2. ETA EVALUATIONS USING WEBER FUNCTIONS

Introduction

So far we have seen some of the methods for providing eta evaluations that appear in the literature and we have seen some of the interesting properties of eta quotients and how they relate to special values of L-series.

However all the methods we have seen so far have certain limitations. Whilst they may tell us that our eta quotients are units and which number fields they lie in or tell us that they relate to such and such an L-series or even give various Galois actions on them, the methods still do not tell us explicitly which units they are.

One way of specifying units precisely is to give minimum polynomials for them. The methods that we demonstrate over the next few pages do precisely that.

In our earlier pages we also noted that many of the existing methods put severe limitations on the class numbers of the underlying imaginary quadratic number fields from which the arguments of our eta evaluations are taken. In this chapter we will overcome this limitation and provide evaluations where the class number is an odd prime and not necessarily 3.

We start by giving a short summary of the work of Weber who, using modular equations for his Weber functions, was able to provide various eta evaluations at certain quadratic irrationals. His techniques do not have a limitation on the class number. Many of the evaluations that appear in the tables at the back of his *Lehrbuch der Algebra* [2] have class numbers which are not a power of two. However Weber's methods do have other limitations.

Firstly Weber can only evaluate eta quotients which are derived from his Weber functions. Secondly he limits himself mostly to evaluations at values $\omega = \sqrt{-m}$ for a positive integer m.

In numerous cases we manage to overcome the second limitation and provide evaluations at other quadratic irrationals. This can only be done because of a number of 'tricks' which Weber could not have employed, mainly due to the fact that computers did not exist in his day to provide the necessary numerical insight required to 'guess' solutions to various equations.

Despite this educated guesswork our methods are worth recording since they seem to work in more cases than one would initially expect and the information that is guessed is only slight and easily follows once the final result is proved. Thus, in this chapter, we are able to present quite a number of new explicit eta evaluations where the class number is five and make progress towards evaluations where the class number is seven.

We do however limit ourselves here to evaluations involving the Weber functions. In later chapters, after an examination of Weber's method for obtaining modular equations, we generalise these functions and their modular equations and this allows us to provide further eta evaluations which Weber could not provide. Strangely there we do not need any numerical information to 'guess' solutions as we do in this chapter.

1. Weber's Own Evaluations

Weber himself provides numerous eta evaluations. He is usually interested however in evaluating one of his function $\mathfrak{f}$ or $\mathfrak{f}_1$ at $\sqrt{-m}$ for some natural number m.

In §128 of his Lehrbuch [2] he uses previously calculated values of the function $\gamma_2(\omega) = \sqrt[3]{j(\omega)}$ and the equation

$$u^3 - \gamma_2(\omega)u - 16 = 0$$

which has roots $\mathfrak{f}(\omega)^8$, $-\mathfrak{f}_1(\omega)^8$ and $-\mathfrak{f}_2(\omega)^8$ to calculate minimum polynomials for $\mathfrak{f}(\omega)$ at certain values of ω.

However this method works only for the situations where one can explicitly calculate the value $\gamma_2(\omega)$ and where it has a nice value, such as an integer. In fact this happens for $\omega = \sqrt{-p}$, where p is a prime or 1 and a discriminant with class number one. Thus Weber calculates minimum polynomials for $\mathfrak{f}(\sqrt{-p})$ when $p = 1$, 3, 7, 11, 19, 43, 67 and 163. He also calculates $\mathfrak{f}_1(\sqrt{-2})$ in a similar way.

In §129, by using transformations of second order (essentially derived from the identity $\mathfrak{f}_1(2\omega)\mathfrak{f}_2(\omega) = \sqrt{2}$) Weber is able to use the above results to compute values of $\mathfrak{f}_1(\sqrt{-m})$ for the composite values $m = 4$, 8, 12, 16, 28 and 32.

By §130 Weber has realised that he can use both the Schläfli modular equations and his modular equations of irrational form to obtain Weber function evaluations. We discuss both kinds of modular equation and the derivations Weber gives for them in a later chapter.

For the reader not yet familiar with these, a Schläfli modular equation provides a polynomial relationship between $u = \mathfrak{f}(\omega)$ and $v = \mathfrak{f}(n\omega)$ or between $u = \mathfrak{f}_1(\omega)$ and $v = \mathfrak{f}_1(n\omega)$, usually where n is prime. These modular equations can be exploited by setting $\omega = \sqrt{-m}$ for some integer m so that we have a polynomial relationship between $\mathfrak{f}(\sqrt{-m})$ and $\mathfrak{f}(\sqrt{-n^2m})$, etc.

This method is particularly useful for cases where the value of $\mathfrak{f}(\sqrt{-m})$ (or $\mathfrak{f}_1(\sqrt{-m})$) is known and is a nice value such as a root of an integer. This is certainly the case for $m = 1$, 2, 3, 4 or 7. With this technique Weber is able to compute the values $\mathfrak{f}(\sqrt{-n^2m})$ where $n^2m = 9$, 25, 27, 49, 63, 75 and 175, and he computes $\mathfrak{f}_1(\sqrt{-n^2m})$ where $n^2m = 18$, 36, 50 and 100.

There are of course higher values of n^2m for which this technique works but one sees the limited applicability of it due to its reliance on a known simple value being given for $\mathfrak{f}(\sqrt{-m})$.

Next Weber makes use of the fact that $\sqrt{-m}/m = -1/\sqrt{-m}$. It is easy to see that if one sets $\omega = \sqrt{-m}/m$ then

$$\mathfrak{f}(\omega) = \mathfrak{f}(-1/\sqrt{-m}) = \mathfrak{f}(\sqrt{-m}) = \mathfrak{f}(m\omega).$$

Using this information, a Schläfli modular equation of degree m involving $\mathfrak{f}(\omega)$ and $\mathfrak{f}(m\omega)$ becomes a minimum polynomial for $\mathfrak{f}(\sqrt{-m})$ once we set $\omega = \sqrt{-m}/m$ in it.

Using this method Weber obtains the additional evaluations $\mathfrak{f}(\sqrt{-m})$ for the values $m = 5$, 13 and 17.

Next Weber takes ω to be a root of $2x^2 + 2rx + n$ where n is odd and r any integer. This has as root $\omega = (-r + \sqrt{-m})/2$ where $m = 2n - r^2$.

Of particular importance is the fact that $-n/\omega = r + \sqrt{-m}$.

The idea is to consider the modular equation of degree n relating $\mathfrak{f}_2(\omega)$ and $\mathfrak{f}_2(\omega/n)$. A simple calculation shows that these have the values $\exp(-r\pi i/24)\sqrt{2}/x$ and

$\exp(-r\pi i/24)x$ respectively where $x = \mathfrak{f}_1(\sqrt{-m})$ or $x = \mathfrak{f}(\sqrt{-m})$ depending on whether r is odd or even.

