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A CRITERION FOR NON-PERSISTENCE OF TRAVELLING BREATHERS FOR PERTURBATIONS OF THE ABLOWITZ–LADIK LATTICE

Abstract. The Ablowitz–Ladik lattice has a two-parameter family of travelling breathers. We derive a necessary condition for their persistence under perturbations of the system. From this we deduce non-persistence for a variety of examples of perturbations. In particular, we show that travelling breathers do not persist under many reversible perturbations unless an additional symmetry is preserved, and we address the case of Hamiltonian perturbations.

1. **Introduction.** The phenomenon of discrete breathers, i.e. spatially localised periodic oscillations in nonlinear lattices, has attracted much recent interest (see e.g. the review articles [2, 4, 6, 8]). One largely unresolved topic in this context is the rigorous mathematical status of *travelling* discrete breathers: although such objects have repeatedly been observed in numerical simulations (e.g. [3, 5, 10]), their existence has been proved for only a few rather special models ([1, 5]). In search of a better mathematical understanding, a natural first question is whether the known travelling breathers persist under certain perturbations of the respective underlying system. In this note we discuss a necessary condition for smooth persistence of travelling discrete breathers under perturbations of the Ablowitz–Ladik (AL) lattice. Our condition establishes non-persistence for a variety of types of perturbations, but it fails to settle the persistence question for some important examples like the Salerno family, which will require more refined methods of analysis.

Although our subsequent discussion of the persistence of travelling breathers applies to a wider class of lattices with hardly any modification, we shall, for the sake of concreteness, exclusively deal with the AL lattice

$$i\dot{u}_k = |u_k|^2(u_{k-1} + u_{k+1}) + u_{k-1} - 2u_k + u_{k+1}\,, \quad k \in \mathbb{Z}\,. \tag{1}$$

For ease of notation, let $\Omega = l^2(\mathbb{Z};\mathbb{C})$ denote the Hilbert space of all square-summable bi-infinite complex-valued sequences; for any $u, v \in \Omega$ and $\lambda \in \mathbb{C}$ we denote by λu, $\overline{u}$, $|u|$, uv and $u+v$ the sequences $(\lambda u_k)_{k\in\mathbb{Z}}$, $(\overline{u}_k)_{k\in\mathbb{Z}}$, $(|u_k|)_{k\in\mathbb{Z}}$, $(u_k v_k)_{k\in\mathbb{Z}}$ and $(u_k + v_k)_{k\in\mathbb{Z}}$, respectively. As usual, the norm of $u \in \Omega$ is symbolized by $\|u\|$. Also, ζ and η with $\zeta(u) = (u_{k+1})_{k\in\mathbb{Z}}$ and $\eta(u) = (\overline{u}_{-k})_{k\in\mathbb{Z}}$ are continuous and one-to-one, and they map Ω onto itself; the map ζ is the (*left*) *shift*, and η

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will be referred to as *skew-flip*. Finally, $R_\vartheta : \Omega \to \Omega$ with $R_\vartheta(u) = e^{i\vartheta} u$ denotes the *rotation* by $\vartheta \in \mathbb{R}$. With these notations, (1) may be rewritten as a differential equation in Ω,

$$i\dot{u} = |u|^2\big(\zeta(u) + \zeta^{-1}(u)\big) + \zeta(u) - 2u + \zeta^{-1}(u)\,. \tag{2}$$

Below we shall deal with functions $S : \Omega \to \mathbb{C}$ which are (real) differentiable with respect to the real and imaginary part of $u_k = v_k + iw_k$, for all $k \in \mathbb{Z}$. Rather than v_k, w_k we use $u_k, \overline{u}_k$ as coordinates, and also partial derivatives of S with respect to the latter,

$$\frac{\partial S}{\partial u_k} = \frac{1}{2}\frac{\partial S^*}{\partial v_k} + \frac{1}{2i}\frac{\partial S^*}{\partial w_k}\,, \quad \frac{\partial S}{\partial \overline{u}_k} = \frac{1}{2}\frac{\partial S^*}{\partial v_k} - \frac{1}{2i}\frac{\partial S^*}{\partial w_k}\,, \quad \text{where } S^*(v_k, w_k) \equiv S(v_k + iw_k)\,.$$

It is a well-known fact (see [1]) that (2) admits a two-parameter family $(u_{\alpha,\beta})$ of *travelling breathers*, i.e. spatially localised solutions of the form

$$u_{\alpha,\beta;k}(t) = e^{-i\sigma t} U(k - ct)\,, \quad k \in \mathbb{Z}, t \geq 0\,. \tag{3}$$

They are given by

$$U(x) = e^{-i\alpha x}\frac{\sinh\beta}{\cosh\beta x}\,, \quad x \in \mathbb{R}\,,$$

with frequency and speed according to

$$\sigma = \sigma(\alpha,\beta) = 2(\cos\alpha\cosh\beta - 1) + \alpha c\,, \quad c = c(\alpha,\beta) = 2\sin\alpha\,\frac{\sinh\beta}{\beta}\,,$$

respectively. Since $u_{\alpha+2\pi,\beta} = u_{\alpha,\beta}$, $u_{\alpha,-\beta} = -u_{\alpha,\beta}$ and $u_{-\alpha,\beta}(-t) = \overline{u}_{\alpha,\beta}(t)$, we may assume $0 \leq \alpha \leq \pi$, $\beta > 0$ without loss of generality. As can be seen from (3), $u_{\alpha,\beta}$ moves with constant velocity c which is nonzero unless $\alpha = 0$ or $\alpha = \pi$.

