Existence and Stability of 3-site Breathers in a Triangular Lattice

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Abstract. We find conditions for existence and stability of various types of Discrete Breather concentrated around three central sites in a triangular lattice of one-dimensional Hamiltonian oscillators with on-site potential and nearest-neighbor coupling. In particular, we confirm that it can support non-reversible breather solutions, despite the time-reversible character of the system. They carry a net energy flux and can be called "vortex breathers". We prove that there are parameter regions for which they are linearly stable, for example in a lattice consisting of coupled Morse oscillators, whereas the related reversible breathers are unstable. Thus non-reversible breathers can be physically relevant.

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1. Introduction

A large amount of work has been done recently in the field of Discrete Breathers i.e. spatially localised time-periodic motions in extended systems of coupled oscillators, since the numerical evidence of existence of this kind of motion in [14] and precursors. Apart from systems with a phase rotation symmetry, the first existence proof of breathers in discrete systems was provided in [9]. This paper concentrated mainly on time-reversible breathers, but a setting which could support non-reversible solutions was also proposed, specifically, a two-dimensional triangular lattice (figure 1) consisting of one degree of freedom Hamiltonian oscillators with nearest-neighbor coupling through the coupling constant ε. The Hamiltonian of the full system is

$$H = H_0 + \varepsilon H_1 = \sum_{i,j=-\infty}^{\infty} \frac{p_{ij}^2}{2} + V(x_{ij}) + \frac{\varepsilon}{2} \sum_{i,j=-\infty}^{\infty} \Big\{ (x_{ij} - x_{i-1,j})^2 + (x_{ij} - x_{i-1,j+1})^2 + \\ + \ (x_{ij} - x_{i,j-1})^2 + (x_{ij} - x_{i,j+1})^2 + (x_{ij} - x_{i+1,j-1})^2 + (x_{ij} - x_{i+1,j})^2 \Big\},$$

and it is assumed that the potential V of the oscillator possesses a stable equilibrium at $(x, p) = (0, 0)$, with $V''(0) = \omega_{ph}^2 > 0$. It is time-reversible with respect to the involution $p \mapsto -p$.

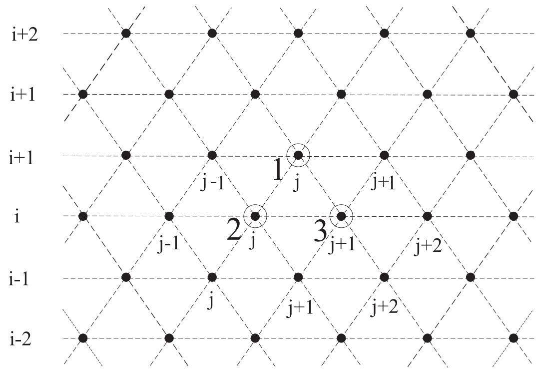

Figure 1. The lattice

It was proposed in [9] that this lattice supports non-reversible space-localised and time-periodic motions, carrying a non-zero mean energy flux, in the form of a rotating wave, which we will call here a "vortex breather". Here we prove this claim, in suitable parameter regions, and furthermore prove that they can be linearly stable. Thus non-reversible breathers, in particular vortex breathers, can be physically significant. Indeed, some have recently been observed in experiments on interacting optical waveguides [12, 4]. Our analytical work complements the numerical work of Cretegny and Aubry who computed more general non-reversible multibreathers [2, 3] and also found them sometimes to be linearly stable. Similar results have been obtained by Kevrekidis and Malomed [5, 11], who called such breathers "discrete vortex solitons", but their analysis is limited to the special case of DNLS systems.

2. The method of proof of existence of breathers

The strategy of existence proof for vortex breathers envisaged in [9] was to restrict attention to functions of time on the lattice with the property that $x_{R(s)}(t) = x_s(t+T/3)$ for all sites s and times t, where R is rotation of the lattice by $2\pi/3$ about the centre of a chosen triangle of sites and T a candidate period, and continue the obvious 3-site solution from the uncoupled limit. This restriction removes the relative phase-shift degeneracy of the general uncoupled multibreather, making the continuation problem non-degenerate.

