1. PROPERTIES OF ETA UNITS

(WITH ROBIN CHAPMAN)

Introduction

This chapter reports on joint work of myself and Robin Chapman, carried out when I visited the University of Exeter in 2002 with the assistance of a Postgraduate Research Fund grant supplied by Macquarie University.

Our aim was to investigate resolvents of logarithms of particular eta units. We start by defining, for a given discriminant d, the values

$$h(a,b,c) = a^{-\frac{1}{4}} \left| \eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|,$$

where (a,b,c) are the coefficients of a reduced binary quadratic form of discriminant $d = b^2 - 4ac$. These values $h(a,b,c)$ are themselves transcendental, but quotients of them are units in the real subfield of the Hilbert class field of the imaginary quadratic number field of discriminant d and turn out to have interesting properties.

The results that we report on here relate these units in a novel way to special values of L-series. In fact we show that the derivatives of the relevant L-series, evaluated at $s = 0$, are each closely related to resolvents formed from single eta units. This is in contrast to Kronecker's limit formula which relates the L-series' to sums over multiple eta units.

Along the way we notice some interesting properties involving Galois actions on our eta units. We are able to prove these and this process provides an excellent opportunity to survey the further results which we have alluded to in the previous chapter.

In particular we look at lower powers of eta quotients than the 24-th power. In the last chapter we saw that quotients of the discriminant function were in the Hilbert class field or various ring class fields, or at least small known extensions of the same. In this chapter we quote results which state that various lower powers of the eta quotients are also in these fields. These quotients are called *class invariants*.

The Galois action on such class invariants can then be described. It is this which leads us to the proofs of the results we mentioned above.

Further in the case of class number five we prove along the way a nice result which expresses the degree to which eta units fail to be fundamental in terms of the class number of the real subfield of the Hilbert class field.

1. L-series and Resolvents of Units

Our starting point is the following expression, which is equation (33) of the preliminaries chapter and was derived using the Kronecker Limit Formula and functional equation for partial zeta functions

$$L'(0,\overline{\chi}) = \frac{-4}{w(d)} \sum_{(a,b,c)} \chi((a,b,c)) \log\left(a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|\right). \tag{1}$$

The values $h(a,b,c)$ defined above are recognizable on the right hand of this expression and indeed it was this equation which motivated us to consider these particular values.

Again we remind the reader of the canonical isomorphism of the form class group and ideal class group of the imaginary quadratic number field of discriminant d, allowing us to consider the character χ as a character of either group. We also recall that $\overline{\chi}$ denotes the complex conjugate of the character χ.

We start with discriminants where the class number is 5. It is easiest to do this with an example. If we set $d = -47$ we have class number 5 and there are three distinct values to consider: $h(1,1,12)$, $h(2,1,6) = h(2,-1,6)$ and $h(3,1,4) = h(3,-1,4)$. We define the two quantities

$$m_1 = \frac{h(1,1,12)^2}{h(2,1,6)^2} \quad \text{and} \quad m_2 = \frac{h(2,1,6)^2}{h(3,1,4)^2}.$$

Of course we could have just as easily defined m_1 and m_2 with the forms $(2,1,6)$ and $(3,1,4)$ swapped. This is equivalent to considering the form class group as being generated by the class of $(3,1,4)$ instead of $(2,1,6)$. In fact this does make a difference to our calculations, especially when we invoke a character of the form class group below. As we noted already, this form class group character corresponds canonically to the character χ of our L-series. But choosing the binary quadratic forms here in a different order will simply correspond to permuting the characters of the form class group with respect to those of the ideal class group. We choose to spare the reader from a lengthy discussion of the explicit canonical isomorphism between the two character groups and keep this permutation of characters in mind when discussing our results.

As for the values m_1 and m_2, we will prove in the next section that they are in fact units and that they are in the real subfield of the Hilbert class field of $\mathbb{Q}(\sqrt{-47})$. We simply assume this for now so that the flow of thought is not broken at this point. In fact we often refer to the values m_i as units for convenience and leave the detailed demonstration till later.

In all cases where the class number is 5 the number of roots of unity $w(d)$ of the associated quadratic number field is 2. Thus, writing $\zeta = e^{\frac{2\pi i}{5}}$, the right hand side of (1) can be written

$$\begin{aligned} (2) \qquad & -2\left[\log(h(1,1,12)) + (\zeta+\zeta^4)\log(h(2,1,6)) + (\zeta^2+\zeta^3)\log(h(3,1,4))\right] \\ &= -\log(m_1) - 2[(1+\zeta+\zeta^4)\log(h(2,1,6)) + (\zeta^2+\zeta^3)\log(h(3,1,4))] \\ &= -\log(m_1) - 2[-(\zeta^2+\zeta^3)(\log(h(2,1,6)) - \log(h(3,1,4)))] \\ &= -\log(m_1) + (\zeta^2+\zeta^3)\log(m_2) \end{aligned}$$

This expression for the L-series is quite general. If we define the units m_1 and m_2 in an analogous way for other discriminants where the class number is 5 then we always have (for some character χ which depends on our specific choice of m_1 and m_2)

$$(3) \qquad L'(0,\overline{\chi}) = -\log(m_1) + (\zeta^2+\zeta^3)\log(m_2) = -\log(m_1) - \frac{1+\sqrt{5}}{2}\log(m_2).$$

A similar computation then shows that

$$(4) \qquad L'(0,\overline{\chi}^2) = -\log(m_1) + (\zeta+\zeta^4)\log(m_2) = -\log(m_1) - \frac{1-\sqrt{5}}{2}\log(m_2).$$

Now we turn to resolvents of logs of our units m_1 and m_2. The Hilbert class field H of the quadratic number field K of discriminant -47 is a Galois extension of $\mathbb{Q}$ of degree 10 with dihedral Galois group generated by complex conjugation and by an automorphism σ which has order 5. The automorphism σ also generates the Galois group of H/K which is cyclic of order 5.

We will be interested in the resolvents

$$\mathrm{res}(m_2, k) = \sum_{i=1}^{5} \zeta^{ik} \log|\sigma^i(m_2)| \quad \text{and} \quad \mathrm{res}(m_1, k) = \sum_{i=1}^{5} \zeta^{ik} \log|\sigma^i(m_1)|,$$

particularly for the values $k = 1, 2$ and where we again have $\zeta = e^{\frac{2\pi i}{5}}$.

It is possible to use the Artin map to write these resolvents in terms of characters which correspond in a canonical way to those of the L-series. However apart from being, as noted above, a complication which is best avoided, the inconsistent way that PARI treats automorphisms would make this a difficult project, even if it did provide any worthwhile benefit here.

As Artin reciprocity is a deep result there is little doubt that out results below indicate by their very nature which resolvents pair with which L-series, for only certain pairings lead to numerically recognizable algebraic values.

Again there is nothing special about the discriminant $d = -47$, for our purposes, except that it has class number 5. All the same expressions can be calculated for any disciminant with that class number.

The following PARI program allows the computation of the resolvent $\mathrm{res}(-, k)$ of an eta unit and the L-series' $L'(0, \overline{\chi}^k)$.

```
\p 500;\ps 500
h(a,b,c) = a^(-1/4)*sqrt(norm(eta((b+sqrt(b^2-4*a*c))/(2*a),1)))
z = exp(2*Pi*I/5)
co(a,d)=local(r);bnf=bnfinit(polred(polcompositum(x^2-d,quadhilbert(d
   ))[1])[5]);aut=nfgaloisconj(bnf);r=polroots(bnf.pol)[1];return(lin
   dep([a,r^9,r^8,r^7,r^6,r^5,r^4,r^3,r^2,r,1]));
pol(a,d) = local(c);c = co(a,d);return(Mod(Pol(c)/(-c[1])+x^10,bnf.po
   l));
s=vector(5)
sig(a,d)=local(p);p=pol(a,d);s[1]=p;s[2]=nfgaloisapply(bnf,aut[2],s[1
   ]);s[3]=nfgaloisapply(bnf,aut[2],s[2]);s[4]=nfgaloisapply(bnf,aut[
   2],s[3]);s[5]=nfgaloisapply(bnf,aut[2],s[4]);return(s);
seval(a,d)=local(r);sig(a,d);r=polroots(bnf.pol)[1];x=r;return([eval(
   s[1].pol),eval(s[2].pol),eval(s[3].pol),eval(s[4].pol),eval(s[5].p
   ol)]);
t=vector(5)
initconjugates(a,d)=t=seval(a,d);kill(x);
l(y)=log(sqrt(norm(y)))
res(i)=l(t[1])+z^i*l(t[2])+z^(2*i)*l(t[3])+z^(3*i)*l(t[4])+z^(4*i)*l(
   t[5])
LOX(j)=-(log(m1)-(z^(2*j)+z^(3*j))*log(m2))
```

