THREE-MANIFOLDS WITH NON-NEGATIVELY PINCHED RICCI CURVATURE

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Abstract. We show that every complete non-compact three-manifold with non-negatively pinched Ricci curvature admits a complete Ricci flow solution for all positive time, with scale-invariant curvature decay and preservation of pinching. Combining with recent work of Lott and Deruelle-Schulze-Simon gives a proof of Hamilton's pinching conjecture without additional hypotheses.

1. introduction

Suppose (M, g) is a smooth complete n-dimensional Riemannian manifold with a positive lower Ricci curvature bound $\mathrm{Ric}(g) \geq (n-1)r^{-2} > 0$. The classical theorem of Bonnet-Myers tells us that the diameter of (M, g) is then bounded above by πr, which is sharp on any round sphere.

In this paper we consider a situation in which the Ricci curvature is still non-negative, but does not have a uniform positive lower bound. Instead it is assumed to be controlled from below in terms of the scalar curvature $\mathcal{R}$. Our goal is to extend recent work of Lott [12] and Deruelle-Schulze-Simon [6] in order to prove Hamilton's pinching conjecture [3, Conjecture 3.39]:

Theorem 1.1 (Hamilton's pinching conjecture)**.** *Suppose (M^3, g_0) is a complete (connected) three-dimensional Riemannian manifold with* $\mathrm{Ric} \geq \varepsilon\mathcal{R} \geq 0$ *for some $\varepsilon > 0$. Then (M^3, g_0) is either flat or is compact.*

This conjecture is the intrinsic analogue of a theorem of Hamilton [8] in which Ric and $\mathcal{R}$ are replaced by the second fundamental form and mean curvature of a hypersurface of Euclidean space, respectively. Partial results towards the conjecture are available in the presence of additional upper and lower curvature bound assumptions. B.-L. Chen and X.-P. Zhu [2] proved that if one assumes additionally that

(1) $|\mathrm{Rm}(g_0)| \leq M$ and
(2) $\mathrm{K}(g_0) \geq 0$, i.e. all sectional curvatures are nonnegative,

then the conjecture is true. Later, J. Lott [12] presented an alternative method to prove the conjecture under the weaker additional assumptions that

(1) $|\mathrm{Rm}(g_0)| \leq M$ and

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(2) $\mathrm{K}(g_0) \geq -\frac{C_0}{d^2_{g_0}(\cdot,p))}$ for some $C_0 > 0$ and $p \in M$, i.e. the sectional curvatures have an inverse quadratic lower bound.

The weakening of the lower sectional curvature bound moving from Chen-Zhu's work to Lott's work is significant because it blocks the use of Hamilton's Harnack estimate, and so new ideas are required, some of which are employed in the final resolution of Hamilton's conjecture. One notable consequence of Lott's work is that a complete, non-compact three-manifold satisfying the Ricci pinching condition $\mathrm{Ric} \geq \varepsilon\mathcal{R} \geq 0$ (for some $\varepsilon > 0$) and with bounded curvature is found either to be flat or to have cubic volume growth.

Very recently, using Lott's work as a foundation, Deruelle-Schulze-Simon [6] have made the decisive step of dramatically weakening the additional hypotheses to only require additionally that

(1) $|\mathrm{Rm}(g_0)| \leq M$.

The purpose of this paper is to demonstrate that hypothesis (1) can be removed, giving the result stated in Theorem 1.1.

All of the prior works on this conjecture use Ricci flow. As part of his approach, Lott proved that under his additional bounded curvature hypothesis one can solve the Ricci flow for all time with scale-invariant curvature decay and while preserving the pinching. In this paper we demonstrate that this is possible without requiring the bounded curvature assumption (1). Precisely, we prove:

Theorem 1.2. *Suppose (M^3, g_0) is a complete non-compact three-dimensional Riemannian manifold with $\mathrm{Ric}(g_0) \geq \varepsilon\mathcal{R}(g_0) \geq 0$ for some $\varepsilon > 0$. Then there exists $a = a(\varepsilon) > 0$ such that the Ricci flow has a complete long-time solution $g(t)$ starting from g_0 so that*

(a) $\mathrm{Ric}(g(t)) \geq \varepsilon\mathcal{R}(g(t)) \geq 0$*;*
(b) $|\mathrm{Rm}(g(t))| \leq at^{-1}$*;*

for all $(x,t) \in M \times (0,+\infty)$.

Remark 1.3. Prior to this result, even short-time existence in this situation was unknown. In the absence of the pinching constraint, short-time existence *remains* an open problem. That is, given a complete three-manifold with nonnegative Ricci curvature, but no assumed overall curvature bound, it is uncertain as to whether there necessarily exists some Ricci flow continuation. Certainly the a/t curvature decay cannot be true in general without the pinching hypothesis.

Although the proof of Theorem 1.2 is independent of the ideas used by Lott to prove the bounded curvature case, there is one crucial additional property of Lott's solution that is automatically inherited by our solution from his work, namely that if there exists at least one point in space-time where the scalar curvature is positive then the flow has *positive* asymptotic volume ratio $v_0 > 0$ at each time. In practice, instead of trying to substitute our existence theory in place of the analogous existence result of Lott in the development of the theory, we can apply our existence theorem to reduce the full Hamilton pinching conjecture to the situation that has already been addressed, as follows.

Proof of Theorem 1.1. Suppose that M^3 is not compact. Then Theorem 1.2 applies to give a complete Ricci flow evolution $g(t)$ that preserves the pinching condition and immediately has bounded curvature. Thus for each $t_0 > 0$ the manifolds $(M^3, g(t_0))$ satisfy the hypotheses of Deruelle-Schulze-Simon [6, Theorem 1.3] and we can deduce that $g(t_0)$ is flat for each $t_0 > 0$, and the Ricci flow is independent of time. In particular, the initial data (M^3, g_0) is flat and Theorem 1.1 follows. □

Thus the significance of our contribution is to extend the applicability of earlier work.

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2. STRATEGY OF THE PROOF

The key step to proving the global existence of Theorem 1.2 will be to obtain local existence, with estimates, for a *uniform* time:

Theorem 2.1 (Special case of Theorem 5.1)**.** *For all $\varepsilon \in (0, \frac{1}{12})$, there exist $T(\varepsilon), a_0(\varepsilon) > 0$ such that the following holds. Suppose (M^3, g_0) is a complete non-compact three-dimensional Riemannian manifold so that*

$$\mathrm{Ric}(g_0) \geq \varepsilon \mathcal{R}(g_0) \geq 0.$$

Then for any $p \in M$, there exists a smooth Ricci flow solution $g(t)$ defined on $B_{g_0}(p,1) \times [0,T]$ such that $g(0) = g_0$, $|\mathrm{Rm}(x,t)| \leq a_0 t^{-1}$ and

$$\mathrm{Ric}(x,t) \geq \varepsilon \mathcal{R}(x,t) - 1 \tag{2.1}$$

for all $(x,t) \in B_{g_0}(p,1) \times (0,T]$.

In order to prove global existence (in space and time) we will exploit the scale invariance of the hypotheses and apply the local existence theorem not to g_0 but to a sequence of blown down metrics $R_i^{-2} g_0$ with $R_i \to \infty$. The resulting Ricci flows can be parabolically scaled back to give Ricci flows $g_i(t)$ on $B_{g_0}(p, R_i) \times [0, TR_i^2]$, still with the same $a_0 t^{-1}$ curvature decay, and now with $g_i(0) = g_0$ where defined. By parabolic regularity, i.e. by Shi's estimates, it will be possible to take a subsequence and extract a smooth complete limit Ricci flow. Moreover, an effect of the rescaling will be that this limit flow will be Ricci pinched, without error, and thus give the flow required by Theorem 1.2.