Weber gives a bewildering array of values m and n for which this method works. He uses it to calculate $\mathfrak{f}(\sqrt{-13})$ and $\mathfrak{f}_1(\sqrt{-m})$ for $m = 10$, 22 and 26.

Lastly Weber sets ω to be the root $(-r+\sqrt{-m})/(2n)$ of $2nx^2+2rx+n$ where $m = 2n^2 - r^2$. This time he considers the values $\mathfrak{f}_2(\omega)$, $\mathfrak{f}_2(\omega/n)$ and $\mathfrak{f}_2(n\omega)$.

Again depending on whether r is odd or even the last two of these can both be expressed in terms of $x = \mathfrak{f}_1(\sqrt{-m})$ or $x = \mathfrak{f}(\sqrt{-m})$.

Combining two Schläfli modular equations of degree n in an obvious way one can then eliminate $\mathfrak{f}_2(\omega)$ leaving only a relation between $\mathfrak{f}_2(\omega/n)$ and $\mathfrak{f}_2(n\omega)$. This leaves of course an equation involving only the expression x.

Weber uses this technique to evaluate $\mathfrak{f}(\sqrt{-41})$.

Now comes Weber's §131 where he makes use of the modular equations of irrational form. These are modular equations that involve all three of his Weber functions $\mathfrak{f}$, $\mathfrak{f}_1$ and $\mathfrak{f}_2$.

For example, consider Weber's modular equations of degree $n \equiv 7 \pmod 8$ of this form. They are polynomial relations between functions of the form

$$2A = \mathfrak{f}(\omega)\mathfrak{f}(n\omega) + (-1)^{\frac{n+1}{8}}(\mathfrak{f}_1(\omega)\mathfrak{f}_1(n\omega) + \mathfrak{f}_2(\omega)\mathfrak{f}_2(n\omega))$$

and

$$B = \frac{2}{\mathfrak{f}_1(\omega)\mathfrak{f}_1(n\omega)} + \frac{2}{\mathfrak{f}_2(\omega)\mathfrak{f}_2(n\omega)} + (-1)^{\frac{n+1}{8}}\frac{2}{\mathfrak{f}(\omega)\mathfrak{f}(n\omega)}.$$

Now one sets $\omega = -1/\sqrt{-n}$ so that $n\omega = \sqrt{-n}$. We set $\sqrt{2}x = \mathfrak{f}(\sqrt{-n})$. We also note that $\mathfrak{f}_1(\omega)\mathfrak{f}_1(n\omega)$ becomes $\mathfrak{f}_2(\sqrt{-n})\mathfrak{f}_1(\sqrt{-n}) = \sqrt{2}/\mathfrak{f}(\sqrt{-n}) = 1/x$, etc. In fact we have

$$A = x^2 + (-1)^{\frac{m+1}{8}}/x \quad \text{and} \quad B = 4x + (-1)^{\frac{m+1}{8}}/x^2,$$

(where we correct a probable typographical error at this point in Weber).

In other words the modular equation of irrational form, being a polynomial relationship between the functions A and B, induces a polynomial equation in the single value x which when factored provides a minimal polynomial for the value $x = \mathfrak{f}(\sqrt{-n})/\sqrt{2}$.

This method allows Weber to obtain new evaluations for $\mathfrak{f}(\sqrt{-n})$ where $n = 23$, 31, 47 and 71.

We mention one final method of Weber's involving his modular equations of irrational form. His idea is somewhat limited in that it applies to a modular equation where the function B does not appear. In the expression for A he picks the value of ω so that the term involving $\mathfrak{f}_2(\omega)\mathfrak{f}_2(\omega/n)$ becomes a constant. The remaining terms involving $\mathfrak{f}$ and $\mathfrak{f}_1$ he evaluates by making use of the following identity of degree 2 which he also derives in §131

$$\mathfrak{f}(\omega)^4\mathfrak{f}(2\omega)^4 + \mathfrak{f}_1(\omega)^4\mathfrak{f}_2(2\omega)^4 = \mathfrak{f}_2(\omega)^6 + 8/\mathfrak{f}_2(\omega)^6.$$

The reason that he can make use of this identity is that he picks ω very carefully. He sets $\omega = (-r+\sqrt{-m})/2$ a root of $2x^2+2rx+n$ for some integer r and $m = 2n-r^2$. The crucial benefit of this is that $2\omega = -n/\omega - 2r$ making the terms on the left hand side of the degree two identity equal to some power of the terms involving $\mathfrak{f}$ and $\mathfrak{f}_1$ of his modular equation of irrational form.

So after taking the appropriate power of his modular equation and substituting in this degree two identity Weber ends up with an expression containing only the value $\mathfrak{f}_2(\omega)$ and various constants. But depending on whether or not r is even $\mathfrak{f}_2(\omega)$

is, up to some constant factors, either $\mathfrak{f}(\sqrt{-m})$ or $\mathfrak{f}_1(\sqrt{-m})$. Thus one obtains a polynomial eqution in one of these two values.

Weber is able to use this process to obtain an evaluation of $\mathfrak{f}_1(\sqrt{-46})$, since he can set $m = 46$, $n = 23$ and $r = 0$.

There is another method which Weber gives in §131, however it makes use of his theory of modular equations of composite degree which we do not examine, so we omit a description of this method. It allows Weber to evaluate $\mathfrak{f}(\sqrt{-39})$.

2. New Class Number Five Evaluations Using Weber Functions

2.1. **Discriminant -47.** We will calculate numerous eta quotients explicitly with arguments in the imaginary quadratic field of discriminant $d = -47$. Of special importance to us is the eta quotient $\left|\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{2}\right)\right|$ which is related to the unit m_1 of the previous chapter. This value is very easy to compute. However we will also calculate other eta quotients where the discriminant is -47, which are not so easy to get at, such as the unit m_2 of the last chapter.

Throughout this section we make extensive use of the Weber function identities mentioned in the previous section and in addition we make use of the following fact without comment

$$\mathfrak{f}_1\left(\frac{b+\sqrt{-d}}{2a}\right) = \overline{\mathfrak{f}_1}\left(\frac{-b+\sqrt{-d}}{2a}\right),$$

where the bar denotes the complex conjugate and the arguments are quadratic irrationals. A similar result holds for the function $\mathfrak{f}_2$. These identities follow straightforwardly from the definition of the Weber functions and the q-series expression for the eta function.

Firstly we begin with Weber's own evaluation of $x = |\mathfrak{f}(\sqrt{-47})|/\sqrt{2}$. In his Lehrbuch [2] he states in the tables at the end that x satisfies the polynomial equation $x^5 - x^3 - 2x^2 - 2x - 1$.