The following fragment of code demonstrates how to use this PARI program. It calculates two quantities

$$q_1 = \frac{L'(0, \overline{\chi})}{\mathrm{res}(m_2, 2)} \quad \text{and} \quad q_2 = \frac{L'(0, \overline{\chi}^2)}{\mathrm{res}(m_2, 1)},$$

for discriminant $d = -47$ and expresses them as linear combinations of $\zeta, \zeta^2, \zeta^3, \zeta^4$.

```
m1 = h(1,1,12)^2/h(2,1,6)^2
m2 = h(2,1,6)^2/h(3,1,4)^2
initconjugates(m2,-47)
q1 = LOX(1)/res(2)
q2 = LOX(2)/res(1)
lindep([q1,z,z^2,z^3,z^4])
lindep([q2,z,z^2,z^3,z^4])
```

The coefficient vectors which are obtained from this computation expressing q_1 and q_2 in terms of powers of ζ are $[-5,-2,2,2,-2]$ and $[-5,2,-2,-2,2]$ respectively. After summing the fifth roots of unity that we have obtained and dividing by 5, we discover that, remarkably,

$$L'(0,\overline{\chi}) = -\frac{2\,\mathrm{res}(m_2,2)}{\sqrt{5}} \quad \text{and} \quad L'(0,\overline{\chi}^2) = \frac{2\,\mathrm{res}(m_2,1)}{\sqrt{5}}. \tag{5}$$

Note that in contrast to the equations (3) and (4) the right hand sides here only involve the unit m_2.

Of course we can do similar computations with the unit m_1 instead of m_2. The same combinations q_1 and q_2 as expressed in the code above are meaningful, with the coefficient vectors being $[-5,4,6,6,4]$ and $[-5,6,4,4,6]$ respectively.

Summing the 5-th roots of unity resulting from these computations yields

$$L'(0,\overline{\chi}) = -(\sqrt{5}+1)\,\frac{\mathrm{res}(m_1,2)}{\sqrt{5}} \quad \text{and} \quad L'(0,\overline{\chi}^2) = (1-\sqrt{5})\,\frac{\mathrm{res}(m_1,1)}{\sqrt{5}}. \tag{6}$$

For future reference we note that if we write all the expressions obtain so far in the form

$$L'(0,\overline{\chi}^j) = \frac{2\alpha}{5}\,\mathrm{res}(m_i,k),$$

then in the first two cases above α takes on the roots of $x^2-5=0$ whilst in the other two cases α takes on the roots of x^2+5x+5.

What is more remarkable perhaps is that there is nothing special about -47. These equations hold if we replace -47 with another discriminant of class number 5 (up to the ambiguity in character mentioned above). In fact we have checked that the previous equations hold, modulo a possible permutation of characters, for $d = -79,-103,-127,-131$ and -179. We conjecture that these are quite general results for class number five.

We also happened to notice numerically that in each case $m_1 = m_2^{\sigma^2+\sigma^3}$. It will be a major aim of this chapter to prove this fact and the relations given above which are actually related.

Before we make any preparations for proving anything, let us extend our computational work into the case where the class number is 7. This time there are seven reduced binary quadratic forms, which we shall denote (a_0,b_0,c_0) and $(a_i,\pm b_i,c_i)$ for $i=1,2,3$.

Here we intend that (a_0,b_0,c_0) represents the identity of the form class group and (a_i,b_i,c_i) composed with (a_1,b_1,c_1) is equivalent to $(a_{i+1},b_{i+1},c_{i+1})$ for $i=1,2$.

There are numerous ways that we could take quotients of the values $h(a_i,b_i,c_i)$ to give units (as usual we call them units since in all cases we quote this is true numerically but we again leave any detailed proofs till later). We will make the following choice

$$m_1 = \frac{h(a_0,b_0,c_0)^2}{h(a_1,b_1,c_1)^2}, \quad m_2 = \frac{h(a_1,b_1,c_1)^2}{h(a_2,b_2,c_2)^2}, \quad m_3 = \frac{h(a_2,b_2,c_2)^2}{h(a_3,b_3,c_3)^2},$$

where we have taken the square so that the values m_i lie in the real subfield of the Hilbert Class Field of the underlying quadratic field of discriminant d. We again defer detailed justification of this till later.

Again using the identity (1), we can write

$$\begin{aligned}(7)\quad L'(0,\overline{\chi}) &= -2\,[\log(h(a_0,b_0,c_0) + (\zeta+\zeta^6)\log(h(a_1,b_1,c_1)) \\ &\quad + (\zeta^2+\zeta^5)\log(h(a_2,b_2,c_2)) + (\zeta^3+\zeta^4)\log(h(a_3,b_3,c_3))] \\ &= -m_1 - 2[(1+\zeta+\zeta^6)\log(h(a_1,b_1,c_1)) \\ &\quad + (\zeta^2+\zeta^5)\log(h(a_2,b_2,c_2)) + (\zeta^3+\zeta^4)\log(h(a_3,b_3,c_3))] \\ &= -m_1 - (1+\zeta+\zeta^6)\,m_2 - 2[(1+\zeta+\zeta^6+\zeta^2+\zeta^5)\log(h(a_2,b_2,c_2)) \\ &\quad + (\zeta^3+\zeta^4)\log(h(a_3,b_3,c_3))] \\ &= -m_1 - (1+\zeta+\zeta^6)\,m_2 + (\zeta^3+\zeta^4)\,m_3,\end{aligned}$$

where we now have $\zeta = e^{-\frac{2\pi i}{7}}$.

We adjust the definition of our resolvents $\mathrm{res}(m_i,k)$ in an obvious way and allow $k \in \{1,2,3\}$. Our Pari program becomes

```
\p 1000;\ps 1000
classnum = 7
h(a,b,c)=a^(-1/4)*sqrt(norm(eta((b+sqrt(b^2-4*a*c))/(2*a),1)))
z=exp(2*Pi*I/classnum)
co(a,d)=local(r);bnf=bnfinit(polred(polcompositum(x^2-d,quadhilbert(d
   ))[1])[7]);aut=nfgaloisconj(bnf);r=polroots(bnf.pol)[1];return(lin
   dep([a,r^13,r^12,r^11,r^10,r^9,r^8,r^7,r^6,r^5,r^4,r^3,r^2,r,1]));
pol(a,d)=local(c);c=co(a,d);return(Mod(Pol(c)/(-c[1])+x^(2*classnum),
   bnf.pol));
s=vector(classnum)
sig(a,d)=local(p);p=pol(a,d);s[1]=p;for(y=1,classnum-1,s[y+1]=nfgaloi
   sapply(bnf,aut[2],s[y]));return(s);
t=vector(classnum)
seval(a,d)=local(r);sig(a,d);r=polroots(bnf.pol)[1];x=r;for(y2=1,clas
   snum,t[y2]=eval(s[y2].pol));return(t);
initconjugates(a,d)=seval(a,d);kill(x);
l(y)=log(sqrt(norm(y)))
res(i)=sum(yval=1,classnum,z^((yval-1)*i)*l(t[yval]))
LOX(j)=-log(m1)-(1+z^(j)+z^(6*j))*log(m2)+(z^(3*j)+z^(4*j))*log(m3)
```

Note that in some later versions of Pari the ordering of automorphisms has been changed. Thus we replace

```
....bnf,aut[2],s[y]....
```

with

```
....bnf,aut[11],s[y]....
```

in the line of code near the centre of this program.