The proof, therefore, will rest on local existence as in Theorem 2.1. Obtaining local existence over a time interval that can depend directly on g_0 is straightforward. One can modify the metric conformally in a boundary layer around the local region in order to make it complete and of bounded curvature, and then flow using Shi's classical existence theorem from [14], cf. [18, 9, 17]. In order to construct a flow whose existence time has no dependence on g_0 other than a dependence on the pinching constant ε, we develop an approach from [9] and [17] that essentially allows us to keep restarting the flow an uncontrollably large number of times until we have the flow over a uniform time interval. In order to make this work we need a collection of a-priori estimates that we compile in Section 3. These quantify how Ricci pinching, curvature decay and roundness are related to each other under Ricci flow. For example, Lemma 3.3 tells us that a local Ricci flow that becomes round where the curvature blows up must satisfy a_1/t curvature decay.

3. A-PRIORI ESTIMATES FOR RICCI FLOW

In this section, we will derive local estimates for the Ricci flow on three-manifolds satisfying a pinching assumption. First, we have the following local persistence of Ricci pinching.

Lemma 3.1. *Suppose $(M^3, g(t)), t \in [0,T]$ is a smooth solution to the Ricci flow such that for some $x_0 \in M$, we have $B_0(x_0,1) \Subset M$ for $t \in [0,T]$. If there exist $a > 0$ and $\frac{1}{12} > \varepsilon > 0$ such that*

(i) *$\mathrm{Ric}(g(0)) \geq \varepsilon \mathcal{R}(g(0)) \geq 0$ on $B_0(x_0,1)$;*
(ii) *$|\mathrm{Rm}(g(t))| \leq a t^{-1}$ on $B_0(x_0,1)$, $t \in (0,T]$;*

then there exists $S_1(a,\varepsilon) > 0$ such that for all $t \in [0, T \wedge S_1]$ we have

$$\mathrm{Ric}(x_0,t) \geq \varepsilon \mathcal{R}(x_0,t) - 1.$$

In particular, $\mathcal{R}(x_0,t) > -4$ for $t \in [0, T \wedge S_1]$.

It is convenient to obtain the lower scalar curvature bound here by tracing the pinching estimate, but we remark that local lower bounds on the scalar curvature are always preserved in the presence of a/t curvature decay owing to an estimate of B.-L. Chen; see [4] and [16, Section 8].

Proof. For $(x,t) \in M \times [0,T]$, define

$$\lambda(x,t) = \inf\{s \geq 0 : \mathrm{Ric}(x,t) - \varepsilon \mathcal{R}(x,t) g(x,t) + s g(x,t) > 0\}. \tag{3.1}$$

Clearly, we have

(a) $\lambda(x,0) = 0$ for $x \in B_0(x_0,1)$;
(b) $\lambda(x,t) \leq Cat^{-1}$ for $t \in (0,T]$ and $x \in B_0(x_0,1)$, with C universal.

It suffices to estimate $\lambda(x_0,t)$ from above for t small. By [2, (74)-(75)], $\lambda(x,t)$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \lambda = h\lambda - A(\lambda_2^2 + \lambda_3^2) + 2B\lambda_2\lambda_3 \tag{3.2}$$

in the barrier sense, where λ_i are eigenvalues of $\mathrm{Ric}(x,t)$ with respect to $g(t)$ with $\lambda_1 \leq \lambda_2 \leq \lambda_3$, h is some linear combination of λ_i, $A = 1 - 2\varepsilon + \frac{\varepsilon}{1-\varepsilon} -$

$2\varepsilon\left(\frac{\varepsilon}{1-\varepsilon}\right)^2$ and $B = 1-\frac{\varepsilon}{1-\varepsilon}+2\varepsilon\left(\frac{\varepsilon}{1-\varepsilon}\right)^2$. Because $0<\varepsilon<\frac{1}{12}$, we have $A \geq B \geq 0$ and hence λ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\lambda \leq C_0|\mathrm{Rm}|\lambda \tag{3.3}$$

for some constant $C_0 > 0$. We may now apply [11, Theorem 1.1] to conclude. This completes the proof. □

The next lemma shows that under almost Ricci pinching, the Ricci flow will be almost Einstein.

Lemma 3.2. *Suppose $(M^3, g(t)), t \in [0,T]$ is a smooth solution to the Ricci flow and that for some $x_0 \in M$, $a > 0$ and $\frac{1}{100} > \varepsilon > 0$ we have*

(i) *$B_t(x_0, 1) \Subset M$ for all $t \in [0,T]$;*
(ii) *$\mathrm{Ric}(g(t)) \geq \varepsilon\mathcal{R}(g(t)) - 1$ for all $x \in B_t(x_0,1)$ and $t \in [0,T]$;*
(iii) *$|\mathrm{Rm}(x,t)| \leq at^{-1}$ for all $x \in B_t(x_0,1)$ and $t \in (0,T]$.*

Then there exist $S_2 = S_2(a,\varepsilon)$, $L_1(\varepsilon) > 0$ and $\sigma(\varepsilon) \in (1,2)$ so that at x_0, for all $t \in (0, T\wedge S_2]$, we have

$$\left|\mathrm{Ric} - \frac{1}{3}\mathcal{R}g\right|^2 \leq \frac{L_1}{t^{2-\sigma}}\cdot(\mathcal{R}+4)^\sigma.$$

Proof. By increasing a if necessary, we may assume that $a > 1$. In the proof, we will use c_i and C_i to denote constants depending only on ε, σ. We will follow closely the computations of Hamilton [7] and Lott [12]. We may assume $S_2 \leq 1$ from the outset, and will constrain S_2 further during the proof.

By taking the trace of (ii), we have

$$\mathcal{R}(g(t)) + 4 > \frac{1}{2} \tag{3.4}$$

on $B_t(x_0,1)$. In what follows, we will always work on $B_t(x_0,1)$ where (3.4) holds. Consider the function

$$f(x,t) = \frac{1}{(\mathcal{R}+4)^\sigma}\left(|\mathrm{Ric}|^2 - \frac{1}{3}(\mathcal{R}+4)^2\right)$$

on $B_t(x_0,1)$ where $1 < \sigma < 2$ is a constant to be chosen later. This is well-defined on on $B_t(x_0,1)$ thanks to (3.4). This is a modified version of the quantity considered in [7, Theorem 10.1]. We will later show that a suitable bound on $f(x,t)$ is equivalent to the conclusion of the lemma.

We first compute the evolution of $|\mathrm{Ric}|^2(\mathcal{R}+4)^{-\sigma}$. By [7, Lemma 10.2], if $\lambda \geq \mu \geq \nu$ denote the eigenvalues of Ric with respect to $g(t)$, then

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|\mathrm{Ric}|^2 = -2|\nabla_i \mathrm{R}_{jk}|^2 + 4(T-L) \tag{3.5}$$

where
(3.6)

$$\begin{cases} T &= \lambda^3 + \mu^3 + \nu^3; \\ L &= \lambda^3 + \mu^3 + \nu^3 - (\lambda^2\mu + \lambda\mu^2 + \lambda\nu^2 + \lambda^2\nu + \mu^2\nu + \mu\nu^2) + 3\lambda\mu\nu. \end{cases}$$

We will use this to compute the evolution of f by following [7, Lemma 10.3] as closely as possible to give

(3.7)