2. **A necessary condition for persistence.** As outlined above, we will discuss whether some of the travelling breathers may persist (possibly slightly altered) if the AL lattice (2) is perturbed to

$$i\dot{u} = |u|^2\big(\zeta(u) + \zeta^{-1}(u)\big) + \zeta(u) - 2u + \zeta^{-1}(u) + \varepsilon X(u)\,, \tag{4}$$

where $\varepsilon > 0$, and $X : \Omega \to \Omega$ represents a continuous, shift- and rotation-equivariant perturbation, i.e. $X \circ \zeta = \zeta \circ X$ and $X \circ R_\vartheta = R_\vartheta \circ X$ for all $\vartheta \in \mathbb{R}$. We restrict our attention to this class of perturbations, else it does not make sense to look for solutions of the form (3), and one would have to generalise the definition of a travelling breather (see [5] for a discussion). Also, we tacitly assume in the sequel that solutions of (4) exist for all times $t \geq 0$.

Fix a function $S : \Omega \to \mathbb{C}$ which is (real) differentiable with respect to all coordinates $u_k, \overline{u}_k$. Later we will impose further conditions on S as appropriate. Let $u = u(t)$ be any solution of (4). For the rate of change of $s : t \mapsto S\big(u(t)\big)$ under (4) we have

$$\begin{aligned}\dot{s}(t) = \frac{d}{dt}S\big(u(t)\big) &= \sum_{k\in\mathbb{Z}}\left(\frac{\partial S}{\partial \overline{u}_k}\dot{\overline{u}}_k(t) + \frac{\partial S}{\partial u_k}\dot{u}_k(t)\right) = \\ &= i\sum_{k\in\mathbb{Z}}\left(\frac{\partial H}{\partial u_k}\frac{\partial S}{\partial \overline{u}_k} - \frac{\partial H}{\partial \overline{u}_k}\frac{\partial S}{\partial u_k}\right)\bigg|_{u(t)}(1 + |u_k(t)|^2) + i\varepsilon\sum_{k\in\mathbb{Z}}\left(\frac{\partial S}{\partial \overline{u}_k}\overline{X}_k - \frac{\partial S}{\partial u_k}X_k\right)\bigg|_{u(t)}\end{aligned} \tag{5}$$

with the Hamiltonian $H : \Omega \to \mathbb{R} \subset \mathbb{C}$ for (1) defined as

$$H(u) := \sum_{k\in\mathbb{Z}}\big(\overline{u}_k(u_{k-1} + u_{k+1}) - 2\log(1 + |u_k|^2)\big)\,. \tag{6}$$

By means of the (non-standard) Poisson bracket for differentiable functions $F, G : \Omega \to \mathbb{C}$,

$$\{F, G\} := i\sum_{k\in\mathbb{Z}}\left(\frac{\partial F}{\partial u_k}\frac{\partial G}{\partial \overline{u}_k} - \frac{\partial F}{\partial \overline{u}_k}\frac{\partial G}{\partial u_k}\right)(1 + |u_k|^2)\,, \tag{7}$$

relation (5) can be written as

$$\frac{d}{dt}S\big(u(t)\big) = \{H, S\}|_{u(t)} + i\varepsilon\sum_{k\in\mathbb{Z}}\left(\frac{\partial S}{\partial \overline{u}_k}\overline{X}_k - \frac{\partial S}{\partial u_k}X_k\right)\bigg|_{u(t)}. \tag{8}$$

In analogy to (3), we ask whether (4) has, for sufficiently small $\varepsilon > 0$, any spatially localised solution of the form

$$u_k^{(\varepsilon)}(t) = e^{-i\sigma_\varepsilon t} U_\varepsilon(k - c_\varepsilon t)\,, \quad k \in \mathbb{Z}, t \geq 0\,, \tag{9}$$

where $c_\varepsilon \neq 0$ and the functions $U_\varepsilon : \mathbb{R} \to \mathbb{C}$ satisfy $\int_{\mathbb{R}} |U_\varepsilon(x)|^2 dx \leq C$ with a constant C not depending on ε. We call such a solution $u^{(\varepsilon)} = \big(u_k^{(\varepsilon)}\big)_{k\in\mathbb{Z}}$ a *continuation* of the unperturbed solution $u_{\alpha,\beta}$ if $\lim_{\varepsilon\to 0} u^{(\varepsilon)}(t) = u_{\alpha,\beta}(t)$ holds in Ω locally uniformly in t. Equivalently, we say that $u_{\alpha,\beta}$ *persists* for $\varepsilon > 0$. It follows from (9) that for $c_\varepsilon \neq 0$