The following is an example of how to use this program for the discriminant $d = -71$ which has class number 7.

```
m1 = h(1,1,18)^2/h(2,1,9)^2
m2 = h(2,1,9)^2/h(4,3,5)^2
m3 = h(4,3,5)^2/h(3,1,6)^2
initconjugates(m1,-71)
```

m_i	j	k	coefficients	α min. poly.
m_1	1	3	$-7, 6, 10, 12, 12, 10, 6$	$x^3 + 14x^2 + 49x + 49$
	2	1	$-7, 12, 6, 10, 10, 6, 12$	
	3	2	$-7, 10, 12, 6, 6, 12, 10$	
m_2	1	3	$-7, 2, 8, 4, 4, 8, 2$	$x^3 + 7x^2 - 49$
	2	1	$-7, 4, 2, 8, 8, 2, 4$	
	3	2	$-7, 8, 4, 2, 2, 4, 8$	
m_3	1	3	$-7, 4, 2, 8, 8, 2, 4$	$x^3 + 7x^2 - 49$
	2	1	$-7, 8, 4, 2, 2, 4, 8$	
	3	2	$-7, 2, 8, 4, 4, 8, 2$	

Table 1.1

```
q=LOX(1)/res(3)
lindep([q,z,z^2,z^3,z^4,z^5,z^6])
```

For the discriminant $d = -71$ the above computation returns the coefficient vector $[-7, 6, 10, 12, 12, 10, 6]$ which, after rearranging, leads us to the following expression

$$L'(0, \overline{\chi}) = \frac{2\alpha}{7} \operatorname{res}(m_1, 3),$$

where α satisfies the equation $x^3 + 14x^2 + 49x + 49 = 0$.

Now there are various other combinations of the unit m_1, m_2, m_3 and values j and k in the expression

```
q = LOX(j)/res(k)
```

that we can use in the code above. A number of these yield meaningful values. We summarise the relevant combinations in the table above.

In summing the linear combinations of 7-th roots of unity which these computations report, we always obtain an expression of the form

$$L'(0, \overline{\chi}^j) = \frac{2\alpha}{7} \operatorname{res}(m_i, k)$$

where α satisfies a cubic equation. The minimal polynomials for α are listed in the table.

It is interesting to note from the table that the minimum polynomial of α only depends on the value m_i which is being used and that in fact for a particular given value m_i, the values α take on all three of the roots of the particular polynomial listed as j and k vary.

We also note the divisibility of the coefficients of the minimum polynomials by 7, although this should not be a suprise given that our character takes 7-th roots of unity as values.

Now the values m_i which we picked are not particularly special. In fact there are numerous other units u which can be expressed as the quotient of two different squares of $h(a, b, c)$ values.

In the following table we give a list of such units along with the various coefficient vectors that result from our program for various values of j and k. The minimum polynomials for the various values of α can easily be determined by comparison with the results in the previous table since, as is clear, they are all equal to plus or minus one of the values from that earlier table.

Now if we replace a particular unit u with its multiplicative inverse we simply obtain the additive inverse for the value α. The reason for this is easy to see from the definition of $\operatorname{res}(u, k)$.

unit	j	k	coefficients
m_2m_3	1	3	$-7, -8, -4, -2, -2, -4, -8$
	2	1	$-7, -2, -8, -4, -4, -8, -2$
	3	2	$-7, -4, -2, -8, -8, -2, -4$
$m_1m_2m_3$	1	3	$-7, 10, 12, 6, 6, 12, 10$
	2	1	$-7, 6, 10, 12, 12, 10, 6$
	3	2	$-7, 12, 6, 10, 10, 6, 12$
m_1m_2	1	3	$-7, -12, -6, -10, -10, -6, -12$
	2	1	$-7, -10, -12, -6, -6, -12, -10$
	3	2	$-7, -6, -10, -12, -12, -10, -6$

Table 1.2

It is also easy to see from the definition of $\text{res}(u, k)$ how the values in this second table can be calculated from those of the previous table, since $\log|\sigma^i(u_1u_2)| = \log|\sigma^i(u_1)| + \log|\sigma^i(u_2)|$ for any two units u_1 and u_2.

What is more interesting perhaps is that we end up with no new values for α other than those in the first table (except, of course, the additive inverses of these values) until we take units which are *not* of the form $u = h(r_1, r_2, r_3)^2/h(s_1, s_2, s_3)^2$. The unit $m_1^2m_2m_3 = h(1, 1, 18)^4/(h(2, 1, 9)\,h(3, 1, 6))^2$ is such an example. In that case the coefficients that result are $[-91, 40, 48, 38, 38, 48, 30]$.

As for the class number five situation, there is nothing special about the discriminant $d = -71$, except that it has class number seven. We have checked that for $d = -151$, which is also class number seven, exactly the same values of α result. Again we speculate that a very general result is at work here.

It has not been possible at this time to extend our computations to the class number 9 and 11 cases, since the memory and computational power required to run our program already becomes prohibitive in these cases.

2. Lower Powers of Eta Quotients

In the previous section we made a number of assertions about our eta quotients which were not proved there. In one such case we claimed that various powers of quotients of the values $h(a, b, c)$ were in the real subfield of the Hilbert class field (or more generally some ring class field) of the related quadratic number field. This section is devoted to proving that result where possible.

The "main theorems" of complex multiplication state that if ω is a root of a binary quadratic form (a, b, c) then

(i) if (a, b, c) has fundamental discriminant $d = b^2 - 4ac < 0$ then $j(\omega)$ is in the Hilbert class field of $\mathbb{Q}(\omega)$ (in fact it generates it); and

(ii) if (a, b, c) has non-fundamental discriminant $D = k^2d = b^2 - 4ac < 0$ then $j(\omega)$ generates the ring class field modulo k of $\mathbb{Q}(\omega)$.

Here $j(\tau)$ is the *absolute modular invariant function* which is well known from the theory of modular forms and elliptic curves.

Now Weber defines three functions

$$\mathfrak{f}(\tau) = \frac{e^{-\frac{\pi i}{24}}\,\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}, \quad \mathfrak{f}_1(\tau) = \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \quad \mathfrak{f}_2(\tau) = \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)} \tag{8}$$

which have been called the *Weber functions*.

He shows that the functions $\mathfrak{f}(\tau)^{24}$, $-\mathfrak{f}_1(\tau)^{24}$ and $-\mathfrak{f}_2(\tau)^{24}$ satisfy the equation

$$(x - 16)^3 = jx,$$

ω	d	eta quotient	class invariant
$\frac{3+\sqrt{d}}{2}$	$d \equiv 5 \pmod 8$	$\sqrt{2}\,\phi = \varepsilon\,\mathfrak{f}_2^3(\omega) \in \mathbb{R}^+,\ \varepsilon^{16} = 1$	$\phi \in \mathrm{Re}\{R_2\}$
	$d \equiv 1 \pmod 8$	$\phi = \varepsilon\,\mathfrak{f}_2^3(\omega) \in \mathbb{R}^+,\ \varepsilon^{16} = 1$	$\phi \in \mathrm{Re}\{H\}$
$\sqrt{d}$	$d \equiv 3 \pmod 4$	$\phi_1 = \mathfrak{f}_1^{12}/\sqrt{2}^3,\ \phi_2 = \mathfrak{f}_2^{12}/\sqrt{2}^3$ $\phi = \mathfrak{f}^{12}/8$	$\phi_1,\ \phi_2 \in \mathrm{Re}\{R_2\}$ $\phi \in \mathrm{Re}\{H\}$
	$d \equiv 7 \pmod 8$	$\phi = \mathfrak{f}^6/\sqrt{2}^3$	$\phi \in \mathrm{Re}\{H\}$
	$d \equiv 2 \pmod 4$	$\phi = \mathfrak{f}_1^6/\sqrt{2}^3$	$\phi \in \mathrm{Re}\{H\}$
	$d \equiv 5 \pmod 8$	$\phi = \mathfrak{f}^3$	$\phi \in \mathrm{Re}\{H\}$
	$d \equiv 1 \pmod 8$	$\phi = \mathfrak{f}^3/\sqrt{2}^3$	$\phi \in \mathrm{Re}\{H\}$
	$d \equiv 12 \pmod{16}$	$\phi = \mathfrak{f}_1^{12}/\sqrt{2}^9$	$\phi \in \mathrm{Re}\{H\}$
	$d \equiv 20 \pmod{32}$	$\phi = \mathfrak{f}_1^{12}/2^5$	$\phi \in \mathrm{Re}\{H\}$
	$d \equiv 4 \pmod{32}$	$\phi = \mathfrak{f}_1^{12}/2^6$	$\phi \in \mathrm{Re}\{H\}$

Table 1.3

where $j = j(\tau)$ is the modular invariant function as above. In other words, $\mathbb{Q}(f(\tau))$, with $f(\tau)$ the 24-th power of any of the Weber functions, is a cubic extension of $\mathbb{Q}(j(\tau))$, *as function fields*.