To allow for computation of their stability, however, and also to make a simultaneous treatment of all symmetry types and to permit potential generalisation to situations with no particular spatial symmetry, here we use a general method for existence of multibreathers, which determines how the coupling resolves the relative phase-shift degeneracy, following the lines of [6, 1, 10, 8].

In the limit $\varepsilon \to 0$, we consider the three encircled oscillators of figure 1 moving in identical periodic orbits with period T but in arbitrary phases (rationally related periods could also be considered, but we intend to make use of the $2\pi/3$ rotation symmetry of the lattice). From now on we call these oscillators "central" and denote them by the indices $1, 2, 3$. This state defines a trivially localised and time-periodic motion with period T. We seek conditions under which this motion can be continued for $\varepsilon \neq 0$ to provide a breather, and we seek to determine the linear stability of the resulting solutions. We apply the action-angle canonical transformation to the central oscillators. The system is described now by the set of variables $(x_{ij}, p_{ij}, w_k, I_k)$ with $k \in \mathbb{S}$ and $i, j \in \mathbb{Z}^2 \setminus \mathbb{S}$ where $\mathbb{S}$ is the set of "central" oscillators. So the above mentioned unperturbed periodic orbit is described at time t by $z_0(t) = (x_{ij}(t), p_{ij}(t), w_k(t), I_k(t))$ with $z_0(t+T) = (x_{ij}(t), p_{ij}(t), w_k(t) + 2\pi, I_k(t))$.

In [1] (extended in [8] and [10]) it is proven that under the non-resonance condition $2\pi n/T \neq \omega_{ph} \forall n \in \mathbb{Z}$ there is an effective Hamiltonian H^{eff} whose critical points correspond to periodic orbits (in fact breathers) of the full system for ε small enough. The effective Hamiltonian is defined by

$$H^{\text{eff}}(J_1, J_2, A, \phi_1, \phi_2) = \frac{1}{T} \oint H \circ z(t) \; \mathrm{d}t,$$

where z is a periodic path in the phase space obtained by a continuation procedure for given relative phases ϕ, relative momenta J and symplectic "area" A. In the lowest order of approximation, the unperturbed orbit z_0 can be taken for z. In our case, this coincides with the averaged Hamiltonian over an angle, for example $w_3 = \omega_3 t + w_{3_0}$, due to the linear relationship of w_3 with t. Since, by construction, the resulting effective Hamiltonian does not depend on the selected angle w_3, a canonical transformation to the "central" oscillators is induced

$$\begin{array}{rclcrcl}
\vartheta &=& w_3 & \qquad & A &=& I_1 + I_2 + I_3 \\
\phi_1 &=& w_1 - w_3 & & J_1 &=& I_1 \\
\phi_2 &=& w_2 - w_3 & & J_2 &=& I_2
\end{array}$$

and, in the lowest order of approximation, the effective Hamiltonian becomes

$$H^{\mathrm{eff}} = H_0(J_1, J_2, A) + \varepsilon\langle H_1\rangle(J_1, J_2, A, \phi_1, \phi_2), \tag{1}$$

where

$$\langle H_1\rangle = \frac{1}{T}\oint H_1 \mathrm{d}t$$

which coincides with $\langle H_1\rangle_{w_3}$, the average value of H_1 over the angle w_3 . Note that since H^{eff} is independent of ϑ, A is a constant of motion.

As we have already mentioned, the critical points of this effective Hamiltonian correspond to breathers. But for non-degenerate critical points, to leading order in ε this condition reduces to the conditions

$$\frac{\partial\langle H_1\rangle}{\partial\phi_i} = 0 \quad i = 1, 2 \ ,$$

which coincides with the result of [6].