$$u_k^{(\varepsilon)}(t + c_\varepsilon^{-1}) = e^{-i\sigma_\varepsilon c_\varepsilon^{-1}} u_{k-1}^{(\varepsilon)}(t)\,, \quad k \in \mathbb{Z}, t \geq 0\,,$$

in other words, $\zeta\big(u^{(\varepsilon)}(t + c_\varepsilon^{-1})\big) = R_{-\sigma_\varepsilon c_\varepsilon^{-1}} u^{(\varepsilon)}(t)$. To utilise (8) we assume more specifically that the function S is shift- and rotation-invariant, i.e. $S \circ \zeta = S$, and $S \circ R_\vartheta = S$ for all $\vartheta \in \mathbb{R}$. Under these assumptions,

$$\int_0^{c_\varepsilon^{-1}} \Big(\frac{\partial S}{\partial \overline{u}_k}\overline{X}_k - \frac{\partial S}{\partial u_k}X_k\Big)\Big|_{u^{(\varepsilon)}(t)} dt \;=\; \int_{-kc_\varepsilon^{-1}}^{(-k+1)c_\varepsilon^{-1}} \Big(\frac{\partial S}{\partial \overline{u}_0}\overline{X}_0 - \frac{\partial S}{\partial u_0}X_0\Big)\Big|_{u^{(\varepsilon)}(t)} dt\,, \quad k \in \mathbb{Z}\,.$$

Consequently, the evaluation of (8) along $u^{(\varepsilon)}$ yields

$$\begin{aligned}
0 \;&=\; S\big(u^{(\varepsilon)}(c_\varepsilon^{-1})\big) - S\big(u^{(\varepsilon)}(0)\big)\\
&=\int_0^{c_\varepsilon^{-1}} \{H,S\}\Big|_{u^{(\varepsilon)}(t)} dt + i\varepsilon \int_0^{c_\varepsilon^{-1}} \sum_{k\in\mathbb{Z}} \Big(\frac{\partial S}{\partial \overline{u}_k}\overline{X}_k - \frac{\partial S}{\partial u_k}X_k\Big)\Big|_{u^{(\varepsilon)}(t)} dt \\
&=\int_0^{c_\varepsilon^{-1}} \{H,S\}\Big|_{u^{(\varepsilon)}(t)} dt + i\varepsilon \int_{\mathbb{R}} \Big(\frac{\partial S}{\partial \overline{u}_0}\overline{X}_0 - \frac{\partial S}{\partial u_0}X_0\Big)\Big|_{u^{(\varepsilon)}(t)} dt\,.
\end{aligned} \tag{10}$$

To allow the passage to the limit $\varepsilon \to 0$ in the last integral (and also to justify the above interchange of integration and summation) we wish to apply dominated convergence to (10). To this end, we assume that the quantities $\frac{\partial S}{\partial u_0}$, $\frac{\partial S}{\partial \overline{u}_0}$ and X_0 are bounded and locally dominated in the following sense: with a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ and a number $N \in \mathbb{N}$ chosen appropriately, the estimate

$$\max\Big\{\Big|\frac{\partial S}{\partial u_0}(u)\Big|, \Big|\frac{\partial S}{\partial \overline{u}_0}(u)\Big|, |X_0(u)|\Big\}^2 \;\leq\; \varphi(\|u\|) \sum_{k=-N}^{N} |u_k|^2 \tag{11}$$

holds for all $u \in \Omega$. If S is conserved under the dynamics of the unperturbed AL lattice, i.e., if $\{H,S\} = 0$, then we obtain from (10) the following necessary condition for the persistence of the travelling breather $u_{\alpha,\beta}$.

Proposition 1. *Assume that $u_{\alpha,\beta}$ persists for $0 < \varepsilon < \varepsilon_0$, and let S and X satisfy* (11)*. If $\{H,S\} = 0$, then*

$$\Phi_{\alpha,\beta}(S;X) := \int_{\mathbb{R}} \Big(\frac{\partial S}{\partial \overline{u}_0}\overline{X}_0 - \frac{\partial S}{\partial u_0}X_0\Big)\Big|_{u_{\alpha,\beta}(t)} dt = 0\,. \tag{12}$$

Remark 2. The term $\int_0^{c_\varepsilon^{-1}}\{H,S\}$ in (10) can be transformed to

$$i\int_{\mathbb{R}} \Big(\frac{\partial H}{\partial u_0}\frac{\partial S}{\partial \overline{u}_0} - \frac{\partial H}{\partial \overline{u}_0}\frac{\partial S}{\partial u_0}\Big)\Big|_{u_{\alpha,\beta}(t)} (1 + |u_{\alpha,\beta;0}(t)|^2)\, dt\,. \tag{13}$$

Setting $\varepsilon = 0$ in (10) we therefore obtain the interesting fact that (13) vanishes for all α, β and for *all* shift- and rotation-invariant functions S, even if they are not conserved under the dynamics of (2). For example, (13) necessarily equals zero when evaluated for the function

$$S = S(u) = \sum_{k\in\mathbb{Z}} u_k \overline{u}_k u_{k+1}^2 \overline{u}_{k+1}^2 = \sum_{k\in\mathbb{Z}} |u_k|^2 |u_{k+1}|^4\,,$$

for which obviously $S \circ \zeta = S$ and $S \circ R_\vartheta = S$, yet $\{H,S\} \neq 0$.