However, when we specialise to a certain value $\tau = \omega$, say, for some root ω of a binary quadratic form, it may be that the resulting $f(\omega)$ is actually *in* $\mathbb{Q}(j(\omega))$. In other words, $\mathbb{Q}(f(\omega)) = \mathbb{Q}(j(\omega))$, *as number fields*. When this happens, Weber calls the value $f(\omega)$ a *class invariant*.

Thus for various values of ω, the 24-th powers of the Weber functions are class invariants. However it may also be that lower powers of the Weber functions become class invariants for various values ω.

Weber himself gave many examples of precisely this phenomenon. Unfortuantely not all of his results were adequately proved.

Much later Birch [1] proved all of Weber's results using *Söhngen's Theorem*. A similar proof could probably be effected using *Shimura's Reciprocity Law*. However the version based on Söhngen's theorem seems to be simple enough.

We will not quote Söhngen's theorem or Birch's proofs here but simply refer the reader to his paper [1]. We will however quote and collate the results themselves which do not seem to be tabulated there.

We state the results in a table with four columns. The first column contains the value of ω in terms of $\sqrt{d}$ for some negative integer d (we assume $d \neq -2, -4$) which provides us with our class invariant. The second column contains a congruence condition on the value d. The third column lists the eta quotient which is a class invariant. Finally the fourth column gives the particular ring class field which the class invariant lies in. We will write H for the Hilbert class field of $\mathbb{Q}(\omega)$ and R_n for its ring class field modulo n and $\mathrm{Re}\{H\}$ for the real subfield of H, etc.

Moreover Birch proves that if 3 does not divide the discriminant of the reduced binary quadratic form of which ω is a root then the same results hold if the functions ϕ, ϕ_1 and ϕ_2 are all replaced with similar functions where all the exponents of $\sqrt{2}$, 2, $\mathfrak{f}$, $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are divided by three.

Birch also states that all the values ϕ, ϕ_1, ϕ_2 that are quoted in the table are in fact units in the fields given. Birch uses some results of Deuring to prove this fact.

We will now use the results in the table to prove that various of our eta quotients are in fact units in the Hilbert class field of the related quadratic number field.

In some cases we must wait until we compute the units explicitly to demonstrate that they are in fact units, as Birch's results and their generalizations do not seem to provide us with this information.

We begin with the value m_1 for the discriminant $d = -47$.

We have that

$$m_1 = \sqrt{2}\left|\frac{\eta((1+\sqrt{-47})/2)}{\eta((1+\sqrt{-47})/4)}\right|^2 = \sqrt{2}\ /\ \left|\mathfrak{f}_1((1+\sqrt{-47})/2)\right|^2.$$

However, note that

$$\left|\mathfrak{f}((1+\sqrt{-47})/2)\right| = \left|\mathfrak{f}_1((-1+\sqrt{-47})/2)\right| = \left|\mathfrak{f}_1((1+\sqrt{-47})/2)\right|.$$

Therefore, since $\mathfrak{f}\,\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2}$ we have that

$$m_1 = \left|\mathfrak{f}_2((1+\sqrt{-47})/2)\right| = \left|\mathfrak{f}_2((3+\sqrt{-47})/2)\right|.$$

According to the results above this is a real unit in the Hilbert class field of $\mathbb{Q}(\sqrt{-47})$.

An identical argument shows that the value m_1 is a unit in the real subfield of the Hilbert class field for the discriminants $d = -79$, -103 and -127. That is, we have that

$$m_1 = \left|\mathfrak{f}_2((3+\sqrt{d})/2)\right|$$

for the given discriminants d.

As for the values m_2 for the discriminants $d = -79$ and -127, we can express them in terms of Weber functions, however it is not possible to change the argument to one of those given in our table. Thus we do not deal with these values here.

Returning to the value m_2 for the discriminant $d = -47$ we find that although it can be expressed in terms of Weber functions it is more fruitful to consider it in terms of some alternative class invariants.

Alice Gee investigated a generalization of the Weber functions in the fourth chapter of her thesis [2]. She defines

$$\mathfrak{g}_0(z) = \frac{\eta\left(\frac{z}{3}\right)}{\eta(z)} \quad \mathfrak{g}_1(z) = \zeta_{24}^{-1}\frac{\eta\left(\frac{z+1}{3}\right)}{\eta(z)} \quad \mathfrak{g}_2(z) = \frac{\eta\left(\frac{z+2}{3}\right)}{\eta(z)} \quad \mathfrak{g}_3(z) = \sqrt{3}\,\frac{\eta(3z)}{\eta(z)}.$$

We will ourselves define functions similar to these in a later chapter.

Gee shows that various class invariants can be constructed out of these functions. Her table 2 lists various cases given by congruence conditions on the fundamental discriminant. Each of these class invariants is given as a function evaluated at $\theta = \sqrt{d/4}$ if $4|d$ or at $\theta = (-1+\sqrt{d})/2$ otherwise.

Firstly we note that

$$\frac{h(1,1,12)^2}{h(3,1,4)^2} = \sqrt{3}\left|\frac{\eta((1+\sqrt{-47})/2)}{\eta((1+\sqrt{-47})/6}\right|^2 = \sqrt{3}\ /\ |\mathfrak{g}_0((1+\sqrt{-47})/2)|^2.$$

But according to Gee's table for $d = -47$ the value $\mathfrak{g}_0^2(\theta)$ is a class invariant. This does not help us with finding a unit in the Hilbert class field but by the theorem 0.5.2 of our preliminaries chapter it is clear that if the value $\sqrt{3}\ /\ |\mathfrak{g}_0((1+\sqrt{-47})/2)|^2$ is actually in the Hilbert class field H of $\mathbb{Q}(\sqrt{-47})$ then it is a real *unit*in H.

Although we are unable to calculate this unit we do later explicitly calculate Gee's class invariant which is of course itself an eta quotient.

But then the value m_2 is the quotient of this real unit and the real unit m_1. Thus it would itself be a real unit in the Hilbert class field of $\mathbb{Q}(\sqrt{-47})$. Unfortunately

Gee's results do not help us to demonstrate that the value $\sqrt{3}\ /\ |\mathfrak{g}_0((1+\sqrt{-47})/2)|^2$ is actually in H.

For the discriminants $d=-131$ and $d=-179$ the values m_1 are of the form

$$m_1=\sqrt{3}\left|\frac{\eta((1+\sqrt{d})/2)}{\eta((1+\sqrt{d})/6}\right|^2=\sqrt{3}\ /\ |\mathfrak{g}_0((1+\sqrt{d})/2)|^2.$$

A similar argument applies to these values, except that for $d=-179$ we use the fact from Gee's table that $\zeta_3\,\mathfrak{g}_0^2(\theta)$ is a class invariant for this discriminant. Again we cannot demonstrate that they are actually units until we can demonstrate that they are actually in the Hilbert class field.

The other units for the class number five cases are neither recognizable as Birch's Weber function class invariants nor those involving Gee's level 3 functions.

We now consider the class number seven cases. The m_1 values for $d=-71$ and -151 are nothing more than examples of the case we dealt with for the m_1 value of $d=-47$ above.

The only other case which can be dealt with is the value $m_1m_2m_3$ for the discriminant $d=-71$. The argument for the level three functions that we gave above works here, since according to Gee's table we are again in a case where $\zeta_3\,\mathfrak{g}_0^2(\theta)$ is a class invariant.

It is likely that a similar analysis to Gee's can be done for level five generalizations of the Weber functions giving further class invariants involving those functions.

As for the values such as $h(2,1,10)^2/h(4,1,5)^2$, that we have not dealt with, it is not clear how to prove that they are class invariants let alone that they are real units in the Hilbert class field. The fact that so many such values exist which cannot be dealt with by Birch or Gee's theorems and that numerically are seen to be units does suggest that an extensive generalization of their results should exist. Unfortunately we have not however been able to find such a theorem or its proof at this time.

3. Galois Actions on Eta Quotients

In section 5 of the preliminaries chapter we quoted a result which shows how K-automorphisms of the Hilbert class field act on quotients of the discriminant function. In fact a much more specific result is available. It is proved in [3] as following from a reciprocity relation given by Stark.

Firstly we need a definition.

Definition 3.0.1. *Let $K=\mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field and let $D=m^2d$ be some not necessarily fundamental discriminant. Let $\mathfrak{m}$ be any integral ideal of K which is not divisible by an integer greater than 1 (primitive). Let its norm be m and let m be prime to $6D$. Suppose $\mathfrak{m}$ has $\mathbb{Z}$-basis $[\omega-t,m]$ for some integer t and $\omega=(-D+\sqrt{-D})/2$.*

We define

$$\eta(\mathfrak{m})=\exp(\pi i m(2-t)/12)\,\eta((t-\overline{\omega})/m).$$

The definition is independent of the choice of the integer t.