The linear stability of the fixed point of H^{eff} determines also the linear stability of the breather. This is proven in [1] for the first order approximation to H^{eff}, under the assumption of distinct eigenvalues of the first order matrix, and in [8] for the general case. The linear stability of this point is determined by the eigenvalues of the matrix $E = \Omega D^2 H^{\mathrm{eff}}$, where $\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and I the 2×2 identity matrix. More specifically, the form of the stability matrix is

$$E = \begin{pmatrix} -\frac{\partial^2 H^{\mathrm{eff}}}{\partial\phi_1\partial J_1} & -\frac{\partial^2 H^{\mathrm{eff}}}{\partial\phi_1\partial J_2} & -\frac{\partial^2 H^{\mathrm{eff}}}{\partial\phi_1^2} & -\frac{\partial^2 H^{\mathrm{eff}}}{\partial\phi_1\partial\phi_2} \\ -\frac{\partial^2 H^{\mathrm{eff}}}{\partial\phi_2\partial J_1} & -\frac{\partial^2 H^{\mathrm{eff}}}{\partial\phi_2\partial J_2} & -\frac{\partial^2 H^{\mathrm{eff}}}{\partial\phi_2\partial\phi_1} & -\frac{\partial^2 H^{\mathrm{eff}}}{\partial\phi_2^2} \\ \frac{\partial^2 H^{\mathrm{eff}}}{\partial J_1^2} & \frac{\partial^2 H^{\mathrm{eff}}}{\partial J_1\partial J_2} & \frac{\partial^2 H^{\mathrm{eff}}}{\partial J_1\partial\phi_1} & \frac{\partial^2 H^{\mathrm{eff}}}{\partial J_1\partial\phi_2} \\ \frac{\partial^2 H^{\mathrm{eff}}}{\partial J_2\partial J_1} & \frac{\partial^2 H^{\mathrm{eff}}}{\partial J_2^2} & \frac{\partial^2 H^{\mathrm{eff}}}{\partial J_2\partial\phi_1} & \frac{\partial^2 H^{\mathrm{eff}}}{\partial J_2\partial\phi_2} \end{pmatrix}$$

or to first order in ε, by taking (1) into consideration,

$$E = \begin{pmatrix} -\varepsilon\frac{\partial^2\langle H_1\rangle}{\partial\phi_1\partial J_1} & -\varepsilon\frac{\partial^2\langle H_1\rangle}{\partial\phi_1\partial J_2} & -\varepsilon\frac{\partial^2\langle H_1\rangle}{\partial\phi_1^2} & -\varepsilon\frac{\partial^2\langle H_1\rangle}{\partial\phi_1\partial\phi_2} \\ -\varepsilon\frac{\partial^2\langle H_1\rangle}{\partial\phi_2\partial J_1} & -\varepsilon\frac{\partial^2\langle H_1\rangle}{\partial\phi_2\partial J_2} & -\varepsilon\frac{\partial^2\langle H_1\rangle}{\partial\phi_2\partial\phi_1} & -\varepsilon\frac{\partial^2\langle H_1\rangle}{\partial\phi_2^2} \\ \frac{\partial^2 H_0}{\partial J_1^2} + \varepsilon\frac{\partial^2\langle H_1\rangle}{\partial J_1^2} & \frac{\partial^2 H_0}{\partial J_1\partial J_2} + \varepsilon\frac{\partial^2\langle H_1\rangle}{\partial J_1\partial J_2} & \varepsilon\frac{\partial^2\langle H_1\rangle}{\partial J_1\partial\phi_1} & \varepsilon\frac{\partial^2\langle H_1\rangle}{\partial J_1\partial\phi_2} \\ \frac{\partial^2 H_0}{\partial J_2\partial J_1} + \varepsilon\frac{\partial^2\langle H_1\rangle}{\partial J_2\partial J_1} & \frac{\partial^2 H_0}{\partial J_2^2} + \varepsilon\frac{\partial^2\langle H_1\rangle}{\partial J_2^2} & \varepsilon\frac{\partial^2\langle H_1\rangle}{\partial J_2\partial\phi_1} & \varepsilon\frac{\partial^2\langle H_1\rangle}{\partial J_2\partial\phi_2} \end{pmatrix}. \tag{2}$$

If the eigenvalues of matrix (2) lie on the imaginary axis and are simple to first order in ε (or have definite "signature", to be explained later) then for small enough ε the discrete breather is linearly stable.