Condition (12) may in particular be checked for each of the well-known independent first integrals $(I_m)_{m\in\mathbb{N}_0}$ of (1), as worked out in [1]; for $m=0,\dots,3$ these integrals are

$$\begin{aligned}
I_0 &= \sum_{k\in\mathbb{Z}} \log(1+|u_k|^2)\,,\\
I_1 &= \sum_{k\in\mathbb{Z}} u_k\overline{u}_{k+1}\,,\\
I_2 &= \sum_{k\in\mathbb{Z}} \Big(\overline{u}_{k-1}u_{k+1}(1+|u_k|^2)+\frac{1}{2}\overline{u}_k^2u_{k+1}^2\Big)\,,\\
I_3 &= \sum_{k\in\mathbb{Z}} \Big(\overline{u}_{k-2}u_{k+1}(1+|u_{k-1}|^2)(1+|u_k|^2)+\\
&\qquad +(\overline{u}_{k-1}^2u_ku_{k+1}+\overline{u}_{k-1}\overline{u}_ku_{k+1}^2)(1+|u_k|^2)+\frac{1}{3}\overline{u}_k^3u_{k+1}^3\Big)\,;
\end{aligned}\tag{14}$$

notice that $H=I_1+\overline{I}_1-2I_0$.

3. **Examples.** Both the usefulness and the limitations of the simple necessary condition (12) can be seen most clearly by means of a few examples. Whenever the perturbation $X_k(u)$ is a polynomial in the coordinates $u_l,\overline{u}_l$, then the determination of $\Phi_{\alpha,\beta}(I_m;X)$ reduces to the computation of integrals of the form $\int_{-\infty}^{\infty}R(e^x)\,dx=\int_0^{\infty}y^{-1}R(y)\,dy$ with R denoting a rational function which has no poles on the real axis; this observation can be helpful for actual computations.

Example 1. Let $X_k(u)=(1+|u_k|^2)(u_{k-1}+u_{k+1})$, $k\in\mathbb{Z}$. It is easily verified that, for all α, β,

$$\Phi_{\alpha,\beta}(S;X)=0\,,$$

if S equals any of the integrals (14). Thus no information is gained from (12). This, however, is not at all surprising because all travelling breathers $u_{\alpha,\beta}$ persist under this particular perturbation: more precisely, $U_\varepsilon=U$ for all ε, and frequency and speed are perturbed according to

$$\sigma_\varepsilon=2\big((1+\varepsilon)\cos\alpha\cosh\beta-1\big)+\alpha c_\varepsilon,\quad c_\varepsilon=2(1+\varepsilon)\sin\alpha\,\frac{\sinh\beta}{\beta}\,.$$

Example 2. For $X_k(u)=-iu_k$, $k\in\mathbb{Z}$, a short calculation yields

$$\Phi_{\alpha,\beta}(I_1;X)=2ie^{i\alpha}\frac{\beta}{\sin\alpha}\,,$$

so that no travelling breather persists in this case. Again, this is not surprising as the system is *dissipative* in a sense: from (5) we deduce that

$$\frac{d}{dt}I_1\big(u^{(\varepsilon)}(t)\big)=-2\varepsilon I_1\big(u^{(\varepsilon)}(t)\big)\,,\quad t\geq 0\,,$$

implying that $I_1\big(u^{(\varepsilon)}(t)\big)=e^{-2\varepsilon t}I_1\big(u^{(\varepsilon)}(0)\big)$. For travelling breathers of the form (9) to exist therefore necessarily $I_1\big(u^{(\varepsilon)}(0)\big)=0$ for $0<\varepsilon<\varepsilon_0$. If $u^{(\varepsilon)}$ is a continuation of $u_{\alpha,\beta}$ then also $I_1\big(u_{\alpha,\beta}(0)\big)=0$, which obviously contradicts

$$I_1\big(u_{\alpha,\beta}(0)\big)=e^{i\alpha}\sum_{k\in\mathbb{Z}}\frac{\sinh^2\beta}{\cosh\beta k\,\cosh\beta(k+1)}\neq 0\,.$$

Example 3. Consider now the family of perturbations

$$X_k^{(\psi)}(u)=4iu_k-e^{i\psi}(u_{k-1}-u_{k+1})(1+|u_k|^2)\,,\quad k\in\mathbb{Z}\,,$$

where $\psi\in\mathbb{R}$. First we calculate $\Phi_{\alpha,\beta}(S;X^{(\psi)})$ with $S=I_0$,

$$\Phi_{\alpha,\beta}\big(I_0;X^{(\psi)}\big)=\frac{4i\beta}{\sin\alpha\cosh\beta}(\sin\alpha\cosh\beta\cos\psi-2)\,.$$