Hajir's theorem can be written as follows

Theorem 3.0.2. *Given two primitive integral ideals $\mathfrak{q}$ and $\mathfrak{m}$ of K prime to $6D$, let $\sigma(\mathfrak{q})$ be the Frobenius automorphism of $\mathfrak{q}$ in $Gal(K^{ab}/K)$, then*

$$\left(\frac{\eta(\mathfrak{m})}{\eta(\mathcal{O})}\right)^{\sigma(\mathfrak{q})} = \frac{\eta(\mathfrak{m}\mathfrak{q})}{\eta(\mathfrak{q})}, \quad \left(\frac{\overline{\eta}(\mathfrak{m})}{\overline{\eta}(\mathcal{O})}\right)^{\sigma(\mathfrak{q})} = \frac{\overline{\eta}(\mathfrak{m}\overline{\mathfrak{q}})}{\overline{\eta}(\overline{\mathfrak{q}})},$$

where $\overline{\eta}(\mathfrak{m}) = \overline{\eta(\mathfrak{m})}$.

We will use this theorem to prove the following result which we already noted numerically

Theorem 3.0.3. *For m_1 and m_2 defined as above in terms of reduced forms of a discriminant of class number five we have*

$$m_1 = m_2^{\sigma^2+\sigma^3} = \left|m_2^{\sigma^2}\right|^2,$$

for some Frobenius automorphism σ in $Gal(K^{ab}/K)$.

Proof: Since m_1 and m_2 are real and $m_2^{\sigma^2}$ is the complex conjugate of $m_2^{\sigma^3}$ the roots of unity in the definition of $\eta(\mathfrak{m})$ are clearly irrelevant to our theorem.

The action of the Frobenius automorphisms on rational integers is trivial. Thus the result of the theorem above can be rewritten

$$\left(m^{-1/4}\,\frac{\eta(\mathfrak{m})}{\eta(\mathcal{O})}\right)^{\sigma(\mathfrak{q})} = m^{-1/4}\,\frac{\eta(\mathfrak{m}\mathfrak{q})}{\eta(\mathfrak{q})}$$

Now we note that

$$m_2 = \frac{h(a_1,b_1,c_1)^2}{h(a_2,b_2,c_2)^2} = \sqrt{\frac{a_2}{a_1}}\left|\frac{\eta(\mathfrak{m})}{\eta(\mathfrak{n})}\right|^2,$$

for some ideals $\mathfrak{m}$ and $\mathfrak{n}$ with norms a_1 and a_2 respectively.

Applying the theorem we obtain

$$m_2^{\sigma^2(\mathfrak{q})} m_2^{\sigma^3(\mathfrak{q})} = \frac{a_2}{a_1}\left|\frac{\eta(\mathfrak{m}\mathfrak{q}^2)}{\eta(\mathfrak{n}\mathfrak{q}^2)}\,\frac{\overline{\eta}(\mathfrak{m}\overline{\mathfrak{q}}^2)}{\overline{\eta}(\mathfrak{n}\overline{\mathfrak{q}}^2)}\,\frac{\eta(\mathfrak{m}\mathfrak{q}^2)}{\eta(\mathfrak{n}\mathfrak{q}^2)}\,\frac{\overline{\eta}(\mathfrak{m}\overline{\mathfrak{q}}^2)}{\overline{\eta}(\mathfrak{n}\overline{\mathfrak{q}}^2)}\right|.$$

However the quantity $a_1^{-1/2}\,|\eta(\mathfrak{m})|^2$ is a real 24-th root of the well known class invariant $F(C) = |\mathbb{N}(\mathfrak{m})^6\Delta(\mathfrak{m})|^2$ and therefore itself depends only on the ideal class C of $\mathfrak{m}$. In fact the previous equation can be expressed in terms of such class invariants.

Now write $\{\mathfrak{m}\}$ for the ideal class of $\mathfrak{m}$. Suppose that $\{\mathfrak{m}\}^2 = \{\mathfrak{n}\}$ and also suppose that the automorphism σ is chosen so that $\{\mathfrak{m}\mathfrak{q}\} = \{\overline{\mathfrak{n}}\}$.

Once this is done, there is cancellation in the expression above and we are simply left with

$$m_2^{\sigma^2(\mathfrak{q})} m_2^{\sigma^3(\mathfrak{q})} = \frac{h(a_0,b_0,c_0)^2}{h(a_1,b_1,c_1)^2} = m_1,$$

as required. □

There are of course similar relationships which hold in the class number seven case. In fact there are a multitude of such relations. We shall prove the following

Theorem 3.0.4. *There is a Frobenius automorphism σ such that*

$$m_3 = |m_1^{\sigma}|^{-2}.$$

Proof: We write

$$m_1 = \frac{h(a_0, b_0, c_0)^2}{h(a_1, b_1, c_1)^2}.$$

We will suppose that the form (a_i, b_i, c_i) corresponds to the ideal $\mathfrak{m}_i$ of norm a_i, for $i \geq 0$. Similarly think of the ideal $\overline{\mathfrak{m}}_i$ as corresponding to the form $(a_i, -b_i, c_i)$.

Write $\{\mathfrak{m}_i\}$ for the class of $\mathfrak{m}_i$. The powers of $\{\mathfrak{m}_1\}$ in order will be $\{\mathfrak{m}_1\}$, $\{\mathfrak{m}_2\}$, $\{\mathfrak{m}_3\}$, $\{\overline{\mathfrak{m}}_3\}$, $\{\overline{\mathfrak{m}}_2\}$, $\{\overline{\mathfrak{m}}_1\}$ and $\{\mathfrak{m}_0\}$.

Also suppose that $\mathfrak{q}$ is an ideal with $\{\mathfrak{q}\} = \{\mathfrak{m}_1\}^3$.

By the theorem of Hajir

$$|m_1^\sigma|^{-2} = \frac{a_0}{a_1}\left|\frac{\eta(\mathfrak{m}_1\mathfrak{q})\,\overline{\eta}(\mathfrak{m}_1\overline{\mathfrak{q}})\,\eta(\mathfrak{m}_1\mathfrak{q}^6)\,\overline{\eta}(\mathfrak{m}_1\overline{\mathfrak{q}}^6)}{\eta(\mathfrak{m}_0\mathfrak{q})\,\overline{\eta}(\mathfrak{m}_0\overline{\mathfrak{q}})\,\eta(\mathfrak{m}_0\mathfrak{q}^6)\,\overline{\eta}(\mathfrak{m}_0\overline{\mathfrak{q}}^6)}\right| = \frac{a_0}{a_1}\left|\frac{\eta(\overline{\mathfrak{m}}_3)\,\overline{\eta}(\overline{\mathfrak{m}}_2)\,\eta(\overline{\mathfrak{m}}_2)\,\overline{\eta}(\overline{\mathfrak{m}}_3)}{\eta(\mathfrak{m}_3)\,\overline{\eta}(\overline{\mathfrak{m}}_3)\,\eta(\overline{\mathfrak{m}}_3)\,\overline{\eta}(\mathfrak{m}_3)}\right|,$$

which after simplification and cancellation leads to the required result as per the previous theorem. □

A vast list of similar relations exist which can be proved by similar methods. We list some of them here without giving the explicit proofs.

Theorem 3.0.5.

$$m_2 = \left|m_1^{\sigma+\sigma^3}\right|^{-2}, \quad m_1 m_3 = |m_2^\sigma|^2, \quad m_3 = \left|m_2^{\sigma+\sigma^2}\right|^2, \quad m_1 = \left|m_2^{\sigma^2}\right|^2,$$
$$m_1 = \left|m_3^{\sigma^2+\sigma^3}\right|^{-2}, \quad m_2 = \left|m_3^{\sigma^3}\right|^{-2}. \tag{9}$$

4. Regulators

In this section we look at regulators of eta units in the Hilbert class field (or in its real subfield). On the one hand we use the analytic class number formula to relate the Dedekind zeta function of the Hilbert class field to such a regulator and on the other hand we express the Dedekind zeta function as a product of L-series which we believe have expressions in terms of resolvents of eta units, as per the first section of this chapter.

We begin by stating a relevant version of the analytic class number formula. It can be derived from the version in the previous chapter by summing over all ideal classes.