3. Analysis for a lattice with a general on-site potential

3.1. Existence of solutions

We consider a lattice like that described in section 1, without specifying the potential of the oscillators. The solution of an uncoupled one-dimensional oscillator is then described by

$$x(t) = \sum_{n=0}^{\infty} A_n(I) \cos nw = \sum_{n=0}^{\infty} A_n(I) \cos\left[n(\omega t + \vartheta)\right] \tag{3}$$

In order to calculate the breather solutions on this lattice we have to calculate first the average value of H_1 along the unperturbed periodic orbit described in the previous section. Now

$$H_1 = \frac{1}{2}\left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_1 - x_3)^2\right] = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_1 x_3 - x_2 x_3.$$

Since the squared terms are independent of ϕ_i we will see they do not contribute, so we deal only with the mixed terms. We calculate first

$$I_1 = \int_0^T x_1 x_3 \mathrm{d}t,$$

and drop the terms which are independent of ϕ_i. Using (3) this becomes

$$\begin{aligned}
I_1 &= \int_0^T \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} A_{1n}(J_1)\cos(nw_1) A_{3s}(J_3)\cos(sw_3)\mathrm{d}t \\
&= \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} A_{1n}(J_1) A_{3s}(J_3) \int_0^T \cos[n(\omega_1 t + w_{1_0})]\cos[s(\omega_3 t + w_{3_0})]\mathrm{d}t \\
&= \sum_{n=1}^{\infty}\sum_{s=1}^{\infty} \frac{A_{1n}A_{3s}}{2} \int_0^T \left(\{\cos[n(\omega_1 t + w_{1_0}) + s(\omega_3 t + w_{3_0})] + \cos[n(\omega_1 t + w_{1_0}) - s(\omega_3 t + w_{3_0})]\right\} \mathrm{d}t
\end{aligned}$$

and since $\omega_1 = \omega_3$ by choice,

$$I_1 = \sum_{n=1}^{\infty} \frac{A_{1n}A_{3n}}{2} \int_0^T \cos[n(w_{1_0} - w_{3_0})]\mathrm{d}t = \sum_{n=1}^{\infty} \frac{A_{1n}A_{3n}T}{2}\cos(n\phi_1),$$

where as we have already mentioned $\phi_1 = w_1 - w_3 = w_{1_0} - w_{3_0}$. In the same way, by putting $\phi_2 = w_2 - w_3 = w_{2_0} - w_{3_0}$ and $\phi_3 = w_2 - w_1 = w_{2_0} - w_{1_0} = \phi_2 - \phi_1$, we calculate

$$I_2 = \sum_{n=1}^{\infty} \frac{A_{2n}A_{3n}T}{2}\cos(n\phi_2)$$

and

$$I_3 = \sum_{n=1}^{\infty} \frac{A_{1n}A_{2n}T}{2}\cos(n\phi_3).$$

So, we finally obtain

$$\langle H_1 \rangle = C(J) - \frac{1}{2}\left\{\sum_{n=1}^{\infty} A_{1n}A_{3n}\cos n\phi_1 + A_{2n}A_{3n}\cos n\phi_2 + A_{1n}A_{2n}\cos n\phi_3\right\}.$$