But we notice that

$$\left|\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{2}\right)\right| = \sqrt{2}/|\mathfrak{f}_1(1+\sqrt{-47})| = \sqrt{2}/|\mathfrak{f}(\sqrt{-47})|.$$

Thus we can calculate this value from the known value which Weber calculates.

Next we notice that

$$\left|\mathfrak{f}\left(\frac{1+\sqrt{-47}}{2}\right)\right| = \left|\mathfrak{f}_1\left(\frac{-1+\sqrt{-47}}{2}\right)\right| = \left|\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{2}\right)\right|.$$

But now from $\mathfrak{f}\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2}$ we calculate that

$$\left|\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{2}\right)\right|^2 = \sqrt{2}\ /\left|\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{2}\right)\right| = |\mathfrak{f}(\sqrt{-47})|.$$

Thus we have that $x = \left|\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{2}\right)\right|^2/\sqrt{2}$ also satisfies $x^5 - x^3 - 2x^2 - 2x - 1 = 0$.

This immediately gives us an evaluation of the unit m_1 of the last chapter for this discriminant.

Next we notice that

$$\left|\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{4}\right)\right| = \sqrt{2}\ /\left|\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{2}\right)\right|.$$

This value is one that we know. In fact we see that

(1) $$y = \left|\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{4}\right)\right|^2/\sqrt{2} \text{ satisfies } y^5+2y^4+2y^3+y^2-1=0.$$

What we will do now is relate this known value to $\left|\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{4}\right)\right|^2/\sqrt{2}$ using a modular equation of degree three for Weber functions. In fact we will show later that this value is precisely the unit $1/m_2$ for the discriminant $d=-47$ which we spoke about in the last chapter.

This is done as follows. Firstly we find

$$\mathfrak{f}_1\left(\frac{-1+\sqrt{-47}}{4}\right) = \mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{12}\right),$$

and

$$\mathfrak{f}_2\left(\frac{-1+\sqrt{-47}}{4}\right) = \mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{12}\right).$$

The arguments which appear here differ in pairs by a factor of three.

We also note that

$$\mathfrak{f}\left(\frac{-1+\sqrt{-47}}{4}\right) = \mathfrak{f}\left(\frac{1+\sqrt{-47}}{12}\right).$$

The type of modular equation which we will use is that of *irrational form* spoken of by Weber in §75 of [2]. The degree three equation of this kind is

$$\mathfrak{f}(\omega)^2\mathfrak{f}(3\omega)^2 = \mathfrak{f}_1(\omega)^2\mathfrak{f}_1(3\omega)^2 + \mathfrak{f}_2(\omega)^2\mathfrak{f}_2(3\omega)^2.$$

If we substitute in the value $\omega = \frac{1+\sqrt{-47}}{12}$ then this becomes

$$|\mathfrak{f}(3\omega)|^4 = \overline{\mathfrak{f}_2}(3\omega)^2\mathfrak{f}_1(3\omega)^2 + \overline{\mathfrak{f}_1}(3\omega)^2\mathfrak{f}_2(3\omega)^2,$$

where we use the fact that $\mathfrak{f}\,\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2}$ to see that $\mathfrak{f}(\omega)$ and $\mathfrak{f}(3\omega)$ are conjugates.

We wish to do away with the asymmetry on the right hand side of this equation. This we can do by repeatedly squaring the equation and rearranging so that all the asymmetry remains on the right hand side. After doing this twice (always replacing $|\mathfrak{f}_1(3\omega)\mathfrak{f}_2(3\omega)|$ with $|\mathfrak{f}_1(3\omega)|$ wherever it occurs) we obtain

$$\begin{aligned}|\mathfrak{f}(3\omega)|^{16}+32/|\mathfrak{f}(3\omega)|^8-16|\mathfrak{f}(3\omega)|^4 &= \mathfrak{f}_1(3\omega)^8\overline{\mathfrak{f}_2}(3\omega)^8+\mathfrak{f}_2(3\omega)^8\overline{\mathfrak{f}_1}(3\omega)^8\\ &= \mathfrak{f}_1(3\omega)^8(\bar{\mathfrak{f}}(3\omega)^8-\overline{\mathfrak{f}_1}(3\omega)^8)+\mathfrak{f}_2(3\omega)^8(\bar{\mathfrak{f}}(3\omega)^8-\overline{\mathfrak{f}_2}(3\omega)^8)\\ &= -|\mathfrak{f}_1(3\omega)|^{16}-|\mathfrak{f}_2(3\omega)|^{16}+|\mathfrak{f}(3\omega)|^{16}.\end{aligned}$$

In other words we have shown

$$16|\mathfrak{f}(\tau)|^4-32/|\mathfrak{f}(\tau)|^8 = |\mathfrak{f}_1(\tau)|^{16}+|\mathfrak{f}_2(\tau)|^{16},$$

where $\tau = \frac{1+\sqrt{-47}}{4}$. Then letting $a=|\mathfrak{f}(\tau)|$, $b=|\mathfrak{f}_1(\tau)|$ and $c=|\mathfrak{f}_2(\tau)|$ so that $abc=\sqrt{2}$ we see that

$$64/(bc)^4-2(bc)^8 = b^{16}+c^{16},$$

which after rearranging and taking the square root becomes

$$8/(bc)^2 = b^8+c^8.$$

It seems logical to now set $l=a^2$, $m=b^2/\sqrt{2}$ and $n=c^2/\sqrt{2}$, since these are the actual units we will be dealing with. Now $lmn=1$ and the equation above has become the very elegant identity

(2) $$m^5n+n^5m=1.$$

But the value n is precisely the value given in the equation (1) above. Therefore we only need to solve the previous equation in order to obtain the eta evaluation for m.

At this point we happen to notice numerically that $l = n + 1$. Thus from $lmn = 1$ we have that $1/m = n(n + 1)$.

Substituting this value into the equation (2) and making use of the fact that $f(n) = 0$ where $f(x)$ is the minimum polynomial for n we find that this value of m is indeed a root. Since the polynomial (2) only has one real root we must have the correct value for m.

Now from the minimal polynomial $f(x)$ for the value n we have that

$$4 + n = 4f(n) + 4 + n.$$

After expanding this out, the right hand side is divisible by n. Dividing through by n we obtain

$$4/n + 1 = 4n^4 + 8n^3 + 8n^2 + 8n + 1.$$

Now the right hands side of this equation is a square. Thus we have

$$\sqrt{4/n+1} = 2n^2 + 2n + 1 = 2/m + 1.$$

In other words, $m = \frac{2}{-1+\sqrt{4/n+1}}$. Expanding this out leads to $n = m^2/(m + 1)$. If we plug this into (2) and factorise we obtain a minimum polynomial for m (it is the only factor of degree 5).