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \frac{|\mathrm{Ric}|^2}{(\mathcal{R}+4)^\sigma} \\ =& \frac{1}{(\mathcal{R}+4)^\sigma}\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|\mathrm{Ric}|^2 + |\mathrm{Ric}|^2\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(\mathcal{R}+4)^{-\sigma} \\ &- 2\langle \nabla|\mathrm{Ric}|^2, \nabla(\mathcal{R}+4)^{-\sigma}\rangle \\ =& (\mathcal{R}+4)^{-\sigma}\left[-2|\nabla\mathrm{Ric}|^2 + 4(T-L)\right] - 2\sigma|\mathrm{Ric}|^4(\mathcal{R}+4)^{-\sigma-1} \\ &- \sigma(\sigma+1)(\mathcal{R}+4)^{-\sigma-2}|\nabla\mathcal{R}|^2|\mathrm{Ric}|^2 + 2\sigma(\mathcal{R}+4)^{-\sigma-1}\langle\nabla|\mathrm{Ric}|^2, \nabla\mathcal{R}\rangle \\ =& (\mathcal{R}+4)^{-\sigma-2}\Big\{ -2(\mathcal{R}+4)^2|\nabla\mathrm{Ric}|^2 + 2\sigma(\mathcal{R}+4)\langle\nabla|\mathrm{Ric}|^2,\nabla\mathcal{R}\rangle \\ &- \sigma(\sigma+1)|\nabla\mathcal{R}|^2|\mathrm{Ric}|^2\Big\} + (\mathcal{R}+4)^{-\sigma-1}\left[4(T-L)(\mathcal{R}+4) - 2\sigma|\mathrm{Ric}|^4\right]. \end{aligned}$$

We now handle the first bracket more carefully. Following the computation in [7, page 284], we see that

(3.8)

$$\begin{aligned} &-2(\mathcal{R}+4)^2|\nabla\mathrm{Ric}|^2 + 2\sigma(\mathcal{R}+4)\langle\nabla\mathcal{R},\nabla|\mathrm{Ric}|^2\rangle - \sigma(\sigma+1)|\mathrm{Ric}|^2|\nabla\mathcal{R}|^2 \\ =& -2|(\mathcal{R}+4)\nabla_i\mathrm{R}_{jk} - \mathcal{R}_i\mathrm{R}_{jk}|^2 + 2(\sigma-1)(\mathcal{R}+4)\langle\nabla\mathcal{R},\nabla|\mathrm{Ric}|^2\rangle \\ &+ [2-\sigma(\sigma+1)]|\mathrm{Ric}|^2|\nabla\mathcal{R}|^2 \\ =& -2|(\mathcal{R}+4)\cdot\nabla_i\mathrm{R}_{jk} - \mathcal{R}_i\mathrm{R}_{jk}|^2 + [2-\sigma(\sigma+1)]|\mathrm{Ric}|^2|\nabla\mathcal{R}|^2 \\ &+ 2(\sigma-1)(\mathcal{R}+4)^{\sigma+1}\left(\left\langle\nabla\mathcal{R},\nabla\frac{|\mathrm{Ric}|^2}{(\mathcal{R}+4)^\sigma}\right\rangle + \frac{\sigma|\nabla\mathcal{R}|^2|\mathrm{Ric}|^2}{(\mathcal{R}+4)^{\sigma+1}}\right) \\ =& -2|(\mathcal{R}+4)\cdot\nabla_i\mathrm{R}_{jk} - \mathcal{R}_i\mathrm{R}_{jk}|^2 + (\sigma-2)(\sigma-1)|\mathrm{Ric}|^2|\nabla\mathcal{R}|^2 \\ &+ 2(\sigma-1)(\mathcal{R}+4)^{\sigma+1}\left\langle\nabla\mathcal{R},\nabla\frac{|\mathrm{Ric}|^2}{(\mathcal{R}+4)^\sigma}\right\rangle. \end{aligned}$$

Hence,

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)\frac{|\mathrm{Ric}|^2}{(\mathcal{R}+4)^\sigma} = -\frac{2}{(\mathcal{R}+4)^{\sigma+2}}|(\mathcal{R}+4)\cdot\nabla_i \mathrm{R}_{jk}-\mathcal{R}_i\mathrm{R}_{jk}|^2+\frac{2(\sigma-1)}{\mathcal{R}+4}\left\langle\nabla\mathcal{R},\nabla\frac{|\mathrm{Ric}|^2}{(\mathcal{R}+4)^\sigma}\right\rangle - \frac{(2-\sigma)(\sigma-1)}{(\mathcal{R}+4)^{\sigma+2}}|\mathrm{Ric}|^2|\nabla\mathcal{R}|^2+\frac{4(T-L)(\mathcal{R}+4)-2\sigma|\mathrm{Ric}|^4}{(\mathcal{R}+4)^{\sigma+1}}. \tag{3.9}$$

On the other hand,

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)(\mathcal{R}+4)^{2-\sigma} &= 2(2-\sigma)(\mathcal{R}+4)^{1-\sigma}|\mathrm{Ric}|^2-(2-\sigma)(1-\sigma)(\mathcal{R}+4)^{-\sigma}|\nabla\mathcal{R}|^2\\ &= \frac{2(2-\sigma)}{(\mathcal{R}+4)^{1+\sigma}}(\mathcal{R}+4)^2|\mathrm{Ric}|^2+\frac{2(\sigma-1)}{\mathcal{R}+4}\left\langle\nabla\mathcal{R},\nabla(\mathcal{R}+4)^{2-\sigma}\right\rangle\\ &\quad -\frac{(2-\sigma)(\sigma-1)}{(\mathcal{R}+4)^2}(\mathcal{R}+4)^{2-\sigma}|\nabla\mathcal{R}|^2.\end{aligned} \tag{3.10}$$

Therefore, the function f satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)f &= \left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)\left[\frac{|\mathrm{Ric}|^2}{(\mathcal{R}+4)^\sigma}-\frac{1}{3}(\mathcal{R}+4)^{2-\sigma}\right]\\ &= \frac{2(\sigma-1)}{\mathcal{R}+4}\langle\nabla\mathcal{R},\nabla f\rangle-\frac{(2-\sigma)(\sigma-1)}{(\mathcal{R}+4)^2}|\nabla\mathcal{R}|^2 f\\ &\quad -\frac{2}{(\mathcal{R}+4)^{\sigma+2}}|(\mathcal{R}+4)\nabla_i\mathrm{R}_{jk}-\mathcal{R}_i\mathrm{R}_{jk}|^2\\ &\quad +\frac{2}{(\mathcal{R}+4)^{\sigma+1}}\left\{(2-\sigma)|\mathrm{Ric}|^2\left[|\mathrm{Ric}|^2-\frac{1}{3}(\mathcal{R}+4)^2\right]-2J\right\}.\end{aligned} \tag{3.11}$$

where $J(\mathrm{Ric}) := |\mathrm{Ric}|^4+(\mathcal{R}+4)(L-T)$. More generally we will regard J as a function of $(0,2)$ tensors, depending only on their eigenvalues, that has this expression when applied to Ric.

We now examine the evolution of f wherever $f>0$. In this case,

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)f \le \frac{\sigma-1}{2-\sigma}\frac{|\nabla f|^2}{f}+\frac{2}{(\mathcal{R}+4)^{\sigma+1}}\left\{(2-\sigma)|\mathrm{Ric}|^2(\mathcal{R}+4)^\sigma f-2J\right\}. \tag{3.12}$$

To control the final term, consider the modified Ricci curvature tensor $\widetilde{\mathrm{Ric}} = \mathrm{Ric} + 2g$ and the corresponding modified scalar curvature tensor $\tilde{\mathcal{R}} = \mathcal{R} + 6$. By the almost pinching assumption for Ric and the fact that $\varepsilon \in (0, \frac{1}{100})$, we have

$$\tilde{\mathcal{R}} > 0 \quad \text{and} \quad \widetilde{\mathrm{Ric}} \geq \varepsilon \tilde{\mathcal{R}}$$

on $B_t(x_0, 1)$ for $t \leq T$.