Note that the three angle variables are not independent. In the following we assume that ϕ_3 depends on the other two according to the relation $\phi_3 = \phi_2 - \phi_1$. Bearing that in mind and using the fact that $I_1 = I_2 = I_3$ and consequently $A_{1n} = A_{2n} = A_{3n} = A_n$, the condition for periodic orbits to leading order in ε becomes

$$\begin{aligned}
\frac{\partial \langle H_1 \rangle}{\partial \phi_1} &= \frac{1}{2}\sum_{n=1}^{\infty} nA_n^2\left(\sin n\phi_1 - \sin n\phi_3\right) &= 0 \\
\frac{\partial \langle H_1 \rangle}{\partial \phi_2} &= \frac{1}{2}\sum_{n=1}^{\infty} nA_n^2\left(\sin n\phi_2 + \sin n\phi_3\right) &= 0.
\end{aligned}$$

This is satisfied for all choices of harmonic content if $\forall n \in \mathbb{N}$,

$$\begin{aligned}
\sin(n\phi_1) - \sin[n(\phi_2 - \phi_1)] &= 0 \\
\sin(n\phi_2) + \sin[n(\phi_2 - \phi_1)] &= 0.
\end{aligned}$$

This system has at least the solutions

$$\begin{array}{ll}
\phi_1 = 0 & \phi_2 = 0 \\
\phi_1 = 0 & \phi_2 = \pi \\
\phi_1 = \pi & \phi_2 = 0 \\
\phi_1 = \pi & \phi_2 = \pi
\end{array}$$

which correspond to time-reversible breather-solutions, and the solutions

$$\begin{array}{ll}
\phi_1 = \frac{2\pi}{3} & \phi_2 = \frac{4\pi}{3} \\
\phi_1 = \frac{4\pi}{3} & \phi_2 = \frac{2\pi}{3}
\end{array}$$

which correspond to non–reversible breather solutions. Under non-degeneracy assumptions, these leading order in ε solutions have unique continuations to true solutions. The true solutions have the same symmetry properties as the leading order ones, by uniqueness of the continuation. Note that due to the symmetry of the system under rotation by an angle $\delta = \frac{2\pi}{3}$, the time-reversible solutions of the system can be grouped in two classes. The first has the "central" oscillators moving in phase and the second has one of the "central" oscillators moving in anti-phase with the other two. Depending on the type of the oscillator (i.e. the sequence (A_n)), we could have other solutions also, but we will not examine that possibility here.

3.2. Stability of the calculated solutions

In order to evaluate the linear stability of the solutions, we need to evaluate the eigenvalues of the stability matrix E. Bearing always in mind that $\phi_3 = \phi_2 - \phi_1$, its various components are calculated as

$$\frac{\partial^2 \langle H_1 \rangle}{\partial \phi_1^2} = \frac{1}{2} \sum_{n=1}^{\infty} n^2 A_n^2 \left(\cos n\phi_1 + \cos n\phi_3 \right) = f_1 + f_3$$

$$\frac{\partial^2 \langle H_1 \rangle}{\partial \phi_2^2} = \frac{1}{2} \sum_{n=1}^{\infty} n^2 A_n^2 \left(\cos n\phi_2 + \cos n\phi_3 \right) = f_2 + f_3$$

$$\frac{\partial^2 \langle H_1 \rangle}{\partial \phi_1 \partial \phi_2} = -\frac{1}{2} \sum_{n=1}^{\infty} n^2 A_n^2 \left(\cos n\phi_3 \right) = -f_3$$

with

$$f_i = f(\phi_i) = \frac{1}{2} \sum_{n=1}^{\infty} n^2 A_n^2 \left(\cos n\phi_i \right).$$