We find in this way that m has minimum polynomial $x^5 + x^4 + x^3 - x^2 - 2x - 1$.

Now as we mentioned earlier this value is in fact the unit $1/m_2$ spoken of in the previous chapter. This can be proved as follows

$$m_2 = h(2,1,6)^2/h(3,1,4)^2 = \sqrt{\frac{3}{2}} \left| \frac{\eta((1+\sqrt{-47})/4)}{\eta((1+\sqrt{-47})/6)} \right|^2$$

$$= \sqrt{\frac{3}{2}} \left| \frac{\eta((1+\sqrt{-47})/4)}{\sqrt{-i(-1+\sqrt{-47})/8}\ \eta((-1+\sqrt{-47})/8)} \right|^2 = \sqrt{2}\ /\ |\mathfrak{f}_1((1+\sqrt{-47})/4)|^2,$$

which is the inverse of the value we have calculated above.

We can continue from our last evaluation and obtain further interesting ones. Firstly we see that

$$\left| \mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{4}\right) \right| = \left| \mathfrak{f}_2\left(\frac{-1+\sqrt{-47}}{12}\right) \right| = \sqrt{2}\ /\ \left| \mathfrak{f}_1\left(\frac{-1+\sqrt{-47}}{6}\right) \right|$$

Thus $z = \left| \mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{6}\right) \right|^2 / \sqrt{2}$ satisfies $z^5 + 2z^4 + z^3 - z^2 - z - 1 = 0$.

Now we do an interesting side calculation. We have

$$\left| \mathfrak{f}\left(\frac{1+\sqrt{-47}}{6}\right) \right| = \left| \mathfrak{f}_1\left(\frac{-5+\sqrt{-47}}{6}\right) \right| = \sqrt{2}\ /\ \left| \mathfrak{f}_2\left(\frac{-5+\sqrt{-47}}{12}\right) \right|$$

$$= \sqrt{2}\ /\ \left| \mathfrak{f}_1\left(\frac{5+\sqrt{-47}}{6}\right) \right| = \sqrt{2}\ /\ \left| \mathfrak{f}\left(\frac{-1+\sqrt{-47}}{6}\right) \right| = \sqrt{2}\ /\ \left| \mathfrak{f}\left(\frac{1+\sqrt{-47}}{6}\right) \right|.$$

Thus we have $\left| \mathfrak{f}\left(\frac{1+\sqrt{-47}}{6}\right) \right|^2 = \sqrt{2}$. Combining this with $|\mathfrak{f}\,\mathfrak{f}_1\mathfrak{f}_2|^2 = 2$ we find

$$\left| \mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{6}\right) \right|^2 = \sqrt{2}\ /\ \left| \mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{6}\right) \right|^2,$$

Eta Evaluations $(d = -47)$	Minimum Polynomial
$\lvert\mathfrak{f}(\sqrt{-47})\rvert/\sqrt{2} = 1 \,/\, \left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{2}\right)\right\rvert$ $= \left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{2}\right)\right\rvert^2/\sqrt{2} = \sqrt{2}\,/\,\left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{4}\right)\right\rvert^2$	$x^5 - x^3 - 2x^2 - 2x - 1$
$\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{4}\right)\right\rvert^2/\sqrt{2} = \sqrt{2}\,/\,\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{6}\right)\right\rvert^2$ $= \left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{6}\right)\right\rvert^2 = 2\,/\,\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{3}\right)\right\rvert^2$	$x^5 + x^4 + x^3 - x^2 - 2x - 1$
$\left\lvert\mathfrak{f}\left(\frac{1+\sqrt{-47}}{6}\right)\right\rvert^2$	$x^2 - 2$

Table 2.1

and therefore satisfies the polynomial equation $x^5 + x^4 + x^3 - x^2 - 2x - 1 = 0$. However since $\left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{6}\right)\right\rvert = \sqrt{2}\,/\,\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{3}\right)\right\rvert$ we have that

$$z = \left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{3}\right)\right\rvert^2/2 \text{ satisfies } z^5 + 2z^4 + z^3 - z^2 - z - 1 = 0.$$

We summarise the eta evaluations that we have completed for the discriminant $d = -47$ in the above table.

2.2. **Discriminant -79.** We move on to the next discriminant of class number 5 which is $d = -79$. Here we particularly wish to evaluate the quantities

$$m_1 = \sqrt{2}\,/\,\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{2}\right)\right\rvert^2 \quad\text{and}\quad m_2 = \sqrt{2}\,/\,\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{4}\right)\right\rvert^2.$$

Recall that the second of these quantities was not even proved to be a unit in the previous chapter. However this will be obvious from its evaluation which we will obtain shortly.

Unfortunately Weber does not compute $x = \lvert\mathfrak{f}(\sqrt{-79})\rvert/\sqrt{2}$, however Ramanujan has essentially calculated it. According to the table of class invariants of 34.2 of [1] it satisfies the equation $x^5 - 3x^4 + 2x^3 - x^2 + x - 1 = 0$.

Now we note that

$$\left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{2}\right)\right\rvert = \sqrt{2}/\left\lvert\mathfrak{f}_1\left(1+\sqrt{-79}\right)\right\rvert = \sqrt{2}/\lvert\mathfrak{f}(\sqrt{-79})\rvert.$$

Also we have

$$\left\lvert\mathfrak{f}\left(\frac{1+\sqrt{-79}}{2}\right)\right\rvert = \left\lvert\mathfrak{f}_1\left(\frac{-1+\sqrt{-79}}{2}\right)\right\rvert = \left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{2}\right)\right\rvert.$$

Therefore from $\mathfrak{f}\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2}$ we find that

$$x = \left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{2}\right)\right\rvert^2/\sqrt{2} \text{ satisfies } x^5 - 3x^4 + 2x^3 - x^2 + x - 1 = 0$$

Also we have

$$\left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{4}\right)\right\rvert = \sqrt{2}\,/\,\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{2}\right)\right\rvert.$$

Thus in particular

$$y = \left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{4}\right)\right\rvert^2/\sqrt{2} \text{ satisfies } y^5 - y^4 + y^3 - 2y^2 + 3y - 1 = 0.$$

We now relate this value to another via a modular equation of degree 5. Firstly we have

$$\mathfrak{f}_2\left(\frac{-1+\sqrt{-79}}{4}\right) = \mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{20}\right)$$

and

$$\mathfrak{f}_1\left(\frac{-1+\sqrt{-79}}{4}\right) = \mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{20}\right)$$

and

$$\mathfrak{f}\left(\frac{-1+\sqrt{-79}}{4}\right) = \mathfrak{f}\left(\frac{1+\sqrt{-79}}{20}\right).$$