Consider the fourth order operator of Ric given by

$$P(\mathrm{Ric}) := |\mathrm{Ric}|^4 + \mathcal{R}(L - T)$$

so that $J(\mathrm{Ric}) = P(\mathrm{Ric}) + 4(L - T)$. Here $P(\mathrm{Ric})$ is the quantity considered by Hamilton in [7, Lemma 10.7]. We instead apply it to the modified Ricci tensor $\widetilde{\mathrm{Ric}}$ and $\tilde{\mathcal{R}}$. By putting $\mathrm{Ric} = \widetilde{\mathrm{Ric}} - 2g$ and $\mathcal{R} = \tilde{\mathcal{R}} - 6$, and treating terms that are less than quartic as error terms, we compute

$$\begin{aligned}
&(2-\sigma)|\mathrm{Ric}|^2\left[|\mathrm{Ric}|^2 - \frac{1}{3}(\mathcal{R}+4)^2\right] - 2J(\mathrm{Ric}) \\
&= (2-\sigma)|\widetilde{\mathrm{Ric}} - 2g|^2\left[|\widetilde{\mathrm{Ric}} - 2g|^2 - \frac{1}{3}(\tilde{\mathcal{R}}-2)^2\right] - 2J(\widetilde{\mathrm{Ric}} - 2g) \\
&\leq (2-\sigma)|\widetilde{\mathrm{Ric}}|^2\left[|\widetilde{\mathrm{Ric}}|^2 - \frac{1}{3}\tilde{\mathcal{R}}^2\right] - 2P(\widetilde{\mathrm{Ric}}) + C_0(|\widetilde{\mathrm{Ric}}|^3 + 1)
\end{aligned} \tag{3.13}$$

for some constant $C_0 > 0$. Here we have used the fact that $T(\mathrm{Ric})$ and $L(\mathrm{Ric})$ are third order operators on Ric. Since $\widetilde{\mathrm{Ric}} > 0$, we have $|\widetilde{\mathrm{Ric}}| \leq \tilde{\mathcal{R}}$ and hence (3.4) implies that

$$\begin{aligned}
&(2-\sigma)|\mathrm{Ric}|^2\left[|\mathrm{Ric}|^2 - \frac{1}{3}(\mathcal{R}+4)^2\right] - 2J(\mathrm{Ric}) \\
&\leq (2-\sigma)|\widetilde{\mathrm{Ric}}|^2\left[|\widetilde{\mathrm{Ric}}|^2 - \frac{1}{3}\tilde{\mathcal{R}}^2\right] - 2P(\widetilde{\mathrm{Ric}}) + C_1(\mathcal{R}+4)^3.
\end{aligned} \tag{3.14}$$

Applying [7, Lemma 10.7] on $\widetilde{\mathrm{Ric}}$, if $2 - \sigma < \varepsilon^2$, then

$$\begin{aligned}
(2-\sigma)|\widetilde{\mathrm{Ric}}|^2\left[|\widetilde{\mathrm{Ric}}|^2 - \frac{1}{3}\tilde{\mathcal{R}}^2\right] - 2P(\widetilde{\mathrm{Ric}}) &\leq -\varepsilon^2|\widetilde{\mathrm{Ric}}|^2\left[|\widetilde{\mathrm{Ric}}|^2 - \frac{1}{3}\tilde{\mathcal{R}}^2\right] \\
&\leq -\frac{1}{3}\varepsilon^2(\mathcal{R}+4)^{2+\sigma} f + C_2(\mathcal{R}+4)^3
\end{aligned} \tag{3.15}$$

for some $C_2 > 0$, where we have used the identity $|\mathrm{Ric}|^2 \geq \frac{1}{3}\mathcal{R}^2$, and where as usual we pay close attention to quartic terms and treat lower order terms as errors. Now we apply the observation in [1, Page 539] to see that $f^{\frac{1}{2-\sigma}} \leq$

$c_0^{-1}(\mathcal{R}+4)$ for some $c_0(\varepsilon)>0$ since

$$\begin{aligned}(\mathcal{R}+4)^\sigma f = |\mathrm{Ric}|^2-\frac{1}{3}(\mathcal{R}+4)^2 &\le |\mathrm{Ric}|^2\\ &= |\widetilde{\mathrm{Ric}}-2g|^2\\ &\le C_3(\mathcal{R}+4)^{2-\sigma}\cdot(\mathcal{R}+4)^\sigma.\end{aligned} \tag{3.16}$$

Here we have used (3.4) and the fact that $\widetilde{\mathrm{Ric}}>0$. Therefore, we conclude that

$$(2-\sigma)|\widetilde{\mathrm{Ric}}|^2\left[|\widetilde{\mathrm{Ric}}|^2-\frac{1}{3}\tilde{\mathcal{R}}^2\right]-2P(\widetilde{\mathrm{Ric}})\le -\frac{c_1}{2}(\mathcal{R}+4)^{1+\sigma}f^{\frac{3-\sigma}{2-\sigma}}+C_2(\mathcal{R}+4)^3 \tag{3.17}$$

for some $c_1(\varepsilon)>0$.

Combining (3.12), (3.14) and (3.17), we conclude that if $2-\varepsilon^2<\sigma<2$, then

$$\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)f\le\frac{\sigma-1}{2-\sigma}\frac{|\nabla f|^2}{f}-c_2f^{\frac{3-\sigma}{2-\sigma}}+C_4(\mathcal{R}+4)^{2-\sigma} \tag{3.18}$$

whenever $f>0$. Equivalently, $F=f^{\frac{1}{2-\sigma}}$ satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)F&\le -c_3F^2+C_5(\mathcal{R}+4)^{2-\sigma}F^{\sigma-1}\\ &\le -\frac{1}{2}c_3F^2+C_6(1+(at^{-1})^{\frac{2(2-\sigma)}{3-\sigma}})\end{aligned} \tag{3.19}$$

whenever $F(x,t)>0$. Here we have used the a/t curvature decay assumption.

If we choose σ sufficiently close to 2 so that

$$\frac{2-\sigma}{3-\sigma}<\frac{1}{4},$$

then if $S_2<a^{-1}$, (recall $a>1\ge S_2\ge t$ also), the function $G=tF$ will satisfy

$$\begin{aligned}t\cdot\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)G&\le t^2\cdot\left(-\frac{1}{2}c_3F^2+C_6(1+(at^{-1})^{1/2})\right)+tF\\ &\le -\frac{1}{2}c_3G^2+C_6\left(1+(at^3)^{1/2}\right)+G\\ &\le -\frac{1}{4}c_3G^2+C_7\end{aligned} \tag{3.20}$$

for $t\in(0,S_2\wedge T]$ whenever $G>0$.

Now we are ready to apply the local maximum principle. Let ϕ be a non-increasing smooth function on $[0,+\infty)$ such that $\phi(s)\equiv 1$ on $[0,\frac{3}{4}]$, vanishing outside $[0,1]$ and satisfying

$$\phi''\ge -10^3\phi \quad\text{and}\quad |\phi'|^2\le 10^3\phi.$$

Let $\Phi(x,t) = e^{-10^3 t}\phi(d_t(x,x_0) + \beta\sqrt{at})$ be a cutoff function on M. The constant β is a universal constant so that

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\Phi \le 0 \tag{3.21}$$

in the barrier sense using [13, Lemma 8.3]. We may assume Φ to be smooth when we apply the maximum principle; for example see [16, Section 7] for a detailed exposition.