Accordingly, no travelling breather persists if $\cos\psi = 0$. If on the other hand $\cos\psi \neq 0$, then $u_{\alpha,\beta}$ may persist if $\sin\alpha\cosh\beta\cos\psi = 2$. For a further analysis, we evaluate (12) with $S = I_1$ and obtain

$$\Phi_{\alpha,\beta}\big(I_1; X^{(\psi)}\big) = 4e^{i\alpha}\cot\alpha\,(\sinh\beta - \beta\cosh\beta)\cos\psi + e^{i\alpha}\cosh\beta\,\Phi_{\alpha,\beta}\big(I_0; X^{(\psi)}\big)\,.$$

For $\cos\psi \neq 0$ therefore $\Phi_{\alpha,\beta}\big(I_1; X^{(\psi)}\big)$ vanishes exactly if

$$\alpha = \alpha_0 = \frac{\pi}{2}\mathrm{sign}(\cos\psi)\,, \quad \beta = \beta_0 = \cosh^{-1}\left(\frac{2}{|\cos\psi|}\right),$$

and it is easily confirmed that u_{α_0,β_0} indeed persists for $\varepsilon > 0$ with $U_\varepsilon \equiv U$ and

$$\sigma_\varepsilon = \alpha_0 c_\varepsilon - 2 + 4\varepsilon\tan\psi\,, \quad c_\varepsilon = 2\sin\alpha_0\frac{\sinh\beta_0}{\beta_0}\,;$$

here, as usual, $\mathrm{sign}(0) = 0$ and $\mathrm{sign}(x) = \frac{x}{|x|}$ for $x \neq 0$. In this example, the simple analysis of $\Phi_{\alpha,\beta}\big(I_1; X^{(\psi)}\big)$ gives a complete picture of the persistence or non-persistence of individual travelling breathers.

Example 4. In general, an answer as satisfactory as in the last example should not be expected. If, for instance,

$$X_k(u) = -|u_k|^2\big(u_{k+1} + u_{k-1} + i(u_{k+1} - u_{k-1})\big) - \frac{4u_k}{1 + |u_k|^2}\,, \quad k \in \mathbb{Z}\,,$$

then again one travelling breather survives in its original form (i.e. with $U_\varepsilon \equiv U$), namely

$$\alpha = \alpha_0 = \frac{\pi}{4}\,, \quad \beta = \beta_0 = \cosh^{-1}\sqrt{2}$$

with $\sigma_\varepsilon = \sigma - 4\varepsilon$, $c_\varepsilon = c$, but for reasons explained in Example 7 below, $\Phi_{\alpha,\beta}(S; X) \equiv 0$ for S equal to any of the first integrals (14). In this case, Proposition 1 does not give any information.

Example 5. Let the perturbation X be *(time-)reversible*, i.e. $X(\overline{u}) = \overline{X(u)}$ for all $u \in \Omega$. For example, the perturbation in Example 1 above is reversible whereas the perturbations in Examples 2, 3 and 4 are not. If X is reversible, and if u is a solution of (4), then so is $\widetilde{u} : t \mapsto \overline{u(-t)}$. Proposition 1 may or may not give useful information in the reversible case. Indeed, for the one-parameter family $X^{(\psi)}$ of reversible, shift- and rotation-equivariant perturbations

$$X_k^{(\psi)}(u) = u_k|u_k|^2 + \psi u_{k+1}|u_{k+1}|^2\,, \quad k \in \mathbb{Z}, \psi \in \mathbb{R}\,,$$

one obtains through an elementary computation

$$\Phi_{\alpha,\beta}\big(I_1; X^{(\psi)}\big) = -\frac{\psi}{24\sin\alpha\cosh^3\beta}\big(2\sinh^3 2\beta + 3e^{2i\alpha}(2\beta - \sinh 2\beta\cosh 2\beta)\big)\,. \tag{15}$$

Since the bracketed expression does not vanish for $\beta > 0$, no travelling breather persists for $\psi \neq 0$. On the other hand, $\Phi_{\alpha,\beta}\big(S; X^{(0)}\big) \equiv 0$ for S equal to any first integral (14).

In the light of (15) it is worth recalling that the calculations leading to Proposition 1 make sense only for *travelling* breathers. Due to the reversibility of $X^{(\psi)}$, *stationary* breathers – corresponding to $\sin\alpha = 0$ here – may well persist ([9]).

Example 6. We now consider perturbations which preserve the Hamiltonian structure of (2). More precisely, we assume that the individual equations in (4) can equivalently be written in the form

$$\dot{u}_k = \{H_\varepsilon, u_k\}_\varepsilon\,, \quad k \in \mathbb{Z}\,, \tag{16}$$

where $0 \leq \varepsilon < \varepsilon_0$, and both the Hamiltonian and the Poisson bracket may (smoothly) depend on ε; clearly, we require that H_0 and $\{\cdot,\cdot\}_0$ equal the unperturbed Hamiltonian H and $\{\cdot,\cdot\}$, respectively, as given by (6) and (7). We are going to show that Proposition 1 generally does not yield any information for this type of perturbation.