We employ Weber's modular equation of irrational form of degree 5. This is given in §75 of [2] as

$$8 = \mathfrak{f}(\omega)^4\mathfrak{f}(5\omega)^4 - \mathfrak{f}_1(\omega)^4\mathfrak{f}_1(5\omega)^4 - \mathfrak{f}_2(\omega)^4\mathfrak{f}_2(5\omega)^4. \tag{3}$$

Once we substitute the particular value $\omega = \frac{1+\sqrt{-79}}{20}$ this can be written

$$8 = |\mathfrak{f}(5\omega)|^8 - \overline{\mathfrak{f}_2}(5\omega)^4\mathfrak{f}_1(5\omega)^4 - \overline{\mathfrak{f}_1}(5\omega)^4\mathfrak{f}_2(5\omega)^4.$$

We apply the same technique as we did for $d = -47$, putting the asymmetric parts to one side and squaring. Doing this just once we obtain

$$\begin{aligned}|\mathfrak{f}(5\omega)|^{16} - 16|\mathfrak{f}(5\omega)|^8 + 64 &= \overline{\mathfrak{f}_2}(5\omega)^8\mathfrak{f}_1(5\omega)^8 + \overline{\mathfrak{f}_1}(5\omega)^8\mathfrak{f}_2(5\omega)^8 + 2|\mathfrak{f}_1(5\omega)\mathfrak{f}_2(5\omega)|^8 \\ &= \overline{\mathfrak{f}_2}(5\omega)^8(\mathfrak{f}(5\omega)^8 - \mathfrak{f}_2(5\omega)^8) + \overline{\mathfrak{f}_1}(5\omega)^8(\mathfrak{f}(5\omega)^8 - \mathfrak{f}_1(5\omega)^8) + 32/|\mathfrak{f}(5\omega)|^8 \\ &= |\mathfrak{f}(5\omega)|^16 - |\mathfrak{f}_1(5\omega)|^16 - |\mathfrak{f}_2(5\omega)|^16 + 32/|\mathfrak{f}(5\omega)|^8.\end{aligned}$$

So writing $l = |\mathfrak{f}(5\omega)|^2$, $m = |\mathfrak{f}_1(5\omega)|^2/\sqrt{2}$ and $n = |\mathfrak{f}_2(5\omega)|^2\sqrt{2}$ so that $lmn = 1$ we have

$$16/(mn)^4 - 64 = 16m^8 + 16n^8 - 32(mn)^4.$$

But adding $64(mn)^4$ to both sides we end up with a square on the left and the right

$$(8(mn)^2 - 4/(mn)^2)^2 = (4m^4 + 4n^4)^2.$$

It is not hard to determine that the correct square root to take is

$$1/(mn)^2 - 2(mn)^2 = m^4 + n^4$$

which after rearranging again and taking the correct square root gives

$$1/(mn) = m^2 + n^2,$$

that is

$$m^3n + mn^3 = 1. \tag{4}$$

Now we happen to note numerically that $1/n = m + 1$. Substituting this into the previous equation and taking the resulting expression modulo the known minimum polynomial for n we see that this indeed gives a root of the equation (4). Since the equation only has one real root, this must be it.

Now we know the minimum polynomial for n. We can easily calculate the minimum polynomial for $1/n$ from it and for $1/n - 1$ for that. But this must be the minimum polynomial for m.

Thus we have that $m = \left|\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{4}\right)\right|^2/\sqrt{2}$ satisfies $m^5 + 2m^4 - 3m^2 - 2m - 1 = 0$.

Eta Evaluations $(d = -47)$	Minimum Polynomial
$\lvert\mathfrak{f}(\sqrt{-79})\rvert/\sqrt{2} = 1 \,/\, \left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{2}\right)\right\rvert$ $= \left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{2}\right)\right\rvert^2/\sqrt{2} = \sqrt{2} \,/\, \left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{4}\right)\right\rvert^2$	$x^5 - 3x^4 + 2x^3 - x^2 + x - 1$
$\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{4}\right)\right\rvert^2/\sqrt{2} = \sqrt{2} \,/\, \left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert^2$ $= \left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert^2$	$x^5 + 2x^4 - 3x^2 - 2x - 1$
$\left\lvert\mathfrak{f}\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert^2$	$x^2 - 2$

Table 2.2

There are a few more evaluations that we can complete with this information. Firstly

$$\left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert = \sqrt{2} \,/\, \left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{4}\right)\right\rvert.$$

In other words $z = \left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert^2/\sqrt{2}$ satisfies $z^5 + 2z^4 + 3z^3 - 2z - 1 = 0$.

However we now compute

$$\left\lvert\mathfrak{f}\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert = \left\lvert\mathfrak{f}_1\left(\frac{-7+\sqrt{-79}}{8}\right)\right\rvert = \sqrt{2} \,/\, \left\lvert\mathfrak{f}_2\left(\frac{-7+\sqrt{-79}}{16}\right)\right\rvert$$
$$= \sqrt{2} \,/\, \left\lvert\mathfrak{f}_1\left(\frac{7+\sqrt{-79}}{8}\right)\right\rvert = \sqrt{2} \,/\, \left\lvert\mathfrak{f}\left(\frac{-1+\sqrt{-79}}{8}\right)\right\rvert = \sqrt{2} \,/\, \left\lvert\mathfrak{f}\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert.$$

Thus we have that

$$\left\lvert\mathfrak{f}\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert^2 = \sqrt{2}.$$

Thus combining this with the above we have that

$$\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-79}}{8}\right)\right\rvert^2 \text{ satisfies } x^5 + 2x^4 - 3x^2 - 2x - 1 = 0.$$

We summarise some of the evaluations we have obtained for $d = -79$ in the table above.

2.3. **Discriminant -103.** Again similar techniques work for $d = -103$ also of class number 5. Firstly we wish to evaluate the expression

$$m_1 = \sqrt{2}\left\lvert\frac{\eta((1+\sqrt{-103})/2)}{\eta((1+\sqrt{-103})/4)}\right\rvert^2 = \sqrt{2} \,/\, \left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-103}}{2}\right)\right\rvert^2$$
$$= \sqrt{2} \,/\, \left\lvert\mathfrak{f}\left(\frac{-1+\sqrt{-103}}{2}\right)\right\rvert^2 = \sqrt{2} \,/\, \left\lvert\mathfrak{f}\left(\frac{1+\sqrt{-103}}{2}\right)\right\rvert^2.$$

However since both of the last expressions in the two rows of this equation are the same, then from $\mathfrak{f}\,\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2}$ we see that these expressions are equal to the value

$$\left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-103}}{2}\right)\right\rvert = \sqrt{2} \,/\, \left\lvert\mathfrak{f}_1\left(1+\sqrt{-103}\right)\right\rvert = \sqrt{2} \,/\, \left\lvert\mathfrak{f}(\sqrt{-103})\right\rvert.$$

Now this evaluation is strangely not made in either [2] or [1], however there does not seem to be any reason why it could not be calculated using Weber's method providing one accepts the use of a computer to calculate the q-series involved. We cheat of course and simply ask Pari to provide us with the minimum polynomial using its *algdep()* function. It does not take long to determine that m_1 has minimum polynomial $x^5 + 2x^4 + 3x^3 + 3x^2 + x - 1$.