Now we apply the maximum principle on $M \times [0, T \wedge S_2]$ to estimate G on $B_t(x_0, \frac{1}{2})$. Owing to the cutoff Φ, the product $\Phi \cdot G$ must attain a maximum at some $(x_1, t_1) \in M \times [0, T \wedge S_2]$. If $t_1 = 0$, the conclusion holds trivially since $G(x_1, 0) = 0$. Thus, we may assume $t_1 > 0$. At (x_1, t_1) where ΦG achieves a maximum, we may assume $G(x_1, t_1) > 0$ so that

$$\begin{aligned}
0 &\le t\Phi\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(\Phi G) \\
&= tG\Phi\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)\Phi + \Phi^2 t\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)G - 2t\Phi\langle\nabla\Phi, \nabla G\rangle \\
&\le \Phi^2\left(-\frac{1}{4}c_3 G^2 + C_7\right) + 2G|\nabla\Phi|^2 \\
&\le -\frac{1}{8}c_3(\Phi G)^2 + C_8.
\end{aligned} \tag{3.22}$$

Here we have used that $\nabla(\Phi G) = 0$ at (x_1, t_1) and $t \le S_2 \le 1$. We also used that $\frac{|\nabla\Phi|^2}{\Phi}$ is bounded by construction.

In particular we find that $\Phi G(x_1, t_1) \le C_9$ and hence,

$$\sup_{M\times[0,T\wedge S_2]} \Phi G \le \Phi G|_{(x_1,t_1)} \le C_9. \tag{3.23}$$

Notice that $B_t(x_0, \frac{1}{2}) \subset \{(x,t) : \Phi = 1\}$ if $t < (16\beta^2 a)^{-1}$. This shows that if we shrink S_2 further, then for all $t \in (0, T \wedge S_2]$,

$$f(x_0, t) = \left(t^{-1}G(x_0,t)\right)^{2-\sigma} \le C_{10}t^{\sigma-2}. \tag{3.24}$$

To recover our conclusion from f, it suffices to point out that from $\widetilde{\mathrm{Ric}} > 0$ and (3.4), we have at x_0 that

$$\begin{aligned}
\left|\mathrm{Ric} - \frac{1}{3}\mathcal{R}g\right|^2 &\le \left|\mathrm{Ric} - \frac{1}{3}(\mathcal{R}+4)g\right|^2 + C_{11}(\mathcal{R}+4) \\
&= (\mathcal{R}+4)^\sigma f + C_{12}(\mathcal{R}+4) \\
&\le C_{10}t^{\sigma-2}(\mathcal{R}+4)^\sigma + C_{12}(\mathcal{R}+4).
\end{aligned} \tag{3.25}$$

Since $\sigma \in (1,2)$, we can absorb the final term, after possibly reducing S_2 one final time, thus completing the proof. □

Next, we show that under the estimate from Lemma 3.2, the Ricci flow gains a curvature decay estimate.

Lemma 3.3. *Suppose M^3 is a non-compact (connected) manifold and $g(t), t \in [0,T]$ is a smooth solution to the Ricci flow on M so that for some $x_0 \in M$,*

(i) $B_t(x_0,1) \Subset M$ *for* $t \in [0,T]$;

(ii) $\mathcal{R}(x,t) > -4$ *for* $x \in B_t(x_0,1)$, $t \in [0,T]$;

(iii) $\left|\mathrm{Ric} - \frac{1}{3}\mathcal{R}g\right|^2 \le Lt^{\sigma-2}(\mathcal{R}+L)^\sigma$ *on* $B_t(x_0,1)$ *for* $t \in [0,T]$ *and some* $L > 0, \sigma \in (1,2)$.

Then there exist $a_1(\sigma,L), S_3(\sigma,L) > 0$ *such that for* $t \in (0, T \wedge S_3]$ *we have*

$$|\mathrm{Rm}(x_0,t)| \le a_1 t^{-1}.$$

Proof. The proof of the curvature estimate follows the point-picking argument from [16, Lemma 2.1].

Suppose the conclusion is false, then there exist $\sigma \in (1,2)$ and $L > 0$ such that for any $a_k \to +\infty$, there exists a sequence of three-dimensional non-compact manifolds M_k, Ricci flows $g_k(t)$, $t \in [0,T_k]$ on M_k and points $x_k \in M_k$ satisfying the hypotheses of the lemma but so that the curvature conclusion fails in an arbitrarily short time. We may assume $a_k T_k \to 0$. By smoothness of each Ricci flow, we can choose $t_k \in (0,T_k]$ so that

(i) $B_{g_k(t)}(x_k,1) \Subset M_k$ for $t \in [0,t_k]$;

(ii) $\mathcal{R}(g_k(t)) > -4$ on $B_{g_k(t)}(x_k,1)$, $t \in [0,t_k]$;

(iii) $\left|\mathrm{Ric}(g_k(t)) - \frac{1}{3}\mathcal{R}(g_k(t))g_k(t)\right|^2 \le Lt^{\sigma-2}(\mathcal{R}(g_k(t)) + L)^\sigma$ on $B_{g_k(t)}(x_k,1)$ for $t \in [0,t_k]$;

(iv) $|\mathrm{Rm}(g_k(t))|(x_k) < a_k t^{-1}$ for $t \in (0,t_k)$;

(v) $|\mathrm{Rm}(g_k(t_k))|(x_k) = a_k t_k^{-1}$.

By (iv) and the fact that $a_k t_k \to 0$, [16, Lemma 5.1] implies that for k sufficiently large, we can find $\beta > 0, \tilde{t}_k \in (0,t_k]$ and $\tilde{x}_k \in B_{g_k(\tilde{t}_k)}(x_k, \frac{3}{4} - \frac{1}{2}\beta\sqrt{a_k\tilde{t}_k})$ such that

$$|\mathrm{Rm}(g_k(x,t))| \le 4|\mathrm{Rm}(g_k(\tilde{x}_k,\tilde{t}_k)| = 4Q_k \tag{3.26}$$

whenever $d_{g_k(\tilde{t}_k)}(x,\tilde{x}_k) < \frac{1}{8}\beta a_k Q_k^{-1/2}$ and $\tilde{t}_k - \frac{1}{8}a_k Q_k^{-1} \le t \le \tilde{t}_k$ where $\tilde{t}_k Q_k \ge a_k \to +\infty$.

Consider the parabolic rescaling centred at $(\tilde{x}_k,\tilde{t}_k)$, namely $\tilde{g}_k(t) = Q_k g_k(\tilde{t}_k + Q_k^{-1}t)$ for $t \in [-\frac{1}{8}a_k, 0]$ so that

(a) $|\mathrm{Rm}_{\tilde{g}_k(0)}(\tilde{x}_k)| = 1$;

(b) $|\mathrm{Rm}_{\tilde{g}_k(t)}| \le 4$ on $B_{\tilde{g}_k(0)}(\tilde{x}_k, \frac{1}{8}\beta a_k) \times [-\frac{1}{8}a_k, 0]$, and

(c) on $B_{\tilde{g}_k(0)}(\tilde{x}_k, \frac{1}{8}\beta a_k)$,

$$\begin{cases} \mathcal{R}(\tilde{g}_k(0)) > -4Q_k^{-1}; \\ \left|\mathrm{Ric}(\tilde{g}_k(0)) - \dfrac{1}{3}\mathcal{R}(\tilde{g}_k(0))\tilde{g}_k(0)\right|^2 \le La_k^{\sigma-2}(\mathcal{R}(\tilde{g}_k(0)) + LQ_k^{-1})^\sigma, \end{cases} \tag{3.27}$$

where we used that $\tilde{t}_k Q_k \geq a_k$ to obtain the final estimate.

