac{1}{N^2}W(v,t)e^{2t}} \left(e^{2t}W_u'(v,t)\right)^2 dvdt \le \\ \frac{1}{N^5} \ \int \ & e^{-\frac{1}{N^2}W(v,t)e^{2t}} \left[\left(e^{2t}W(v,t)\right)_t\right]^2 dvdt \end{aligned}$$

Now it is clear that if W is not zero identically then the left side is of order $\frac{1}{N^3}$ while the right side is of order $\frac{1}{N^5}$ as $N \to \infty$. This proves then that $W \equiv 0$ identically. The proof of the theorem is completed. □

Proof of Lemma 2

By the definition of K we have

$$\begin{aligned} &\dot\omega \le K^2e^{2t} - \omega^2 \quad \text{for} \quad t \le T \quad \text{and} \\ &\dot\omega = -\omega^2 \quad \text{for} \quad t > T \end{aligned}$$

The proof is based on the following elementary

Fact: Any solution of the inequality

$$\dot\omega \le B^2 - \omega^2, \quad \omega\left(t_0\right) = \omega_0$$

blows up if $|\omega_0| > B$. Moreover, the blow up time t_* is estimated as follows

$$\begin{aligned} &\text{for} \quad \omega_0 > B, \quad t_0 + \Delta < t_* < t_0 \\ &\text{for} \quad \omega_0 < -B, \quad t_0 < t_* < t_0 + \Delta \\ &\text{where} \quad \Delta = \frac{1}{2B}ln\left(\frac{\omega_0 - B}{\omega_0 + B}\right) \end{aligned}$$

To prove (i) one takes $B = Ke^T$ and obtains the required estimate. In the same manner one obtains (ii) and the right hand side of (iii).

Let us prove the rest of (iii). Pick two moments of time $\tau_0 < \tau_1 < 0$. On the segment $\tau \in [\tau_0, \tau_1]$ we have $\dot{\omega} \leq B^2 - \omega^2$ with $B = Ke^{\tau_1}$. If $\omega(\tau_0) = \omega_0$ is too negative then the blow up happens before τ_1 and this is impossible. Thus one obtains

$$\Delta = \frac{1}{2B} ln\left(\frac{\omega_0 - B}{\omega_0 + B}\right) > \tau_1 - \tau_0.$$

This implies

$$\omega_0 \geq -B\frac{1 + e^{2Bd}}{e^{2Bd} - 1}$$

Choose $\tau_1 = \tau_0/2$, then this inequality can be written in the form:

$$\omega_0 \geq -e^{\tau_0/4} \cdot f(\tau_0)$$

where the function

$$f(\tau_0) = Ke^{\tau_0/4} \cdot \frac{1 + e^{-K\tau_0 e^{\tau_0/2}}}{1 + e^{-K\tau_0 e^{\tau_0/2}}}$$

An easy calculation shows that for $\tau_0 \to -\infty$, $f \to 0$ and thus $f(\tau_0)$ is bounded from above by some positive constant $\tilde{K}$. Since τ_0 was arbitrary, (iii) follows. $\square$

Proof of Lemma 3

Any point (p, u, t) with $|u| > U$ is situated outside the support Π and so moves in the straight line $u(t) = u + pt$ unless it hits Π. If it does not touch Π then ω vanishes identically. This is the case in both (4.8) and (4.11).

In the cases of (4.6 – 4.7) and (4.9 – 4.10) the line $u(t) = u + pt$ hits Π in backward time. Since for the free motion $\dot{\omega} = -\omega^2$, it follows that $\omega(p, u, t)$ has to be non-negative and can be bounded from above by $Ke^{\tilde{t}}$, where $\tilde{t}$ is the time of entering Π.

This completes the proof of Lemma 3. $\square$

Though by Theorem 4 there are always solutions with conjugate points one cannot claim that they appear on those radial solutions which are regular at zero. The next example shows that all regular solutions may remain minimal.

Example 4.13

Consider two compactly supported functions $\Phi(u)$ and $\Psi(t)$. Define the function

$$W(u,t) = -e^{-2t}\left(\dot{\Psi}(t)\Phi(u) + \frac{1}{2}\Psi(t)\left(\dot{\Phi}(u)\right)^2\right)$$

Then $W(u,t)$ is compactly supported and one can easily verify that all the solutions of

$$\dot{u} = f(u,t) \quad \text{with} \quad f(u,t) = \dot{\Phi}(u)\Psi(t) \tag{4.16}$$

are the solutions of the Newton equation $\ddot{u} = -e^{2t}W'_u(u,t)$.

The solutions of (4.16) form a foliation of $\mathbb{R}(u)\times\mathbb{R}(t)$ and then by Weierstrass' theorem of calculus of variations (see e.g. [11] §23) are globally minimal. It is clear from the construction that all corresponding radial solutions $u(r)$ of (1.1) are regular at zero.

5 Discussion and open problems

The results of this paper leave open some very natural questions. We have formulated our results in the context of compactly supported potentials V on $\mathbb{R}\times\mathbb{R}^n$, but most of them have generalizations to some non-compactly supported cases. Theorem 1 and its proof apply verbatim to V of non-compact support, in particular to V periodic in u, but periodicity in x is excluded by each of the hypotheses (A) and (B).

Theorem 2 can be generalized to V periodic in u and compactly supported in x, but it is not clear whether it can be extended to other cases.

It would be very interesting to obtain results for potentials periodic in both u and x because of the fundamental papers [15], [16] by J Moser, which for our equation imply the existence of minimal laminations for all irrational slopes which for certain slopes are foliations. In this context our main question looks as follows: are there other periodic potentials except those with $V'_u(u,x) = 0$ such that for any slope there is a smooth foliation of $\mathbb{T}^{n+1}$ by minimal solutions? We refer the reader to the survey article by V Bangert [1] for detailed discussions of this and related questions.

The proof of Theorem 2 is based on the reduction to the case of Hamiltonian systems and so cannot be generalised in a straight-forward way to non-radial potentials V. However, it might be that the result remains true. For example, it is very reasonable to expect that the equation perturbed by a compactly supported W

$$\Delta u = -V'_u(u,r) + \varepsilon W'_u(u,x_1,x_2)$$

always has non-minimal solutions for ε sufficiently small. In the case of Hamiltonian systems it would follow from the theorem on continuous dependence of solutions on the initial values.

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