Now it will be convenient to write m_1 in a slightly different form. From the first equation for m_1 above we see that

$$m_1 = \sqrt{2} \Big/ \left| \mathfrak{f}_1 \left(\frac{1+\sqrt{-103}}{2} \right) \right|^2 = \left| \mathfrak{f}_2 \left(\frac{1+\sqrt{-103}}{4} \right) \right|^2 \Big/ \sqrt{2}.$$

For consistency we will denote this unit by n for consistency with the earlier cases we dealt with.

Now we also wish to calculate the value m_2 for the discriminant $d = -103$. In fact we have

$$\begin{aligned} 1/m_2 &= \left| \frac{\eta((3+\sqrt{-103})/8)}{\eta((1+\sqrt{-103})/4)} \right|^2 \Big/ \sqrt{2} = \left| \frac{\eta((-3+\sqrt{-103})/8)}{\eta((-3+\sqrt{-103})/4)} \right|^2 \Big/ \sqrt{2} \\ &= \left| \mathfrak{f}_1 \left(\frac{-3+\sqrt{-103}}{4} \right) \right|^2 \Big/ \sqrt{2} = \left| \mathfrak{f} \left(\frac{1+\sqrt{-103}}{4} \right) \right|^2 \Big/ \sqrt{2}. \end{aligned}$$

Again for consistency we denote this value by l. Of course we also denote

$$m = \left| \mathfrak{f}_1 \left(\frac{1+\sqrt{-103}}{4} \right) \right|^2,$$

so that $lmn = 1$.

We happen to notice numerically that $l = n + 1$.

Now in order to make use of a modular equation we must the values l, m and n in terms of values with argument $\tau = \frac{3+\sqrt{-103}}{28}$. In fact by making simple transformations of the expressions above we find that $m = |\mathfrak{f}(7\tau)|^2$, $n = |\mathfrak{f}_2(7\tau)|^2/\sqrt{2}$ and $l = |\mathfrak{f}_1(7\tau)|^2/\sqrt{2}$.

Now we note that

$$\mathfrak{f}(\tau) = \mathfrak{f}\left(\frac{3+\sqrt{-103}}{28} \right) = \mathfrak{f}\left(\frac{-3+\sqrt{-103}}{4} \right) = \overline{\mathfrak{f}}(7\tau).$$

Similarly we find that

$$\mathfrak{f}_1(\tau) = \overline{\mathfrak{f}_2}(7\tau), \quad \mathfrak{f}_2(\tau) = \overline{\mathfrak{f}_1}(7\tau).$$

Now the appropriate modular equations is Weber's modular equation of irrational form of degree 7. This is given by Weber in §75 of [2] as

$$\mathfrak{f}(\tau)\mathfrak{f}(7\tau) - \mathfrak{f}(\tau)\mathfrak{f}(7\tau) - \mathfrak{f}_1(\tau)\mathfrak{f}_1(7\tau) = 0,$$

which, for the specific value of τ that we have chosen, becomes

$$\overline{\mathfrak{f}}(7\tau)\mathfrak{f}(7\tau) - \overline{\mathfrak{f}_2}(7\tau)\mathfrak{f}_1(7\tau) - \overline{\mathfrak{f}_1}(7\tau)\mathfrak{f}_2(7\tau) = 0.$$

We remove the asymmetry from this equation as we did in earlier cases by repeatedly putting all the asymmetric terms on one side of the equation and squaring until all the asymmetric terms contain 8-th powers. These we replace in exactly the same manner as before and we finally end up with

$$(l^4 + n^4)^2 = 8/m + m^5 - 5m^2.$$

Of course we can use the expression $lmn = 1$ to change this equation so that it is only in terms of two of the values l, m and n.

Eta Evaluations $(d = -103)$	Minimum Polynomial
$\sqrt{2} \,/\, \left\| \mathfrak{f}_1\left(\frac{1+\sqrt{-103}}{2}\right)\right\|^2 = \sqrt{2} \,/\, \left\|\mathfrak{f}\left(\sqrt{-103}\right)\right\|$ $= \left\|\mathfrak{f}_2\left(\frac{1+\sqrt{-103}}{4}\right)\right\| / \sqrt{2}$	$x^5 + 2x^4 + 3x^3 + 3x^2 + x - 1$
$\left\|\mathfrak{f}_1\left(\frac{3+\sqrt{-103}}{4}\right)\right\|^2 / \sqrt{2} = \left\|\mathfrak{f}\left(\frac{1+\sqrt{-103}}{4}\right)\right\|^2 / \sqrt{2}$	$x^5 - 3x^4 + 5x^3 - 4x^2 + x - 1$

Table 2.3

Firstly we subtract $4/m^4 = 4l^4m^4$ from each side and multiply through by m^4. After rearranging we obtain

$$m^4(l^4 + n^4)^2 = (m^3 - 1)(m^3 - 2)^2.$$

Dividing through by m^9 and using $1/m = ln$ we obtain

$$(ln)^5(l^4 - n^4)^2 = (1 - (ln)^3)(1 - 2(ln)^3)^2.$$

By putting $l = n + 1$ into this and using the minimum polynomial for n which we know, we see that indeed $l = n + 1$ is a root of this equation.

However, if instead we substitute $n = l - 1$ then we end up with a polynomial which l satisfies. It has only two quintic factors (and some irrelevant factors of lower degree). One of these quintic factors we recognize as the minimum polynomial of $-n$. Since l cannot equal $-n$ (otherwise l and n would be rational) and since $-n$ is the only real root of this quintic factor then l must have the other quintic factor as minimum polynomial. That is l satisfies the quintic equation $l^5 - 3l^4 + 5l^3 - 4l^2 + l - 1 = 0$.

We summarise these eta evaluations in Table 2.3.

2.4. **Discriminant -127.** Finally for class number five we look at the discriminant $d = -127$. The process for this discriminant is very similar to that for $d = -79$ so we simply summarise the steps involved. Firstly we wish to evaluate the unit which we call m_1 in the last chapter. It has the value

$$m_1 = \sqrt{2}\left|\frac{\eta((1+\sqrt{-127})/2)}{\eta((1+\sqrt{-127})/4}\right|^2 = \sqrt{2} \,/\, \left|\mathfrak{f}_1\left(\frac{1+\sqrt{-127}}{2}\right)\right|^2$$
$$= \sqrt{2} \,/\, \left|\mathfrak{f}\left(\frac{1+\sqrt{-127}}{2}\right)\right|^2 = \sqrt{2} \,/\, \left|\mathfrak{f}(\sqrt{-127})\right|.$$

But this is also equal to $\left|\mathfrak{f}_2((1+\sqrt{-127})/4)\right|^2 / \sqrt{2}$.

