Evaluation of the Dedekind Eta Function

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Introduction

Over the years a vast number of mathematicians have made contributions to the area of mathematics known as *class field theory*. In a nutshell, class field theory attempts to classify the Abelian extensions of an algebraic number field based on information contained in the base field itself.

In the early years outstanding contributions were made by none less than Kronecker, Weber, Hilbert and others of equal fame. What may be added then, it may be asked, to this now very large and classical body of knowledge? But recent years have seen renewed interest in that part of the theory which is called *explicit class field theory*.

I say renewed interest since the topic is not new. People such as Weber who were instrumental in developing the general theory were also interested in making it explicit. But the advent of computer technology has seen a large number of papers published on the topic. This has led to new developments in actually computing class field extensions.

This thesis revisits some of the elementary techniques that have been devised to explicitly generate class fields of imaginary quadratic number fields. This part of the theory is rich due to the connection with elliptic curves having complex multiplication. More specifically this thesis is dedicated to the explicit determination of class field generators arising from the evaluation of the Dedekind eta function at points in imaginary quadratic number fields. By explicit evaluation is meant, ultimately, the computation of a minimum polynomial of such a generator.

But again one asks, what can be added to a field so well established as complex multiplication? Did not even Weber master this subject? However it is precisely the work of Weber which motivates the study which follows - that and some more recent developments arising from the work of Alf van der Poorten (the author's doctoral supervisor), Kenneth Williams and Robin Chapman.

Chapter zero discusses the latter work, effectively introducing the topic of eta evaluations and recounting what has been accomplished in recent years. This starting point for our investigations revolves around the development of generalizations of the Chowla-Selberg formula. This formula expresses a product of eta functions over a set of 'inequivalent' points in an imaginary quadratic number field in terms of a product of gamma functions. By inequivalent here we mean that the eta functions are evaluated at roots of a complete set of reduced binary quadratic forms of a given discriminant.

The literature contains numerous more specific versions of the Chowla-Selberg formula including a version which covers just a single genus of reduced binary quadratic forms.

From this starting point Williams and van der Poorten attempted to obtain an expression for an eta evaluation made at a root of a single binary quadratic form. This they were essentially able to do, breaking the Chowla-Selbeg formula up altogether. However the expression they obtained involved various L-series which needed to be evaluated.

Chapman and van der Poorten eventually found ways of evaluating some of these L-series and their result expressed eta values in terms of a product of gamma functions and various units lying in certain class fields of the underlying imaginary quadratic number field. Furthermore if quotients of eta functions were then taken, the gamma factors cancelled and left behind only the algebraic units. Thus began this project which we can say more specifically is dedicated to evaluation of eta quotients at quadratic irrationalities.

The main novelty of chapter zero is an idealic version of the main part of Chapman and van der Poorten's result. This version of events is more convenient and is referred to in later chapters.

Also provided of course are statements of numerous well known results from the literature which are referred to in later chapters. Finally for this preliminaries chapter an interesting analogy with L-series associated to cyclotomic number fields and units thereof is noted. By the end of chapter zero many of the key concepts in this research, L-series, the Kronecker limit formula and eta quotients, have all featured.

Chapter one is essentially a study devoted to obtaining specific information about the units which come from eta quotients by making use of properties of the associated L-series' and the limit formula. The results of this chapter, above all, give us a feel for what it is we are trying to evaluate.

Perhaps the most interesting development is an apparent relationship between resolvents of logarithms of eta units and various L-series. The specific conjectures made here are proved at least in the case where the class number of the underlying imaginary quadratic field is five.

Next class invariants and some associated results are introduced which enable a proof that certain of our eta quotients are actually units in the Hilbert class field of an associated imaginary quadratic number field.

Some interesting relations involving Galois actions on our eta units are stated and proved and finally we determine how far short our eta units are from being the full group of units in the field in which they lie. By this stage one has a good concept of why evaluation of eta quotients is so worthwhile and the scene is set for what follows.

Chapter two begins the real meat of the thesis. Weber's modular equations for his *Weber functions* are used to actually explicitly evaluate some eta quotients. This allows one to improve upon the results from the literature which were mentioned in chapter zero. There it was noted that the available literature is almost always limited to cases where the class number of the underlying imaginary quadratic number field was a power of two. The main thrust of this thesis is to break this restriction and do evaluations in cases where the class number is not limited in such a way.