Theorem 4.0.6. *The zeta function of a field K has the first non-zero coefficient in its Taylor series expansion at $s = 0$ given by*

$$\lim_{s\to 0} s^{-r_1-r_2+1}\zeta_K(s) = -\frac{h_K R_K}{w_K}$$

where w_K is the number of roots of unity in K, h_K is the class number and R_K is the regulator.

Now we consider the well known expression for the zeta function in terms of L-series.

Theorem 4.0.7. *Let L/K be a finite Abelian extension with Galois group $G = Gal(L/K)$. Let the character group of G be denoted $\hat{G}$, then*

$$\zeta_L(s) = \zeta_K(s) \prod_{1\neq\chi\in\hat{G}} L(s,\chi),$$

where the product is over all primitive L-series for the extension L/K.

If we take the field L to be the Hilbert class field H of an imaginary quadratic field K whose class number h_K is prime then we note that $r_1 + r_2 - 1 = h_K - 1$ (since $r_1 = 0$ and $2r_2 = h_k$). But the cardinality of the character group $\hat{G}$, minus its identity element, must also be $h_k - 1$. Thus applying the limit which appears in the first theorem above to this last theorem and evaluating the zeta function of K we have

Theorem 4.0.8. *Let H be the Hilbert class field of the imaginary quadratic field K with Galois group G of prime order. Then*

$$\prod_{1 \neq \chi \in \hat{G}} L'(0, \chi) = \frac{h_H R_H}{h_K},$$

where h_H and h_K are the class numbers of the Hilbert class field and of K respectively and R_H is the regulator of the Hilbert class field.

Before this result becomes of particular use to us we need to express the regulator of the Hilbert class field H in terms of the regulator of its real subfield $\mathrm{Re}\{H\}$. To do this we use one of the Brauer relations which appear in [4]. First we need a couple of definitions.

Definition 4.0.9. *For a subgroup T and element γ of some other group let T^γ denote the conjugate subgroup $\gamma^{-1}T\gamma$. A group Γ is called a* Frobenius group *if it has a proper subgroup T such that $T^\gamma \bigcap T = (1)$ whenever $\gamma \notin T$.*

If Γ is a Frobenius group, then it is a theorem of frobenius that Γ has a normal subgroup Σ such that $T \bigcap \Sigma = (1)$ and $\Gamma = \Sigma T$.

This will be a useful definition, for the Galois group of the Hilbert class field H over $\mathbb{Q}$ is a Frobenius group. This is clear since the Galois group of $H/\mathbb{Q}$ is dihedral and generated by an automorphism σ and complex conjugation τ. In particular, if we let $T =< \tau >$ then the Galois group satisfies the conditions of the definition.

The subgroup Σ of Frobenius' theorem for our Galois group is $< \sigma >$.

Definition 4.0.10. *For any number field K define the quantity*

$$\xi(K) = \frac{h_K R_K}{\omega_K \sqrt{|d_K|}},$$

where h_k, R_K, ω_K and d_K are the class number, regulator, number of roots of unity and discriminant of the field K respectively.

Now for a Galois number field L and a subgroup T of its Galois group, let L^T denote the subfield of L fixed by the automorphisms in T and $|T|$ denote the order of T.

With these definitions the theorem which interests us is given as theorem 75 in §VIII.7 of [4] which we repeat here

Theorem 4.0.11. *Let L be a Galois number field with Galois group $\Gamma = Gal(L/\mathbb{Q})$. If Γ is a Frobenius group as in the definition above, then*

$$\xi(\mathbb{Q})^{|T|}\xi(L) = \xi(L^T)^{|T|}\xi(L^\Sigma).$$

If we let N denote the real subfield $\mathrm{Re}\{H\}$ of the Hilbert class field H then we have immediately the following result.

Theorem 4.0.12. *If H is the Hilbert class field of the imaginary quadratic field K and if N denotes the real subfield $Re\{H\}$ of H, then*

$$\xi(\mathbb{Q})^2\xi(H) = \xi(N)^2\xi(K).$$

Substituting the expressions for ξ into this equation and simplifying we obtain

$$\frac{h_H R_H}{\sqrt{|d_H|}} = \frac{h_K R_N^2 h_N^2}{|d_N|\sqrt{|d_K|}}. \tag{10}$$

But now if we let $\mathcal{N}_{L/M}$ denote the relative norm and $D(L/K)$ the relative discriminant from the field L to K then we have the following result (see [4] III.2)

Theorem 4.0.13. *Given a tower of number fields $N/L/K$, we have*

$$D(N/K) = \mathcal{N}_{L/K}(D(N/L))D(L/K)^{[N:L]}.$$

Applying this to the tower of fields $H/N/\mathbb{Q}$ reduces equation (10) to the following

Theorem 4.0.14. *If N is the real subfield of H, the Hilbert class field of an imaginary quadratic number field K, then*

$$h_H R_H = h_K R_N^2 h_N^2.$$

In particular when the class numbers of the Hilbert class field and its real subfield N are 1 then this relation even becomes

$$R_H = h_K R_N^2.$$

Pari reveals that this is indeed the case for all the examples that were dealt with in the first part of this chapter.

Either way, combining the results of theorems (4.0.8) and (4.0.14), we obtain

Theorem 4.0.15. *Let H be the Hilbert class field of the imaginary quadratic field K with Galois group G of prime order. Then*

$$\prod_{1\neq\chi\in\hat{G}} L'(0,\chi) = R_N^2 h_N^2,$$

where R_N and h_N are the regulator and class number of the real subfield of the Hilbert class field.

Now we turn to the class number five situation which we encountered in the first part of this chapter. The eta units there m_1 and m_2 are in the real subfield N of the Hilbert class field (a fact which we have actually proved for the discriminant $d = -47$) and they are clearly independent.

The unit rank of N is 2. It may be that our 2 eta units m_1 and m_2 are not fundamental but we can still calculate the regulator of N as though they were. This will simply introduce some integer factor which we shall denote by C. In fact C is equal to the index of the group of units generated by ± 1, m_1 and m_2 in the full unit group of N (see [5] Lemma 4.15 for a proof).

We note that in the class number five situation $L'(0,\chi^k) = L'(0,\overline{\chi}^k)$ for $k = 1, 2$.

Thus if we let $R_N(m_1, m_2)$ denote the regulator with respect to the units m_1 and m_2 in N then from the theorem above we have

$$L'(0,\chi)L'(0,\chi^2) = C^{-1} h_N R_N(m_1, m_2).$$

But we can calculate the regulator in this expression

$$\begin{aligned} R_N(m_1,m_2) &= \left|\det\begin{pmatrix} \log|m_1| & \log|m_2| \\ 2\log|m_1^{\sigma^2}| & 2\log|m_2^{\sigma^2}| \end{pmatrix}\right| \\ &= \log|m_1|\log\left|m_2^{\sigma^2}\right|^2 - \log|m_2|\log\left|m_1^{\sigma^2}\right|^2, \end{aligned} \tag{11}$$

where σ is the an automorphism of order 5 in the Galois group of the Hilbert class field but which acts as an embedding of N into $\mathbb{C}$.

But now we use the relationship

$$m_1 = \left|m_2^{\sigma^2}\right|^2 = m_2^{\sigma^2} m_2^{\sigma^3}$$

which we proved in an earlier section of this chapter. This leads to

$$\left|m_1^{\sigma^2}\right|^2 = m_1^{\sigma^2} m_1^{\sigma^3} = m_2^2 m_2^{\sigma^4} m_2^{\sigma} = m_2/m_1.$$

Thus the regulator can be expressed

$$R_N(m_1, m_2) = \log^2|m_1| + \log|m_1|\log|m_2| - \log^2|m_2|.$$

However if we multiply the expressions for $L'(0,\chi)$ and $L'(0,\chi^2)$ that we have given in equations (3) and (4) this is precisely the expression that we get. Thus we have shown

Theorem 4.0.16. *If m_1 and m_2 are the two eta values computed for a discriminant of class number five as above and they are both units in the real subfield N of the Hilbert class field then along with ± 1 they generate a group of units whose index in the full group of units of N is equal to the class number of N.*

We have also shown of course that if m_1 and m_2 are units in N then

$$L'(0,\chi)L'(0,\chi^2) = R_N(m_1, m_2).$$

Finally we prove the relationship between the resolvents of m_2 and the L-series that was noted numerically in (5) of the first section. In order to do this we will show that the product of the left hand sides equals the product of the right hand sides and likewise for the differences.