Likewise we find that the unit m_2 is given by

$$1/m_2 = \left|\frac{\eta((1+\sqrt{-127})/8)}{\eta((1+\sqrt{-127})/4}\right|^2 / \sqrt{2} = \left|\mathfrak{f}_1\left(\frac{1+\sqrt{-127}}{4}\right)\right|^2 / \sqrt{2}.$$

We call the first of these values above n, the second m and to be consistent with what has gone before we denote

$$l = \left|\mathfrak{f}\left(\frac{1+\sqrt{-127}}{4}\right)\right|^2,$$

so that as usual $lmn = 1$.

Again neither Weber nor Ramanujan/Berndt evaluate $\left|\mathfrak{f}(\sqrt{-127})\right|$, however we believe that Weber's method can still be applied. We ourselves ask PARI to do the evaluation numerically which has that n satisfies $n^5 - n^4 - 2n^3 + n^2 + 3n - 1 = 0$.

Eta Evaluations $(d = -127)$	Minimum Polynomial
$\sqrt{2}\,/\,\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-127}}{2}\right)\right\rvert^2 = \sqrt{2}\,/\,\left\lvert\mathfrak{f}\left(\sqrt{-127}\right)\right\rvert$ $= \left\lvert\mathfrak{f}_2\left(\frac{1+\sqrt{-127}}{4}\right)\right\rvert/\sqrt{2}$	$x^5 - x^4 - 2x^3 + x^2 + 3x - 1$
$\left\lvert\mathfrak{f}_1\left(\frac{1+\sqrt{-127}}{4}\right)\right\rvert^2/\sqrt{2}$	$x^5 - x^3 - 4x^2 + 4x - 1 = 0$

Table 2.4

Again we happen to notice numerically that $m = 1/l + 1$ (we must by now suspect some kind of general relation of this form for class number five at least when the Weber functions are involved, though we still do not have a specific conjecture and certainly no idea of a proof of such a result).

Even better we notice that $n + 1/m = 1$. Thus if this result were proved, from the minimal polynomial for n by substituting $n = 1 - 1/m$ we can obtain a minimal polynomial for m.

However as usual, to verify that the result is indeed true we go via a modular equation. We write

$$m = \left|\mathfrak{f}_1\left(\frac{-7+\sqrt{-127}}{4}\right)\right|^2/\sqrt{2}, \quad n = \left|\mathfrak{f}_2\left(\frac{-7+\sqrt{-127}}{4}\right)\right|^2/\sqrt{2}, \quad \text{etc.}$$

Now we can make use of the modular equation of degree 11, since

$$\mathfrak{f}_2\left(\frac{-7+\sqrt{-127}}{4}\right) = \mathfrak{f}_1\left(\frac{7+\sqrt{-127}}{44}\right), \quad \mathfrak{f}_1\left(\frac{-7+\sqrt{-127}}{4}\right) = \mathfrak{f}_2\left(\frac{7+\sqrt{-127}}{44}\right),$$

and so on.

In a manner similar to our previous arguments, the modular equation for degree 11 becomes

$$\bar{\mathfrak{f}}(11\tau)^2\mathfrak{f}(11\tau)^2 - \overline{\mathfrak{f}_2}(11\tau)^2\mathfrak{f}_1(11\tau)^2 - \overline{\mathfrak{f}_1}(11\tau)^2\mathfrak{f}_2(11\tau)^2 = 4,$$

for the value $\tau = (7 + \sqrt{-127})/44$.

After twice putting the asymmetric part on one side of the equation and squaring we obtain

$$l^6 - 6l^4 + 17l^2 - 24 + 16/l^2 - 2/l^4 = m^8 + n^8.$$

After rearranging we can take the square root

$$l^4 - 3l^2 + 4 = l(m^4 + n^4).$$

Now we replace l with $1/(mn)$, multiply through by $(mn)^4$ and we have

$$4(mn)^4 - 3(mn)^2 + 1 = (mn)^3(m^4 + n^4).$$

Substituting in $m = 1/(1-n)$ and using the minimum polynomial for n we see that this is indeed a root of the equation. As usual we prove that our relation between m and n holds.

Finally substituting $n = 1-1/m$ into this equation gives us a minimum polynomial for m after rearranging and factorizing. In fact the resulting expression has only one quintic factor so we find that m satisfies $m^5 - m^3 - 4m^2 + 4m - 1 = 0$. Again we summarise our evaluations in a table. This is Table 2.4 above.

3. Other Discriminants and Class Numbers

Unfortunately our run of luck with class number five now comes to an end. The next two fundamental discriminants with this class number are $d = -131$ and $d = -179$. However the units that are associated with these discriminants are no longer Weber function eta quotients.

It seems that we need some new sets of functions which are eta quotients similar to the Weber function but where the arguments of the numerator and denominator differ by a factor of 3, 5 or even 7. In a later chapter we find exactly such sets of functions and find that they satisfy modular equations. Using these we are able to provide new evaluations which the Weber functions do not provide.

Another direction we would like to take things is to higher class numbers. The first discriminant of class number seven is $d = -71$. There are now three units which we would like to evaluate

$$\begin{aligned} m_1 = \sqrt{2}\left|\frac{\eta((1+\sqrt{-71})/2)}{\eta((1+\sqrt{-71})/4)}\right|^2 &= \sqrt{2} \Big/ \left|\mathfrak{f}_1\left(\frac{1+\sqrt{-71}}{2}\right)\right|^2 \\ &= \left|\mathfrak{f}_2\left(\frac{1+\sqrt{-71}}{4}\right)\right|^2 /\sqrt{2} = \left|\mathfrak{f}_2\left(\frac{3+\sqrt{-71}}{4}\right)\right|^2 /\sqrt{2}, \end{aligned}$$

$$\begin{aligned} m_2 = \sqrt{2}\left|\frac{\eta((1+\sqrt{-71})/4)}{\eta((3+\sqrt{-71})/8)}\right|^2 &= \sqrt{2} \Big/ \left|\mathfrak{f}\left(\frac{-1+\sqrt{-71}}{4}\right)\right|^2 \\ &= \sqrt{2} \Big/ \left|\mathfrak{f}\left(\frac{1+\sqrt{-71}}{4}\right)\right|^2 = \sqrt{2} \Big/ \left|\mathfrak{f}_1\left(\frac{3+\sqrt{-71}}{4}\right)\right|^2 . \end{aligned}$$

and

$$\begin{aligned} m_3 &= \sqrt{\frac{4}{3}}\left|\frac{\eta((3+\sqrt{-71})/8)}{\eta((1+\sqrt{-71})/6)}\right|^2 = \sqrt{\frac{4}{3}}\left|\frac{\eta((-5+\sqrt{-71})/8)}{\eta((-5+\sqrt{-71})/6)}\right|^2 \\ &= \sqrt{2}\left|\frac{\eta((5+\sqrt{-71})/8)}{\eta((5+\sqrt{-71})/16)}\right|^2 = \sqrt{2} \Big/ \left|\mathfrak{f}_1\left(\frac{5+\sqrt{-71}}{8}\right)\right|^2 = \sqrt{2} \Big/ \left|\mathfrak{f}\left(\frac{3+\sqrt{-71}}{8}\right)\right|^2 \end{aligned}$$