The inspiration for this chapter was a long time in coming and it seemed like the end of a thesis problem when the author managed evaluations in a class number five case. However there was much more to come.

Chapter three is a short article which has been accepted for publication in the Australia Mathematical Society Gazette summarizing and revisiting the investigations that eventually inspired the rest of this thesis. It seemed appropriate to include it at this point.

It is said that no real mathematics has been been produced without first studying the masters. Therefore chapter four is a detailed study of Weber's method of obtaining modular equations for his Weber functions.

It is detailed yet remains a summary. Both of these properties are beneficial since Weber often omits details or uses antiquated notions or notations but at the same time he often approaches a result the long way around and the current author has spent no small amount of time in trying to pare back what is not essential.

This study led in particular to some new theorems and a new path through the material, significantly shortening Weber's approach. Of course the other benefit of chapter four is that it is in English, although it has been noted by some in passing that it is not the German which is the major obstacle with Weber.

But why study this one hundred year old material? What is to be gained from something so classical? Well, Weber is a treasure trove of ideas and it is not even clear that all of his results have yet been established with rigour. In addition it is realised fairly quickly that if Weber functions are useful for evaluating a certain class of eta quotients, then perhaps these functions can be generalized allowing one to make further eta evaluations.

Chapter five is precisely this kind of generalization. Amazingly various analogues of Weber's functions do exist and he even mentions them at one point as have others since.

But what is more amazing perhaps is that no one seems to have developed these functions far in the literature and obtained modular equations for them. The reason is probably simple. It took the author more than a year of painstaking work and multiple attempts to find the exact generalization of these functions and their modular equations. The work is incredibly fiddly and an enormous number of calculations needed to be performed en route to achieve the final result. In addition computer assistance was essential for the manipulation of the q-series involved and in some cases considerable computing power was required.

However to the author's astonishment one of the classes of modular equations developed, already existed in the literature. In fact Ramanujan has beaten us all to it! The level three functions yield modular equations which are identical to those of Ramanujan's alternative cubic theory of the hypergeometric function!

Given the completely different approach in this thesis this is all quite a suprise and leads one to ask a question about whether a satisfactory generalization of Ramanujan's theory exists for signatures 5, 7 and 13 as it is shown that it does for Weber's theory in this thesis.

Of course it has been possible with the generalized Weber function approach and by the use of a computer to add a number of modular equations to Ramanujan's list even for the signature three case which he does deal with.

Chapters five and six are intended to be submitted as journal publications at some point in the future. The latter contains a summary of all the new techniques for evaluating the Dedekind eta function which have been developed over the course of this project. The current author worked closely with Robin Chapman for part of this project and some of the early results in chapter six are joint work with him or simply embellishments of techniques he developed entirely.

Chapter six culminates in an actual evaluation involving the generalized Weber theory of chapter five.

Finally chapter seven takes the theory to its logical conclusion and provides numerous eta evaluations in class number five and seven cases making use of the generalizations of Weber's work described earlier.

This chapter and the thesis finish with a curious set of modular equations which appear to be all of the same form, irrespective of degree. Unable to make use of these in eta evaluations at this point, future directions of research are mentioned and the current study is closed.

It should be noted again by the reader that chapters three, five and six are intended as publications and so are not introduced in the same way as the other chapters. This also affects the flow of ideas to a small degree between chapters but we hope that the reader will not be overly affected by this.

Here seems to be an appropriate place to thank the many people who have made this research possible and interesting. In particular I would like to thank my doctoral supervisor Alf van der Poorten who seems to know the exact mix of moral advice, impromptu lectures and leaving well alone to elicit the greatest quantity of mathematics from a student. Of course he also provided the very interesting problem on which I have worked. I thank Robin Chapman, Peter Stevenhagen and Chris Cummins for their valuable input to the project. I thank Gerry Myerson of Macquarie University for patiently discussing a whole chapter of this thesis at a time when there were subtle errors in my working which I had not been able to locate. He also helped in finding typographical errors in some of the final versions of the sections.

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This thesis has not been submitted as part of any degree or course at any other institution.

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