We have also

$$\begin{aligned} \frac{\partial^2 H_0}{\partial J_1^2} &= 2c \\ \frac{\partial^2 H_0}{\partial J_1 \partial J_2} &= c \\ \frac{\partial^2 H_0}{\partial J_2^2} &= 2c \end{aligned}$$

with

$$c = \frac{\mathrm{d}\omega}{\mathrm{d}J}.$$

On the other hand if we put

$$g_i = g(\phi_i) = \sum_{n=1}^{\infty} n A_n \frac{\partial A_n}{\partial I} \sin n\phi_i$$

then,

$$\begin{aligned} \frac{\partial^2 \langle H_1 \rangle}{\partial \phi_1 \partial J_1} &= g_3 \\ \frac{\partial^2 \langle H_1 \rangle}{\partial \phi_1 \partial J_2} &= g_1 + g_3 \\ \frac{\partial^2 \langle H_1 \rangle}{\partial \phi_2 \partial J_1} &= g_2 - g_3 \\ \frac{\partial^2 \langle H_1 \rangle}{\partial \phi_2 \partial J_2} &= -g_3 \end{aligned}$$

So the stability matrix (2) becomes

$$E = \begin{pmatrix} -\varepsilon g_3 & -\varepsilon(g_1 + g_3) & -\varepsilon(f_1 + f_3) & \varepsilon f_3 \\ -\varepsilon(g_2 - g_3) & \varepsilon g_3 & \varepsilon f_3 & -\varepsilon(f_2 + f_3) \\ 2c + \varepsilon \frac{\partial^2 \langle H_1 \rangle}{\partial J_1^2} & c + \varepsilon \frac{\partial^2 \langle H_1 \rangle}{\partial J_1 \partial J_2} & \varepsilon g_3 & \varepsilon(g_2 - g_3) \\ c + \varepsilon \frac{\partial^2 \langle H_1 \rangle}{\partial J_1 \partial J_2} & 2c + \varepsilon \frac{\partial^2 \langle H_1 \rangle}{\partial J_2^2} & \varepsilon(g_1 + g_3) & -\varepsilon g_3 \end{pmatrix} \qquad (4)$$

The eigenvalues of the matrix for the first class of time-reversible solutions with $\phi_1 = \phi_2 = 0$ are to leading order of approximation

$$\lambda_{1,2,3,4} = \pm\sqrt{-3cf(0)}\sqrt{\varepsilon} + O(\varepsilon).$$

Since

$$f(0) = \frac{1}{2}\sum_{n=1}^{\infty} n^2 A_n^2 > 0$$

the sign of the anharmonicity c determines the linear stability of the breather. If $\varepsilon c < 0$ the specific solution is unstable, while for $\varepsilon c > 0$ the leading order calculation suggests linear stability but since up to this order the eigenvalues have multiplicity two, which can lead to complex instability, we have to perform a symplectic signature analysis as is described in [7]. For the specific case, with $f(0) = f$ it is

$$D^2 H^{\text{eff}} = \begin{pmatrix} 2c & c & 0 & 0 \\ c & 2c & 0 & 0 \\ 0 & 0 & 2\varepsilon f & -\varepsilon f \\ 0 & 0 & -\varepsilon f & 2\varepsilon f \end{pmatrix},$$

which leads to the quadratic form

$$\delta^2 H = \frac{3}{2}c\left(J_1 + J_2\right)^2 + \frac{1}{2}c\left(J_1 - J_2\right)^2 + \frac{1}{2}\varepsilon f\left(\phi_1 + \phi_2\right)^2 + \frac{3}{2}\varepsilon f\left(\phi_1 - \phi_2\right)^2,$$

which is definite if $\varepsilon c > 0$, and thus the breather remains linearly stable for all small perturbations.

For the other class of solutions with at least one $\phi_i = \pi$ the eigenvalues of the stability matrix E are

$$\lambda_{1,2} = \pm\sqrt{-c(2f(0) + f(\pi))}\sqrt{\varepsilon} + O(\varepsilon), \quad \lambda_{3,4} = \pm\sqrt{-3cf(\pi)}\sqrt{\varepsilon} + O(\varepsilon).$$

Since

$$f(\pi) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n n^2 A_n^2$$

we cannot know its sign in general, but from the above we conclude that

$$2f(0) + f(\pi) > 0.$$

So in order to have both pairs of eigenvalues in the imaginary axis, i.e. this class of solutions to be linearly stable, we need $f(\pi)$ and $\varepsilon c > 0$.