The first of these units can be expressed in terms of $\left|\mathfrak{f}(\sqrt{-71})\right|$. Firstly we note that

$$\left|\mathfrak{f}_1\left(\frac{1+\sqrt{-71}}{2}\right)\right| = \left|\mathfrak{f}\left(\frac{1+\sqrt{-71}}{2}\right)\right|,$$

and so

$$\left|\mathfrak{f}_2\left(\frac{1+\sqrt{-71}}{2}\right)\right| = \sqrt{2} \Big/ \left|\mathfrak{f}_1\left(\frac{1+\sqrt{-71}}{2}\right)\right|^2 = m_1.$$

But now

$$m_1 = \sqrt{2}/\left|\mathfrak{f}_1\left(1+\sqrt{-71}\right)\right| = \sqrt{2}/\left|\mathfrak{f}(\sqrt{-71})\right|.$$

But Weber has computed this last value in the tables at the end of his Lehrbuch [2]. We see in fact that m_1 has minimum polynomial $x^7+x^6-x^5-x^4-x^3+x^2+2x-1$.

Now we wish to relate m_1 and m_2 with a modular equation of degree 5. We note that

$$\mathfrak{f}_1\left(\frac{-3+\sqrt{-71}}{4}\right) = \mathfrak{f}_2\left(\frac{3+\sqrt{-71}}{20}\right), \quad \text{etc.}$$

The modular equation of degree five has already been given as equation (3) and it must lead to the same relation (4) as before except that this time we note that it must relate m_1 with $1/m_2$, i.e.

$$m_1^3/m_2 + m_1/m_2^3 = 1. \tag{5}$$

Numerically we note that $m_1 = 1 - m_2 - 1/m_2 + 1/m_2^2$. Substituting this in gives a prospective minimal polynomial for m_2, which is $x^7 - 2x^6 + 4x^5 - 4x^4 + 5x^3 - 4x^2 + 2x - 1$.

Forget our original definition of m_2 for a moment and define it to be the (only) real root of this polynomial. But if this is so then $1/m_2 = m_2^6 - 2m_2^5 + 4m_2^4 - 4m_2^3 + 5m_2^2 - 4m_2 + 2$. We square this to obtain a similar expression for $1/m_2^2$.

Now we plug these into $m_1' = 1 - m_2 - 1/m_2 + 1/m_2^2$ and reduce modulo the minimal polynomial for m_2. We obtain $m_1' = m_2^6 - m_2^5 + 2m_2^4 + m_2^2 - 1$. Now m_1' is real and we easily verify that it satisfies the minimal polynomial for m_1. Thus it in fact equals m_1.

Thus we have proven that $m_1 = 1 - m_2 - 1/m_2 + 1/m_2^2$ where m_2 is the real root of $x^7 - 2x^6 + 4x^5 - 4x^4 + 5x^3 - 4x^2 + 2x - 1$.

But since this really is only the purported m_2 let us now denote it m_2'. Thus we have a proven expression for m_1 in terms of m_2'. Plugging this into (5) we obtain a proven expression for m_2 in terms of m_2'. Since m_2 is determined uniquely by this equation it is sufficient to show that $m_2 = m_2'$ is a solution. This we do, thus showing that m_2 and m_2' are one and the same.

It is easy to see that these evaluations are not as easy as the class number five ones. The situation does not improve much for the unit m_3.

From the definitions above, we see that m_3 is equal to

$$\sqrt{2} \,/\, \left| \mathfrak{f}_1\left(\frac{5+\sqrt{-71}}{8}\right)\right|^2 = \sqrt{2} \,/\, \left| \mathfrak{f}_1\left(\frac{11+\sqrt{-71}}{8}\right)\right|^2$$

whilst m_2 becomes the value

$$\sqrt{2} \,/\, \left| \mathfrak{f}_1\left(\frac{3+\sqrt{-71}}{4}\right)\right|^2 = \left| \mathfrak{f}_2\left(\frac{3+\sqrt{-71}}{8}\right)\right|^2 /\sqrt{2} = \left| \mathfrak{f}_2\left(\frac{11+\sqrt{-71}}{8}\right)\right|^2 /\sqrt{2}.$$

But now we note that

$$\mathfrak{f}_2\left(\frac{-11+\sqrt{-71}}{8}\right) = \mathfrak{f}_2\left(\frac{11+\sqrt{-71}}{24}\right), \quad \text{etc.}$$

Therefore we can use the modular equation of degree three.

After a struggle similar to that above we find that m_3 has minimum polynomial $x^7 + x^6 + 2x^5 + 2x^4 + x^3 - 2x^2 - 3x - 1$.

4. Conclusion

We have been able to evaluate eta quotient units when the class number is five, by making use of interesting numerical coincidences. This works for the first four discriminants of class number five and it is interesting to note that it is the modular equations of degree 3, 5, 7 and 11 which are employed, in that order, for these cases.

After this we run into trouble expressing our eta units in terms of Weber functions. This gives the impetus for our development of generalizations of the Weber functions and their modular equations, which are described in the later chapters.

For class number seven, things are not so simple. It may be possible to do better by using our higher level Weber functions of later chapters. We defer a further discussion of eta quotient evaluations until then.

Over the next few chapters we look at Weber's development of modular equations for his Weber functions and we finally generalize these and their modular equations in preparation for further evaluation work. This generalization of the Weber functions will in fact be the workhorse of this document and represents the most significant part of the research that we undertake.

For the next chapter however we include a short (light) article written for the Australian Mathematical Society *Gazette* which recapitulates in succinct form the motivations which lead to the later work, as just described.

References

[1] Bruce Berndt, *Ramanujan's Notebooks Part V*, Springer-Verlag, 1998
[2] Heinrich Weber, *Lehrbuch der Algebra*, Dritter Band, Third Edition. Chelsea N.Y.