We first assume that $\{\cdot,\cdot\}_\varepsilon = \{\cdot,\cdot\}_0$ for $0 \le \varepsilon < \varepsilon_0$, whereas the Hamiltonian H is perturbed according to $H_\varepsilon = H + \varepsilon H^{(1)}$ with $H^{(1)} : \Omega \to \mathbb{R}$ being a differentiable shift- and rotation-invariant function. Consequently, $X_k(u) = \frac{\partial H^{(1)}}{\partial \overline{u}_k}(1+|u_k|^2)$, and an evaluation of $\Phi_{\alpha,\beta}(I_0;X)$ leads to

$$\Phi_{\alpha,\beta}(I_0;X) = \int_{\mathbb{R}} \left(u_0 \frac{\partial H^{(1)}}{\partial u_0} - \overline{u}_0 \frac{\partial H^{(1)}}{\partial \overline{u}_0} \right) \Big|_{u_{\alpha,\beta}(t)} dt = \int_0^{c(\alpha,\beta)^{-1}} \sum_{k\in\mathbb{Z}} \left(u_k \frac{\partial H^{(1)}}{\partial u_k} - \overline{u}_k \frac{\partial H^{(1)}}{\partial \overline{u}_k} \right) = 0$$

for all α, β with $c(\alpha,\beta) \neq 0$, because $H^{(1)}$ is real-valued and rotation-invariant. In order to calculate $\Phi_{\alpha,\beta}(I_m;X)$ for $m \ge 1$ we assume $H^{(1)}$ to be locally determined in the sense that

$$H^{(1)}(u) = \sum_{k\in\mathbb{Z}} f(u_k, \dots, u_{k+N}), \quad u \in \Omega, \tag{17}$$

with some $N \in \mathbb{N}$ and a rotation-invariant function $f : \mathbb{C}^{N+1} \to \mathbb{R}$. Since f together with its first derivatives can be uniformly approximated by polynomials on any compact set in $\mathbb{C}^{N+1}$, we focus on the special case

$$f(u_k, \dots, u_{k+N}) = u_k^{p_0} \overline{u}_k^{q_0} \dots u_{k+N}^{p_N} \overline{u}_{k+N}^{q_N} + \ c.c.,$$

where p_l, q_l denote nonnegative integers not all of which are zero; to ensure rotation-invariance, $\sum_{l=0}^N (p_l - q_l) = 0$. As an example, we verify that $\Phi_{\alpha,\beta}(I_1;X) \equiv 0$. In fact, as a result of an elementary calculation,

$$\begin{aligned}
\Phi_{\alpha,\beta}(I_1;X) &= \int_{\mathbb{R}} \left(u_{-1} \frac{\partial H^{(1)}}{\partial u_0} - \overline{u}_1 \frac{\partial H^{(1)}}{\partial \overline{u}_0} \right) \Big|_{u_{\alpha,\beta}(t)} (1 + |u_{\alpha,\beta;0}(t)|^2)\, dt \\
&= -\ e^{i\alpha(1+\psi)} \int_{\mathbb{R}} \sum_{l=0}^N \left(p_l \frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l-1}(t)|} - q_l \frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l+1}(t)|} \right) \Psi(t)\, dt\ + \\
&\quad +\ e^{i\alpha(1-\psi)} \int_{\mathbb{R}} \sum_{l=0}^N \left(p_l \frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l+1}(t)|} - q_l \frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l-1}(t)|} \right) \Psi(t)\, dt
\end{aligned}$$

where $\psi := \sum_{l=0}^N l(p_l - q_l)$ and $\Psi := \prod_{l=0}^N |u_{\alpha,\beta;l}|^{p_l+q_l}$. Scrutinising for example the integral multiplied by $e^{i\alpha(1+\psi)}$, we see that it is the sum of the two expressions,

$$\int_{\mathbb{R}} \sum_{l=0}^N \frac{p_l \pm q_l}{2} \left(\frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l-1}(t)|} \mp \frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l+1}(t)|} \right) \Psi(t)\, dt,$$

which have to be read, respectively, with upper and lower signs only. As a matter of fact, both expressions vanish:

$$\begin{aligned}
&\int_{\mathbb{R}} \sum_{l=0}^N \frac{p_l + q_l}{2} \left(\frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l-1}(t)|} - \frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l+1}(t)|} \right) \Psi(t)\, dt = \\
&\qquad -\frac{1}{2\sin\alpha} \int_{\mathbb{R}} \sum_{l=0}^N (p_l + q_l) \frac{|u_{\alpha,\beta;l}|'}{|u_{\alpha,\beta;l}|} \Psi(t)\, dt = -\frac{1}{2\sin\alpha} \int_{\mathbb{R}} \Psi'(t)\, dt = 0,
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}} \sum_{l=0}^N \frac{p_l - q_l}{2} \left(\frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l-1}(t)|} + \frac{|u_{\alpha,\beta;l}(t)|}{|u_{\alpha,\beta;l+1}(t)|} \right) \Psi(t)\, dt = \\
&\qquad \int_{\mathbb{R}} \sum_{l=0}^N (p_l - q_l) \cosh\beta\, \Psi(t)\, dt = \cosh\beta \sum_{l=0}^N (p_l - q_l) \int_{\mathbb{R}} \Psi(t)\, dt = 0.
\end{aligned}$$