The eigenvalues of the matrix for the non-reversible solutions are for both cases

$$\lambda_{1,2,3,4} = \pm i\sqrt{3cf}\sqrt{\varepsilon} + O(\varepsilon)$$

with

$$f = f\left(\frac{2\pi}{3}\right) = f\left(\frac{4\pi}{3}\right) = \sum_{n=1}^{\infty}(-1)^n n^2 A_n^2 \cos\left(\frac{n\pi}{3}\right)$$

and

$$g = g\left(\frac{2\pi}{3}\right) = -g\left(\frac{4\pi}{3}\right) = \sum_{n=1}^{\infty}(-1)^{n+1} n A_n \frac{\partial A_n}{\partial I}\sin\left(\frac{n\pi}{3}\right)$$

The leading order of these eigenvalues implies linear stability if $\varepsilon c f > 0$, but since, up to this order of approximation, the multiplicity of the eigenvalues is two we perform a symplectic signature analysis. Now

$$\left.\begin{array}{l} \phi_2 = \dfrac{4\pi}{3} \\ \phi_1 = \dfrac{2\pi}{3} \end{array}\right\} \Rightarrow \phi_3 = \phi_2 - \phi_1 = \frac{2\pi}{3},$$

so for this solution

$$f_1 = f_2 = f_3 = f, \quad g_1 = g_3 = g \quad \text{and} \quad g_2 = -g.$$

So $D^2 H^{\text{eff}}$ becomes

$$D^2 H^{\text{eff}} = \begin{pmatrix} 2c & c & \varepsilon g & -2\varepsilon g \\ c & 2c & 2\varepsilon g & -\varepsilon g \\ \varepsilon g & 2\varepsilon g & 2\varepsilon f & -\varepsilon f \\ -2\varepsilon g & -\varepsilon g & -\varepsilon f & 2\varepsilon f \end{pmatrix}.$$

Consequently the corresponding quadratic form is

$$\delta^2 H = 2cJ_1^2 + 2cJ_1J_2 + 2cJ_2^2 + 2\varepsilon g J_1\phi_1 + 4\varepsilon g J_2\phi_1 + 2\varepsilon f\phi_1^2 - 4\varepsilon g J_1\phi_2 - 2\varepsilon g J_2\phi_2 - 2\varepsilon f\phi_1\phi_2 + 2\varepsilon f\phi_2^2,$$

which can be expressed as

$$\begin{aligned} \delta^2 H = {} & \frac{c}{2}\left(2J_1 + J_2 + \frac{\varepsilon g}{c}(\phi_1 - 2\phi_2)\right)^2 + \frac{3c}{2}\left(J_2 + \frac{\varepsilon g\phi_1}{c}\right)^2 + \frac{\varepsilon}{2}\left(f - \frac{\varepsilon g^2}{c}\right)(\phi_1 + \phi_2)^2 \\ & + \frac{3\varepsilon}{2}\left(f - \frac{\varepsilon g^2}{c}\right)(\phi_1 - \phi_2)^2, \end{aligned}$$

which is definite if $\varepsilon f c > 0$ and ε is small enough, and so the discrete breather remains linearly stable for all small perturbations. Note that g is not used to extract our results, so we don't need to compute it, as we can see also in the following example.