Now we just obtained an expression for the product of the L-series. Replacing m_1 with $|m_2^{\sigma^2}|^2$ throughout we get

$$L'(0,\chi)L'(0,\chi^2) = 4\log^2|m_2^{\sigma^2}| + 2\log|m_2^{\sigma^2}|\log|m_2| - \log^2|m_2|$$

Now we compute the product of the right hand sides of (5).

We have

$$\begin{aligned} 2\,\mathrm{res}(m_2,1) &= 2\log|m_2| + 2(\zeta+\zeta^4)\,\log|m_2^{\sigma}| + 2(\zeta^2+\zeta^3)\,\log\left|m_2^{\sigma^2}\right| \\ &= \sqrt{5}\left(\frac{\sqrt{5}-1}{2}\log|m_2| - 2\log\left|m^{\sigma^2}\right|\right) \end{aligned} \tag{12}$$

Similarly

$$2\,\mathrm{res}(m_2,2) = \sqrt{5}\left(\frac{\sqrt{5}+1}{2}\log|m_2| + 2\log\left|m^{\sigma^2}\right|\right)$$

Multiplying out we find that the product of the right hand sides of (5) is identical to the product of the L-series' given above.

Now we must subtract the L-series. Referring to (3) and (4) we obtain

$$L'(0,\chi^2) - L'(0,\chi) = \sqrt{5}\log|m_2|.$$

Similarly subtracting the right hand sides of (5) gives

$$\frac{2}{\sqrt{5}}(\mathrm{res}(m_2,1) + \mathrm{res}(m_2,2)) = 2\log|m_2| - \log|m_2^{\sigma}| - \log\left|m_2^{\sigma^2}\right| = \sqrt{5}\log|m_2|$$

as required.

A similar argument to the above can also be used to demonstrate the relations between the L-series and the resolvents of the unit m_1 as given in equation (6).

It is not clear how one would prove the similar resolvent/L-series relations for class number seven since in that case one cannot just compare the sum and product of the expressions as there are now three L-series and three resolvents.

We would like to extend the result of theorem (4.0.16) to class numbers greater than 5. We take this as far as we are able to here.

We call the units generated by the eta units and ± 1 the group of *elliptic units*. Thus given an imaginary quadratic number field K whose Hilbert class field H is an extension of prime degree we would like to show that the index of our group of elliptic units in the full group of units of the real subfield N of the Hilbert class field is always equal to the class number of N.

So, let us consider the situation where the class number of the imaginary quadratic field we are considering is p for a fixed odd prime p. We have $(p-1)/2$ units $m_i = h(a_{i-1}, b_{i-1}, c_{i-1})^2 / h(a_i, b_i, c_i)^2$ for $1 \le i \le (p-1)/2$ and where (a_1, b_1, c_1) is being thought of as a generator of the form class group.

From the above arguments it is clear that to establish our result it is only necessary to show that the regulator $R_N(m_1, m_2, \ldots, m_{(p-1)/2})$ is actually equal to the expression obtained from the Kronecker limit formula for the product of L-series with non-trivial and non-conjugate characters.

The first step in the process is to generalize the result of theorem (3.0.3). This we will do by considering the action of the Frobenius automorphisms on each of the units m_i.

Firstly we will assume that the Frobenius automorphism $\sigma = \sigma(\mathfrak{q})$ of theorem (3.0.2) is that of a fixed prime ideal $\mathfrak{q}$ in the ideal class which canonically corresponds to the form class of (a_1, b_1, c_1). This will enable us to determine the action of σ^k on any of our units m_i easily for any integer k.

Again we think of (a_i, b_i, c_i) as corresponding to an ideal $\mathfrak{m}_i$ of norm a_i for $0 \le i \le (p-1)/2$. We will also define the ideal $\mathfrak{m}_i$ of norm a_i for $(p+1)/2 \le i \le p-1$ to be the ideal $\overline{\mathfrak{m}}_{p-i}$ of norm a_{p-i} which corresponds to the binary quadratic form $(a_{p-i}, -b_{p-i}, c_{p-i})$. Thus m_i for $0 \le i \le p-1$ will be a complete set of representatives of the set of ideal classes.

We also extend our set of units as follows. We define m_i for $(p+3)/2 \le i \le p-1$ to be equal to $1/m_{p-i+1}$ and $m_{(p+1)/2} = 1$ and $m_0 = 1/m_1$.

With these carefully chosen definitions and by a similar argument to that which we gave previously for the class number five case we can show the following

Theorem 4.0.17. *We have*

$$\left| m_i^{\sigma^k(\mathfrak{q})} \right|^2 = m_{i+k} m_{i-k},$$

where the subscripts are taken modulo p.

Proof: We have

$$\left| m_i^{\sigma^k(\mathfrak{q})} \right|^2 = m_i^{\sigma^k(\mathfrak{q}) + \sigma^{p-k}(\mathfrak{q})} \tag{13}$$
$$= \sqrt{\frac{a_{i+k} a_{i-k}}{a_{i+k-1} a_{i-k-1}}} \left| \frac{\eta(\mathfrak{m}_{i+k-1}) \overline{\eta}(\mathfrak{m}_{i+k-1}) \eta(\mathfrak{m}_{i-k-1}) \overline{\eta}(\mathfrak{m}_{i-k-1})}{\eta(\mathfrak{m}_{i+k}) \overline{\eta}(\mathfrak{m}_{i+k}) \eta(\mathfrak{m}_{i-k}) \overline{\eta}(\mathfrak{m}_{i-k})} \right|,$$

where the subscripts are taken modulo p.

Now so long as we don't have $i+k \equiv 0 \pmod p$ or $i-k \equiv 0 \pmod p$ then this can be written

$$\left|m_i^{\sigma^k(\mathfrak{q})}\right|^2 = |m_{i+k}m_{i-k}|,$$

where the subscripts are again considered modulo p.

Clearly we cannot have both of the exceptional cases at the same time. Therefore we consider them separately. When $i+k \equiv 0 \pmod p$, looking at the expression above, we see that we end up with

$$\left|m_i^{\sigma^k(\mathfrak{q})}\right|^2 = |m_{i-k}/m_1|,$$

and when $i+k \equiv 0 \pmod p$ we have

$$\left|m_i^{\sigma^k(\mathfrak{q})}\right|^2 = |m_{i+k}/m_1|,$$

as required. □

Now we compute the regulator $R = R_N(m_1, m_2, \ldots, m_{(p-1)/2})$. We assume of course that the units m_i are independent units. However, if this were not the case, then the regulator would simply be zero, a situation which is impossible, if we prove our result, due to the non-vanishing of the L-series involved.

By definition the regulator R is the absolute value of

$$\begin{vmatrix} \log|m_1| & \log|m_2| & \ldots & \ldots & \log|m_{(p-1)/2}| \\ \log|m_1^\sigma|^2 & \log|m_2^\sigma|^2 & \ldots & \ldots & \log\left|m_{(p-1)/2}^\sigma\right|^2 \\ \ldots & & & & \\ & & & & \\ \ldots & & & & \\ \log\left|m_1^{\sigma^{(p-3)/2}}\right|^2 & \log\left|m_2^{\sigma^{(p-3)/2}}\right|^2 & \ldots & \ldots & \log\left|m_{(p-1)/2}^{\sigma^{(p-3)/2}}\right|^2 \end{vmatrix}.$$

We sketch the proof that this regulator is equal to the product of L-series in analogy with the class number five case which we demonstrated above. The proof is long and involved and so we leave some of the trivial computations up to the reader to check for themself.

Firstly we can apply the previous theorem to the entries of each of the rows of this determinant except the first. Then each new entry in the rows following the first has a product of two m values. We refer to this version of the matrix as the old version.

Next we add to each row in the matrix all the rows which came above it in the old version of the matrix. So for example the new final row will be the sum of all the old rows of the matrix. We note that we have not changed the determinant by adding the rows in this way.

Having done this, each entry in the determinant now involves the product of a string of successive m_i's.

However now we recall that $m_i = h(a_{i-1}, b_{i-1}, c_{i-1})^2/h(a_i, b_i, c_i)^2$ for $1 \le i \le (p-1)/2$, etc. Thus after cancellation we can express each product of successive m_i's as a square of the quotient of two $h(a_i, b_i, c_i)$ values.