As can be seen clearly, the reason for the first equality to hold is the algebraic relations among the individual components of $u_{\alpha,\beta}$ and their derivatives, whereas the second equality follows from $\sum_{l=0}^N (p_l - q_l) = 0$, i.e., from the rotation-invariance of f. An analogous yet increasingly laborious computation shows that $\Phi_{\alpha,\beta}(I_m;X) \equiv 0$ for all first integrals (14). Since $|u_k(t)| \le \sinh\beta$ for all $k \in \mathbb{Z}$, $t \geq 0$, the aforementioned approximation argument shows that $\Phi_{\alpha,\beta}(I_m; X) \equiv 0$ for all $m \in \mathbb{N}_0$ whenever the perturbation X is derived from the Hamiltonian (17).

A well-known Hamiltonian perturbation of the AL lattice is the so-called Salerno family

$$X_k(u) = -|u_k|^2(u_{k-1} - 2u_k + u_{k+1}), \quad k \in \mathbb{Z},$$

introduced in [11]; with $0 \leq \varepsilon < 1$ and

$$H_\varepsilon(u) = \sum_{k\in\mathbb{Z}} \left(\overline{u}_k(u_{k-1} + u_{k+1}) - \frac{2}{(1-\varepsilon)^2} \log(1 + (1-\varepsilon)|u_k|^2) + \frac{2\varepsilon}{1-\varepsilon}|u_k|^2 \right),$$

$$\{F, G\}_\varepsilon = i \sum_{k\in\mathbb{Z}} \left(\frac{\partial F}{\partial u_k}\frac{\partial G}{\partial \overline{u}_k} - \frac{\partial F}{\partial \overline{u}_k}\frac{\partial G}{\partial u_k} \right) \left(1 + (1-\varepsilon)|u_k|^2\right),$$

the individual equations in (4) are indeed equivalent to (16). This perturbation is more general than the ones studied above because the Poisson bracket also depends on ε. However, it is readily confirmed that this variation of the Poisson bracket can be compensated for by a modification of the Hamiltonian function. More precisely, one has

$$\dot{u}_k = \{H_\varepsilon, u_k\}_\varepsilon = \{\widetilde{H}_\varepsilon, u_k\}_0, \quad k \in \mathbb{Z}, \tag{18}$$

where the modified Hamiltonian $\widetilde{H}_\varepsilon$ is given by

$$\widetilde{H}_\varepsilon(u) = H_0(u) + \varepsilon \sum_{k\in\mathbb{Z}} \overline{u}_k(u_{k-1} - 2u_k + u_{k+1})h(|u_k|^2) =: H_0(u) + \varepsilon \widetilde{H}^{(1)}(u),$$

with the C^∞ function h defined as

$$h(x) := \begin{cases} x^{-1}\log(1+x) - 1 & \text{if } x > -1, x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Evidently, $\widetilde{H}^{(1)}$ is of the form (17), and consequently Proposition 1 does not give any information for the Salerno family: $\Phi_{\alpha,\beta}(S; X) \equiv 0$ for all first integrals (14).

In general, it may not be possible to represent a Hamiltonian perturbation through a perturbation of the Hamiltonian function alone as in (18), even up to $O(\varepsilon^2)$ terms. However, the prospects of fruitfully applying Proposition 1 are, as a rule, not improved by this generalisation. Specifically, let

$$H_\varepsilon = H_0 + \varepsilon H^{(1)} + O(\varepsilon^2), \quad \{\cdot,\cdot\}_\varepsilon = \{\cdot,\cdot\}_0 + \varepsilon[\cdot,\cdot] + O(\varepsilon^2).$$

Comparing (4) and (16) yields, for all $k \in \mathbb{Z}$,

$$\begin{aligned} X_k(u) &= \frac{\partial H^{(1)}}{\partial \overline{u}_k}(1 + |u_k|^2) + i[H_0, u_k] + i\varepsilon[H^{(1)}, u_k] + O(\varepsilon^2) \\ &=: X_k^{(1)}(u) + X_k^{(2)}(u) + \varepsilon X_k^{(3)}(u) + O(\varepsilon^2), \end{aligned}$$

so that X_k itself depends on ε. As explained above, $\Phi_{\alpha,\beta}\big(I_m; X^{(1)}\big) \equiv 0$ holds for all $m \in \mathbb{N}_0$ under fairly mild assumptions. If in addition $\Phi_{\alpha,\beta}\big(I_m; X^{(2)}\big) = 0$, then the ε-dependence of X rules out any conclusion in the spirit of Proposition 1 to be drawn from (10), because the different orders of ε may no longer be distinguished clearly. To make use of (10) one would have to independently evaluate