It what follows, we tread carefully to accommodate the fact that we have no uniform lower bound on the injectivity radius of $\tilde{g}_k(0)$. By (b), we can pick a universal $\rho > 0$ so that the conjugate radius is always larger than ρ. Therefore we can lift $\left(B_{\tilde{g}_k(0)}(\tilde{x}_k, \rho), \tilde{g}_k(t)\right)$ to $(B_{euc}(\rho), \tilde{g}_k(t))$ by the exponential map of $\tilde{g}_k(0)$ at $\tilde{x}_k$. By Shi's estimate [14], we may assume $\tilde{g}_k(t)$ converges to $\tilde{g}_\infty(t)$ uniformly locally smoothly on the $B_{euc}(\rho) \times (-\infty, 0]$ after passing to a subsequence. Using (c), we conclude that

$$\begin{cases} \mathcal{R}(\tilde{g}_\infty(0)) & \geq 0; \\ \mathrm{Ric}(\tilde{g}_\infty(0)) & = \frac{1}{3}\mathcal{R}(\tilde{g}_\infty(0))\tilde{g}_\infty(0) \end{cases} \tag{3.28}$$

and hence $\tilde{g}_\infty(0)$ is Einstein with non-negative Einstein constant on the Euclidean ball of radius ρ, thanks to the Bianchi identity, and thus has constant non-negative sectional curvature. By (a) and the smooth convergence, the constant sectional curvature must be positive and hence $\mathrm{Ric}(\tilde{g}_\infty(0)) = 3\alpha$ on the Euclidean ball of radius ρ for some $\alpha > 0$ that could be computed explicitly.

We now extend this control to a larger region. For any fixed $r > \rho$ and each k, pick a maximal disjoint collection of balls $B_{\tilde{g}_k(0)}(y_k^j, \frac{\rho}{100})$ within $B_{\tilde{g}_k(0)}(\tilde{x}_k, r)$, indexed by j. By volume comparison, the number of points y_k^j is bounded uniformly in k, and so by passing to a subsequence we may assume that j always ranges from 1 to some $J \in \mathbb{N}$. We thus have J sequences y_k^j that can be substituted for $\tilde{x}_k$ in the previous process of lifting by the exponential map and passing to a subsequence to give a limit that is of constant sectional curvature. After passing to all J subsequences, because of the overlaps between the covering balls $B_{\tilde{g}_k(0)}(y_k^j, \rho)$ we deduce that each limit has the *same* constant sectional curvature. In particular, after deleting finitely many elements of the sequence we may assume that $\mathrm{Ric}(\tilde{g}_k(0)) > 2\alpha$ on $B_{\tilde{g}_k(0)}(\tilde{x}_k, r)$ for all $k \in \mathbb{N}$.

But it is well understood that one cannot have a large region of a manifold with controllably positive Ricci curvature: Set $L := 2\pi/\sqrt{\alpha}$ and take the subsequence constructed above corresponding to $r = 2L$. Then $\mathrm{Ric}(\tilde{g}_1(0)) > 2\alpha$ on $B_{\tilde{g}_1(0)}(\tilde{x}_1, 2L)$. Suppose there exists $z \in B_{\tilde{g}_1(0)}(\tilde{x}_1, 2L)$ such that $d_{\tilde{g}_1(0)}(\tilde{x}_1, z) = L$. Let γ be a minimizing geodesic realizing $d_{\tilde{g}_1(0)}(\tilde{x}_1, z) = L$; then γ must lie inside $B_{\tilde{g}_1(0)}(\tilde{x}_1, 2L)$ where the Ricci curvature is bounded from below by 2α. By the proof of the Bonnet-Myers theorem, the Ricci curvature's lower bound implies $d_{\tilde{g}_1(0)}(\tilde{x}_1, z) \leq \pi/\sqrt{\alpha}$ which is impossible. And hence, $B_{\tilde{g}_1(0)}(\tilde{x}_1, L) = M_1$ which contradicts that $B_{g_1(\tilde{t}_1)}(x_1, 1) \Subset M_1$ and the non-compactness of M_1. □

4. Ricci flow lemmas

In this section we recall some useful results about Ricci flow. The first of these is a result of Hochard that allows us to modify an incomplete Riemannian metric at its extremities in order to make it complete, without increasing the curvature too much, and without changing it in the interior.

Lemma 4.1 (Hochard [10, Corollaire IV.1.2]). *There exists $\sigma(n) > 1$ such that given a Riemannian manifold (N^n, g) with $|\mathrm{Rm}(g)| \le \rho^{-2}$ throughout for some $\rho > 0$, there exists a complete Riemannian metric h on N such that*

(1) *$h \equiv g$ on $N_\rho := \{x \in N : B_g(x, \rho) \Subset N\}$, and*
(2) *$|\mathrm{Rm}(h)| \le \sigma\rho^{-2}$ throughout N.*

An immediate consequence of Hochard's lemma, via Shi's existence theorem for Ricci flow starting with complete initial metrics of bounded curvature [14], is the following local existence result for Ricci flow.

Proposition 4.2. *There exist constants $\alpha(n) \in (0, 1]$ and $\Lambda(n) > 1$ so that the following is true. Suppose (N^n, h_0) is a smooth manifold (not necessarily complete) that satisfies $|\mathrm{Rm}(h_0)| \le \rho^{-2}$ throughout, for some $\rho > 0$. Then there exists a smooth Ricci flow $h(t)$ on N for $t \in [0, \alpha\rho^2]$, with the properties that*

(i) *$h(0) = h_0$ on $N_\rho = \{x \in N : B_{h_0}(x, \rho) \Subset N\}$;*
(ii) *$|\mathrm{Rm}(h(t))| \le \Lambda\rho^{-2}$ throughout $N \times [0, \alpha\rho^2]$.*

When applying this lemma, we will only be interested in the restriction of $h(t)$ to the subset N_ρ.

We also recall the shrinking balls lemma, which is one of the local ball inclusion results based on the distance distortion estimates of Hamilton and Perelman from [13, Lemma 8.3].

Lemma 4.3 ([16, Corollary 3.3]). *There exists a constant $\beta = \beta(n) \ge 1$ depending only on n such that the following is true. Suppose $(N^n, g(t))$ is a Ricci flow for $t \in [0, S]$ and $x_0 \in N$ with $B_0(x_0, r) \Subset N$ for some $r > 0$, and $\mathrm{Ric}(g(t)) \le a/t$ on $B_{g_0}(x_0, r)$ for each $t \in (0, S]$. Then*

$$B_t\left(x_0, r - \beta\sqrt{at}\right) \subset B_{g_0}(x_0, r).$$

5. Existence of Ricci flow

In this section, we will construct the Ricci flow starting from complete non-compact manifolds with non-negatively pinched Ricci curvature. More generally, we will construct the partial Ricci flow which is based on the ideas from the work of Hochard [9], Simon and the second named author [17].

Theorem 5.1. *For all $\frac{1}{12} > \varepsilon > 0$, there exist $T(\varepsilon), a_0(\varepsilon) > 0$ such that the following holds. Suppose (M^3, g_0) is a three-dimensional manifold and $p \in M$ so that*

(i) *$B_{g_0}(p, R + 4) \Subset M$ for some $R \ge 1$;*
(ii) *$\mathrm{Ric}(g_0) \ge \varepsilon\mathcal{R}(g_0) \ge 0$ on $B_{g_0}(p, R + 4)$.*

Then there exists a smooth Ricci flow solution $g(t)$ defined on $B_{g_0}(p, R) \times [0, T]$ such that $g(0) = g_0$, $|\mathrm{Rm}(x, t)| \le a_0 t^{-1}$ and

$$\mathrm{Ric}(x, t) \ge \varepsilon\mathcal{R}(x, t) - 1 \tag{5.1}$$

for all $(x,t) \in B_{g_0}(p,R) \times (0,T]$.