To make the notation neater we will write $h_i = h(a_i, b_i, c_i)$ for $1 \le i \le (p-1)/2$ and we define h_i for the other values of i by

$$h_{p-i} = h_i.$$

After these definitions our regulator becomes the absolute value of

$$\begin{vmatrix} \log\left|\frac{h_0}{h_1}\right|^2 & \log\left|\frac{h_1}{h_2}\right|^2 & \dots & \dots & \log\left|\frac{h_{(p-3)/2}}{h_{(p-1)/2}}\right|^2 \\ \log\left|\frac{h_{p-1}}{h_2}\right|^2 & \log\left|\frac{h_0}{h_3}\right|^2 & \dots & \dots & \log\left|\frac{h_{(p-5)/2}}{h_{(p+1)/2}}\right|^2 \\ \dots & & & & \\ & & & & \\ \dots & & & & \\ \log\left|\frac{h_{(p+3)/2}}{h_{(p-1)/2}}\right|^2 & \log\left|\frac{h_{(p+5)/2}}{h_{(p+1)/2}}\right|^2 & \dots & \dots & \log\left|\frac{h_0}{h_{p-2}}\right|^2 \end{vmatrix}.$$

In fact it is now clear that if we write $a_i = \log|h_i|^2$ then our determinant can be written conveniently as

$$\begin{vmatrix} a_0 - a_1 & a_1 - a_2 & \dots & \dots & a_{\frac{p-3}{2}} - a_{\frac{p-1}{2}} \\ a_1 - a_2 & a_0 - a_3 & \dots & \dots & a_{\frac{p-5}{2}} - a_{\frac{p-1}{2}} \\ \dots & & & & \\ & & & & \\ \dots & & & & \\ a_{\frac{p-3}{2}} - a_{\frac{p-1}{2}} & a_{\frac{p-5}{2}} - a_{\frac{p-1}{2}} & \dots & \dots & a_0 - a_2 \end{vmatrix}.$$

Each entry has a part with positive sign and a part with negative sign. One sees that all the positive parts of the entries are the same along diagonals going one way across the determinant whilst the negative parts are all the same along diagonals going the other way across the determinant.

To evaluate this $(p-1)/2$ by $(p-1)/2$ determinant we first multiply the rows and columns by various factors.

Letting $\zeta = \zeta_p$ the p-th root of unity $\exp(2\pi i/p)$ we multiply the first row by $b_1 = -(\zeta + \zeta^2 + \dots + \zeta^{p-1}) = 1$, the second row by $b_2 = -(\zeta^2 + \zeta^3 + \dots + \zeta^{p-2})$, the third row by $b_3 = -(\zeta^3 + \zeta^4 + \dots + \zeta^{p-3})$, etc.

Next we multiply the first column by $1/b_1$, the second column by $1/b_2$, the third column by $1/b_3$, etc. Note that by doing thall these multiplications we have not changed the value of our regulator at all. We thus have a new version of the determinant which turns out to be easier to evaluate which we obtained from the original version.

Next we sum together all the rows vectors in the new version of the determinant. Note that a_0 occurs precisely once in each column and that the a_0 in row i has been multiplied by $b_i/b_i = 1$ by the above process. Thus we have a contribution of a_0 to the sum at each of the coordinates of the vector row sum.

Next we need to express $1/b_i$ as a sum of roots of unity. A simple computation shows that

$$1/b_i = \zeta^{p-i} \sum_{j=0}^{k_i-1} \zeta^{j(p-2i+1)},$$

where we choose k_i to be the smallest natural number such that $k_i(p-2i+1) \equiv 1 \pmod{p}$ (simply multiply the expression for $1/b_i$ by that for b_i to see that this is so).

Now we notice that each a_j appears twice in every column i except a_0 which we have already dealt with and $a_{(p-1)/2-i+1}$. If $p \equiv 1 \pmod 4$ then this value appears with a minus sign in the original determinant in row $(p-1)/2-2(i-1)$ up to about half way, or more precisely for columns $1 \le i \le (p-1)/4$ and with a positive sign in row $2(i-(p-1)/4)-1$ for the remaining columns, and if $p \equiv -1 \pmod 4$ then it appears with a minus sign in row $(p-1)/2 - 2(i-1)$ for columns $1 \leq i \leq (p+1)/4$ and with a positive sign in row $2(i - (p+1)/4)$ for the remaining columns.

We carefully compute the constants that these a_j, which appear only once in a column, have been multiplied by in the new version of our determinant and we find that such an a_j always becomes $(\zeta^j + \zeta^{p-j})\, a_j$ in all cases. Thus when the rows vectors are summed in the new version of the determinant then we end up with a contribution of precisely $(\zeta^j + \zeta^{p-j})\, a_j$ for each of these a_j that only appear once in their column.

Now all the remaining values of a_j appear twice in every column. So when the row vectors are summed there are two components to the sum. Once the rows and columns have been multiplied by the values b_i, etc, that we have above, then if the value a_j appears in the rows k and l of column i then when the rows are summed there is a contribution of $a_j(\pm b_k \pm b_l)/b_i$ to the i-th coordinate of the row sum, where the signs depend on what signs the a_j are endowed with in the original version of the determinant.

Firstly we calculate where the a_j appear in the original version of the determinant and with what signs. After some study one realises that the a_j values appear with a positive sign in whichever of rows $i + j$ and $i - j$ exist and with a minus sign in whichever of rows $((p - 2i + 2) \pm (p - 1) - 2j + 1)/2$ exist.

Now by carefully calculating the constants $(\pm b_k \pm b_l)/b_i$ that end up in our row vector sum for the a_j's from column i where k and l are the row numbers in that column where they appear, we find than in every case we again have that the coefficient of a_j in the sum is $(\zeta^j + \zeta^{p-j})$.

Thus each of the columns in the new version of the determinant sum up to precisely $a_0 + (\zeta + \zeta^{p-1})\, a_1 + (\zeta^2 + \zeta^{p-2})\, a_2 + \cdots + (\zeta^{(p-1)/2} + \zeta^{(p+1)/2})\, a_{(p-1)/2}$. In other words if we replace the first row of the determinant with the sum of all the rows we find that each entry in the row becomes precisely this expression. Thus we find that this expression is actually a factor of our determinant.

Now if we redo the whole argument again with ζ replaced with ζ^2 throughout we will find that $a_0 + (\zeta^2 + \zeta^{p-2})\, a_1 + (\zeta^4 + \zeta^{p-4})\, a_2 + \cdots + (\zeta^{p-1} + \zeta)\, a_{(p-1)/2}$ is a factor, and so on.

We see immediately that these are precisely the expressions given by the Kronecker limit formula for the L-series of the theorem we are trying to prove (the factor of $4/w(d) = 2$ which appears at the front of the expression (1) is obtained by taking a factor 2 out of each of the expressions we obtain above, since $a_i = \log|h_i|^2 = 2\log|h_i|$).

Of course it is easy to see that the product of all the factors that we have obtained above is actually *equal* to the regulator we are calculating. In other words there are no other factors which we have missed.

Thus from the argument we have given earlier we have proved the following theorem

Theorem 4.0.18. *If m_i for $1 \leq i \leq (p-1)/2$ are eta units belonging to an imaginary quadratic discriminant d having prime class number p, as described earlier, and they are all in the real subfield N of the Hilbert class field associated to the underlying quadratic field of discriminant d then along with ± 1 they generate a group of units whose index in the full group of units of N is equal to the class number of N.*

We note that this theorem is similar to others along the same lines which appear in the literature. Although we are unable to find precisely this version of the theorem or the proof we have given it has some similarity to results mentioned in [3] which we have made use of throughout.

The main innovation in our case is the fact that the units are simple eta quotients of just two distinct h_i values and the eta units are compared to the full group of units of the real subfield of the Hilbert class field H rather than of H itself.

References

[1] B. J. Birch *Weber's Class Invariants* Mathematika 16 (1969), pp. 283–294.
[2] A. Gee *Class Invariants from Dedekind's Eta Function* Thesis, Thomas Stieltjes Institute for Mathematics, 2001
[3] F. Hajir *Elliptic Units of Cyclic Unramified Extensions of Complex Quadratic Fields* Acta Arith. LXIV.1 (1993) pp. 69 – 85.
[4] A. Frölich, M. J. Taylor *Algebraic Number Theory* Camb. Stud. in Adv. Math. 27, Camb. Uni. Press 1994
[5] L. C. Washington *Introduction to Cyclotomic Fields* Grad. Texts in Math., Springer-Verlag, 1980