4. The Morse oscillator

We apply the previous analysis to a triangular lattice consisting of Morse oscillators. The Hamiltonian of the Morse oscillator is

$$H_M = \frac{p^2}{2} + \left(e^{-x} - 1\right)^2.$$

The solution of the corresponding system in the domain of bounded motion is

$$x(t) = \ln\left\{\frac{1 - \sqrt{E}\cos\left(\sqrt{2(1-E)}t + \vartheta\right)}{1 - E}\right\}.$$

The Fourier series of this solution is, [13]

$$x(t) = C_0 - 2\sum_{s=1}^{\infty} s^{-1} e^{-sa} \cos\left[s(\sqrt{2(1-E)}\ t + \vartheta)\right],$$

where E is the pertinent value of the total energy for the specific orbit, $\cosh a = E^{-\frac{1}{2}}$ and C_0 a factor independent of t. By applying the method of the previous section, we obtain for the average value of H_1

$$\langle H_1 \rangle = 2\left\{\int \arctan\frac{\sin\phi_1}{z_1 - \cos\phi_1}\mathrm{d}\phi_1 + \int \arctan\frac{\sin\phi_2}{z_2 - \cos\phi_2}\mathrm{d}\phi_2 + \int \arctan\frac{\sin\phi_3}{z_3 - \cos\phi_3}\mathrm{d}\phi_3\right\} + C(J),$$

where as before $\phi_3 = \phi_2 - \phi_1$, $C(J)$ is independent of ϕ, and

$$z_1 = e^{a_1 + a_3}, z_2 = e^{a_2 + a_3}, z_3 = e^{a_1 + a_2}.$$

Since the central oscillators are moving in identical orbits we have $a_i = a$ and $z_i = z = e^{2a}$. The only critical points of the corresponding effective Hamiltonian are those we mentioned in the previous section. Next, we calculate the components of the stability matrix for the specific case. Let

$$f_i = 2\frac{z\cos\phi_i - 1}{1 + z^2 - 2z\cos\phi_1},$$

then

$$\begin{aligned}
\frac{\partial^2 \langle H_1 \rangle}{\partial \phi_1^2} &= f_1 + f_3 \\
\frac{\partial^2 \langle H_1 \rangle}{\partial \phi_1 \partial \phi_2} &= -f_3 \\
\frac{\partial^2 \langle H_1 \rangle}{\partial \phi_2^2} &= f_2 + f_3.
\end{aligned}$$

The form of the unperturbed part of the Hamiltonian, in action-angle variables, is

$$H_0(I_1, I_2, I_3) = \frac{1}{2}(2\sqrt{2}I_1 + 2\sqrt{2}I_2 + 2\sqrt{2}I_3 - I_1^2 - I_2^2 - I_3^2).$$

So,

$$\begin{aligned}
\frac{\partial^2 H_0}{\partial J_1^2} &= -2 \\
\frac{\partial^2 H_0}{\partial J_1 \partial J_2} &= -1 \\
\frac{\partial^2 H_0}{\partial J_2^2} &= -2.
\end{aligned}$$

Then the form of the stability matrix for the Morse oscillator coincides with that of (4). The eigenvalues for the reversible solutions are to leading order of approximation, for the first class of time-reversible solutions

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{6}{z-1}}\sqrt{\varepsilon} + O(\varepsilon),$$

while for the second they are

$$\lambda_{1,2} = \pm\sqrt{2\frac{z+3}{z^2-1}}\sqrt{\varepsilon} + O(\varepsilon),\ \ \lambda_{3,4} = \pm i\sqrt{\frac{6}{1+z}}\sqrt{\varepsilon} + O(\varepsilon).$$

This means that the time-reversible solutions are unstable. The eigenvalues of the matrix E for the non-reversible solutions are

$$\lambda_{1,2,3,4} = \pm i\sqrt{3\frac{2+z}{1+z+z^2}}\sqrt{\varepsilon} + O(\varepsilon),$$

which means, using also the results of the previous section, that these solutions are linearly stable.

5. Conclusions

We proved the existence of various 3-site breathers in this triangular lattice, both time-reversible and non-reversible; the latter are “vortex breathers”. After that we examined their linear stability and extracted general stability criteria. In particular, we found that the rotating wave solutions in the Morse-triangular lattice are linearly stable, while the time-reversible ones are unstable. So, in that specific case, the non-reversible solutions are the important ones.

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