Proof. We begin by specifying the constants that will be used in the proof. For the given pinching constant ε, we will use:

- $\Lambda(3) > 1$ and $\alpha(3) \in (0,1]$: from Proposition 4.2;
- $\beta(3)$: from Lemma 4.3;
- $L_1(\varepsilon) > 0, \sigma(\varepsilon) \in (1,2)$: from Lemma 3.2;
- $a_1(\sigma, L_1), S_3(\sigma, L_1)$: from Lemma 3.3;
- $a_0 := \max\{\Lambda\alpha, \Lambda(a_1+\alpha), 1\}$;
- $S_1(a_0,\varepsilon)$: from Lemma 3.1;
- $S_2(a_0,\varepsilon)$: from Lemma 3.2;
- $\Lambda_0 := 16 \cdot \max\{(a_0a_1)^{-1/2}, \beta, S_1^{-1/2}, S_2^{-1/2}, S_3^{-1/2}\}$;
- $\mu = \sqrt{1+\alpha a_1^{-1}} - 1 > 0$.

Choose $1 \geq \rho > 0$ sufficiently small so that for all $x \in B_{g_0}(p, R+4)$, we have $|\mathrm{Rm}(g_0)| \leq \rho^{-2}$. By Proposition 4.2, applied with $N = B_{g_0}(p, R+4)$, we can find a smooth solution $g(t)$ to the Ricci flow on (a superset of) $B_{g_0}(p, R+3) \times [0, \alpha\rho^2]$ with $|\mathrm{Rm}(g(t))| \leq \Lambda\rho^{-2}$ and (once restricted to $B_{g_0}(p, R+3)$) with initial data g_0. Because $a_0 \geq \Lambda\alpha$, the curvature bound can be weakened to $|\mathrm{Rm}(g(t))| \leq a_0 t^{-1}$.

We now define sequences of times t_k and radii r_k inductively as follows:

(a) $t_1 = \alpha\rho^2,\ r_1 = R+3$, where ρ is obtained from above;
(b) $t_{k+1} = (1+\mu)^2 t_k = (1+\alpha a_1^{-1})t_k$ for $k \geq 1$;
(c) $r_{k+1} = r_k - \Lambda_0\sqrt{a_0 t_k}$ for $k \geq 1$.

Let $\mathcal{P}(k)$ be the following statement: there is a smooth Ricci flow solution $g(t)$ defined on $B_{g_0}(p, r_k) \times [0, t_k]$ with $g(0) = g_0$ such that $|\mathrm{Rm}(g(t))| \leq a_0 t^{-1}$. Noted that we choose ρ so that $\mathcal{P}(1)$ is true. Our goal is to show that $\mathcal{P}(k)$ is true for all k so that $r_k > 0$.

We now perform an inductive argument. Suppose $\mathcal{P}(k)$ is true, that is to say that $\exists g(t)$ on $B_{g_0}(p, r_k) \times [0, t_k]$ with $|\mathrm{Rm}(g(t))| \leq a_0 t^{-1}$. We want to show that $\mathcal{P}(k+1)$ is true provided that $r_{k+1} > 0$.

Let $x \in B_{g_0}(p, r_{k+1} + \frac{1}{2}\Lambda_0\sqrt{a_0 t_k})$ so

$$B_{g_0}\left(x, \frac{1}{4}\Lambda_0\sqrt{a_0 t_k}\right) \Subset B_{g_0}(p, r_k). \tag{5.2}$$

Consider the rescaled Ricci flow $\tilde{g}(t) = \lambda_1^{-2} g(\lambda_1^2 t), t \in [0, 16\Lambda_0^{-2}a_0^{-1}]$ where $\lambda_1 = \frac{1}{4}\Lambda_0\sqrt{a_0 t_k}$ so that $B_{g(0)}(x, \frac{1}{4}\Lambda_0\sqrt{a_0 t_k}) = B_{\tilde{g}(0)}(x, 1)$. Since $\mathrm{Ric}(\tilde{g}(0)) \geq \varepsilon\mathcal{R}(\tilde{g}(0))$ on $B_{\tilde{g}(0)}(x,1)$ and $|\mathrm{Rm}(\tilde{g}(t))| \leq a_0 t^{-1}$ on $B_{\tilde{g}(0)}(x,1)$ for $t \in (0, 16\Lambda_0^{-2}a_0^{-1}]$, we can apply Lemma 3.1 to $\tilde{g}(t)$ to conclude that

$$\mathrm{Ric}(\tilde{g}(x,t)) \geq \varepsilon\mathcal{R}(\tilde{g}(x,t)) - 1$$

for $t \in [0, 16\Lambda_0^{-2} a_0^{-1}]$ since $16\Lambda_0^{-2} \le S_1$ and $a_0 \ge 1$. Rescaling back to $g(t)$, we see that on $B_{g_0}\left(p, r_{k+1} + \frac{1}{2}\Lambda_0\sqrt{a_0 t_k}\right) \times [0, t_k]$, we have

$$\mathrm{Ric}(g(t)) \ge \varepsilon \mathcal{R}(g(t)) - \left(\frac{1}{4}\Lambda_0\sqrt{a_0 t_k}\right)^{-2}. \tag{5.3}$$

Next we aim to prove the estimate (5.4) below on the ball $B_{g_0}(p, r_{k+1} + \frac{1}{4}\Lambda_0\sqrt{a_0 t_k})$. Fix $x \in B_{g_0}(p, r_{k+1} + \frac{1}{4}\Lambda_0\sqrt{a_0 t_k})$. By the shrinking balls lemma 4.3 and the assumption $\Lambda_0 \ge 8\beta$, we have the inclusions

$$B_{g(t)}\left(x, \frac{1}{8}\Lambda_0\sqrt{a_0 t_k}\right) \subset B_{g_0}\left(x, \frac{1}{4}\Lambda_0\sqrt{a_0 t_k}\right) \Subset B_{g_0}\left(p, r_{k+1} + \frac{1}{2}\Lambda_0\sqrt{a_0 t_k}\right)$$

for all $t \in [0, t_k]$, where (5.3) holds. Consider the rescaled Ricci flow $\hat{g}(t) = \lambda_2^{-2} g(\lambda_2^2 t), t \in [0, 8^2\Lambda_0^{-2} a_0^{-1}]$ where $\lambda_2 = \frac{1}{8}\Lambda_0\sqrt{a_0 t_k}$ so that $B_{g(\lambda_2^2 t)}(x, \frac{1}{8}\Lambda_0\sqrt{a_0 t_k}) = B_{\hat{g}(t)}(x, 1)$. Under the rescaling, the pinching estimate (5.3) becomes

$$\mathrm{Ric}(\hat{g}(t)) \ge \varepsilon \mathcal{R}(\hat{g}(t)) - \frac{1}{4} \ge \varepsilon \mathcal{R}(\hat{g}(t)) - 1.$$

Keeping in mind the a_0/t curvature decay we can apply Lemma 3.2 over the whole time interval $[0, 8^2\Lambda_0^{-2} a_0^{-1}]$ because the assumptions $8^2\Lambda_0^{-2} \le S_2$ and $a_0 \ge 1$ imply that $S_2 \ge 8^2\Lambda_0^{-2} a_0^{-1}$. Rescaling the conclusion of Lemma 3.2 back to $g(t)$ shows that

$$\left|\mathrm{Ric} - \frac{1}{3}\mathcal{R} g\right|^2 \le \frac{L_1}{t^{2-\sigma}} \cdot \left[\mathcal{R} + 4\left(\frac{1}{8}\Lambda_0\sqrt{a_0 t_k}\right)^{-2}\right]^{\sigma} \tag{5.4}$$

on $B_{g_0}(p, r_{k+1} + \frac{1}{4}\Lambda_0\sqrt{a_0 t_k}) \times [0, t_k]$ as required.