$$\frac{\partial}{\partial \varepsilon} \int_{\mathbb{R}} \left(\frac{\partial S}{\partial \overline{u}_0}\big(\overline{X^{(1)}}_0 + \overline{X^{(2)}}_0\big) - \frac{\partial S}{\partial u_0}\big(X_0^{(1)} + X_0^{(2)}\big) \right)$$

at $\varepsilon = 0$, which clearly is impossible without further information about $u^{(\varepsilon)}$. For the concrete example of the Salerno family,

$$H^{(1)}(u) = -4\sum_{k\in\mathbb{Z}} \log(1 + |u_k|^2) + 2\sum_{k\in\mathbb{Z}} \frac{2|u_k|^2 + |u_k|^4}{1 + |u_k|^2}$$

$$[F, G] = -i\sum_{k\in\mathbb{Z}} \left(\frac{\partial F}{\partial u_k}\frac{\partial G}{\partial \overline{u}_k} - \frac{\partial F}{\partial \overline{u}_k}\frac{\partial G}{\partial u_k} \right)|u_k|^2$$

and thus

$$X_k^{(2)}(u) = -\frac{\partial H_0}{\partial \overline{u}_k}|u_k|^2\,, \quad X_k^{(3)}(u) = -\frac{\partial H^{(1)}}{\partial \overline{u}_k}|u_k|^2\,, \quad k \in \mathbb{Z}\,.$$

It is easy to check that $\Phi_{\alpha,\beta}\big(I_m; X^{(2)}\big) \equiv 0$ for all $m \in \mathbb{N}_0$. Even if $\Phi_{\alpha,\beta}\big(I_1; X^{(3)}\big)$ did not vanish, this fact could not be used to deduce non-persistence of any travelling breather.

Thus to pursue the question of persistence of travelling breathers under Hamiltonian perturbations one will need to use another method. Probably results can be obtained by considering the existence problem as one of homoclinic orbits to a saddle-centre equilibrium for an associated advance-delay differential equation, as proposed by the second author several years ago. Such an approach has been implemented by Iooss and Kirchg"assner for small amplitude travelling solutions of a Klein–Gordon chain, but the analysis presented in [7] does not reach far enough to obtain conditions for solutions that go to zero at plus and minus infinity. The expected conditions are that certain quantities which describe the interaction with phonons travelling at the same phase velocity must all vanish.

Example 7. It is an all-important property of the breather solutions $u_{\alpha,\beta}$ of (1) that they are invariant under the skew-flip combined with a time-reversal, that is,

$$u_{\alpha,\beta}(t) = \eta u_{\alpha,\beta}(-t)$$

holds for all α, β and $t \in \mathbb{R}$. Also, if S equals any of the first integrals (14) then $S \circ \eta = S$. This latter property implies that

$$\frac{\partial S}{\partial u_k} \circ \eta = \frac{\partial S}{\partial \overline{u}_{-k}}\,, \quad \frac{\partial S}{\partial \overline{u}_k} \circ \eta = \frac{\partial S}{\partial u_{-k}}\,, \quad k \in \mathbb{Z}\,.$$

Assume now that the perturbation X satisfies $X \circ \eta = \eta \circ X$. In this case

$$\Phi_{\alpha,\beta}(S;X) = \int_{\mathbb{R}} \left(\left(\frac{\partial S}{\partial u_0} X_0 \right) \circ \eta - \frac{\partial S}{\partial u_0} X_0 \right) \Big|_{u_{\alpha,\beta}(t)} = \int_{\mathbb{R}} \frac{\partial S}{\partial u_0} X_0|_{u_{\alpha,\beta}(-t)} - \int_{\mathbb{R}} \frac{\partial S}{\partial u_0} X_0|_{u_{\alpha,\beta}(t)} = 0$$

for all α, β, provided that $S \circ \eta = S$. Therefore, if the function S and the perturbation X are, respectively, invariant and equivariant under the skew-flip, then Proposition 1 does not provide any information. It is easily checked that in this case, with u being a solution of (4), the Ω-valued function

$$\widetilde{u} : t \mapsto \eta u(-t)$$

defines a solution of (4), too. To any travelling breather thus corresponds a breather of skew-flipped shape which travels into the opposite direction with the same speed. These two solutions cannot be distinguished by means of $\Phi_{\alpha,\beta}(S;X)$. As in the case of Hamiltonian perturbations, ruling out the persistence of travelling breathers in this more symmetric (and hence more degenerate) situation will require additional tools. This observation pertains to Examples 1, 4, the $\psi = 0$ case in Example 5 as well as the Salerno family.

Remark 3. For the perturbations studied in this article, $\Phi_{\alpha,\beta}(S;X)$ vanishes identically for any first integral S from (14) only if either X is Hamiltonian or $X \circ \eta = \eta \circ X$. (Both properties hold simultaneously for the Salerno family.) Thus besides Hamiltonian perturbations of (2) skew-flip equivariant perturbations apparently play a very special role for the AL lattice. In this context it is interesting to note that skew-flip symmetry is analogous to CPT symmetry in quantum field theory (see [13]), believed to be an inviolable symmetry of nature.

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