Next we would like to establish a_1/t decay of curvature on the ball $B_{g_0}(p, r_{k+1} + \frac{1}{8}\Lambda_0\sqrt{a_0 t_k})$ for all $t \in (0, t_k]$. Pick an arbitrary point $x \in B_{g_0}(p, r_{k+1} + \frac{1}{8}\Lambda_0\sqrt{a_0 t_k})$ where we would like this decay to hold. By the shrinking balls lemma 4.3 and the assumption $\Lambda_0 \ge 16\beta$, we have the inclusions

$$B_{g(t)}\left(x, \frac{1}{16}\Lambda_0\sqrt{a_0 t_k}\right) \subset B_{g_0}\left(x, \frac{1}{8}\Lambda_0\sqrt{a_0 t_k}\right) \Subset B_{g_0}\left(p, r_{k+1} + \frac{1}{4}\Lambda_0\sqrt{a_0 t_k}\right)$$

for all $t \in [0, t_k]$. Thus rescaling the flow to $\lambda_3^{-2} g(\lambda_3^2 t)$, where $\lambda_3 = \frac{1}{16}\Lambda_0\sqrt{a_0 t_k}$, which lives for $t \in [0, (16)^2\Lambda_0^{-2} a_0^{-1}]$, we can apply Lemma 3.3, using (5.4) and (5.3), to deduce that

$$|\mathrm{Rm}(x, t)| \le a_1 t^{-1} \tag{5.5}$$

for all $t \in (0, t_k]$ provided that $S_3 \ge (16)^2\Lambda_0^{-2} a_0^{-1}$, and this indeed holds by our assumptions that $16^2\Lambda_0^{-2} \le S_3$ and $a_0 \ge 1$.

Denote $\Omega = B_{g_0}\left(p, r_{k+1} + \frac{1}{8}\Lambda_0\sqrt{a_0 t_k}\right)$ so that for $h_0 = g(t_k)$, estimate (5.5) gives $\sup_\Omega |\mathrm{Rm}(h_0)| \le \rho^{-2}$ where $\rho = \sqrt{t_k a_1^{-1}}$. Moreover, for $x \in B_{g_0}(p, r_{k+1})$, the assumptions $\Lambda_0\sqrt{a_0a_1} \geq 16$ and $\Lambda_0 \geq 16\beta$, together with the shrinking balls lemma 4.3 (needing only the weaker a_0/t decay rather than the a_1/t decay that we have proved) give

$$B_{g(t_k)}(x,\rho) \subset B_{g(t_k)}\left(x, \frac{1}{16}\Lambda_0\sqrt{a_0t_k}\right) \subset B_{g_0}\left(x, \frac{1}{8}\Lambda_0\sqrt{a_0t_k}\right) \Subset \Omega. \tag{5.6}$$

This shows that $B_{g_0}(p, r_{k+1}) \subset \Omega_\rho$, where Ω_ρ is computed with respect to $g(t_k)$. Hence, we may apply Proposition 4.2 to find a Ricci flow $g(t)$ on (a superset of) $B_{g_0}(p, r_{k+1}) \times [t_k, t_k + \alpha\rho^2]$, extending the existing $g(t)$ on this smaller ball, with

$$|\mathrm{Rm}(g(t))| \leq \Lambda\rho^{-2} = \Lambda a_1 t_k^{-1} \leq a_0 t^{-1} \tag{5.7}$$

since $\Lambda(a_1 + \alpha) \leq a_0$ and $t_k + \alpha\rho^2 = t_k\left(1 + \alpha a_1^{-1}\right) = t_{k+1}$. This shows that $\mathcal{P}(k+1)$ is true provided that $r_{k+1} > 0$.

Since $\lim_{j\to+\infty} r_j = -\infty$, there is $i \in \mathbb{N}$ such that $r_i \geq R+1$ and $r_{i+1} < R+1$. In particular, $\mathcal{P}(i)$ is true since $r_i > 0$. We now estimate t_i.

$$\begin{aligned} R+1 > r_{i+1} &= r_1 - \Lambda_0\sqrt{a_0}\cdot\sum_{k=1}^{i}\sqrt{t_k} \\ &\geq R+3-\Lambda_0\sqrt{a_0t_i}\cdot\sum_{k=0}^{\infty}(1+\mu)^{-k} \\ &= R+3-\sqrt{t_i}\cdot\frac{\Lambda_0\sqrt{a_0}(1+\mu)}{\mu}. \end{aligned} \tag{5.8}$$

This implies

$$t_i > \frac{4\mu^2}{a_0\Lambda_0^2(1+\mu)^2} =: T(\varepsilon). \tag{5.9}$$

In other words, there exists a smooth Ricci flow solution $g(t)$ defined on $B_{g_0}(p, R+1) \times [0,T]$ so that $g(0) = g_0$ and $|\mathrm{Rm}(g(t))| \leq a_0t^{-1}$. That the almost pinching estimate (5.1) holds at an arbitrary point $x_0 \in B_{g_0}(p,R)$ follows immediately from an application of Lemma 3.1 on $B_{g_0}(x_0, 1)$ provided we allow ourselves to reduce $T > 0$. This completes the proof. □

By an exhaustion argument, we can prove Theorem 1.2 now.

Proof of Theorem 1.2. By reducing $\varepsilon > 0$ if necessary, we may assume that $\varepsilon \in (0, \frac{1}{100})$. Let $R_i \to +\infty$ and denote $h_{i,0} = R_i^{-2}g_0$ so that $\mathrm{Ric}(h_{i,0}) \geq \varepsilon\mathcal{R}(h_{i,0})$ on M. By Theorem 5.1, there is a Ricci flow solution $h_i(t)$ on $B_{h_{i,0}}(p,1)\times[0,T]$ with

(a) $|\mathrm{Rm}(h_i(t))| \leq a_0t^{-1}$;
(b) $\mathrm{Ric}(h_i(t)) \geq \varepsilon\mathcal{R}(h_i(t)) - 1$.

Define $g_i(t) = R_i^2 h_i(R_i^{-2} t)$ which is a Ricci flow solution on $B_{g_0}(p, R_i) \times [0, TR_i^2]$ with

$$\begin{cases} g_i(0) = g_0; \\ |\mathrm{Rm}(g_i(t))| \le a_0 t^{-1}; \\ \mathrm{Ric}(g_i(t)) \ge \varepsilon \mathcal{R}(g_i(t)) - R_i^{-2} \end{cases} \tag{5.10}$$

on each $B_{g_0}(p, R_i) \times (0, TR_i^2]$.

By [4, Corollary 3.2] (see also [15]) and a modification of Shi's higher order estimate given in [5, Theorem 14.16], we infer that for any $k \in \mathbb{N}$, $S > 0$ and $\Omega \Subset M$, we can find $C(k, \Omega, g_0, \varepsilon, S) > 0$ so that for sufficiently large i we have

$$\sup_{\Omega \times [0,S]} |\nabla^k \mathrm{Rm}(g_i(t))| \le C(k, \Omega, g_0, \varepsilon, S). \tag{5.11}$$

By working in coordinate charts and applying the Ascoli-Arzelà Theorem, we may pass to a subsequence to obtain a smooth solution $g(t) = \lim_{i \to +\infty} g_i(t)$ of the Ricci flow on $M \times [0, +\infty)$ so that $g(0) = g_0$, $|\mathrm{Rm}(x, t)| \le a_0 t^{-1}$ and

$$\mathrm{Ric}(x, t) \ge \varepsilon \mathcal{R}(x, t) \tag{5.12}$$

for all $(x, t) \in M \times [0, +\infty)$. Moreover, it is a complete solution by Lemma 4.3. By tracing this pinching estimate, we deduce that $\mathcal{R} \ge 0$. This completes the proof. □

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