A RIGIDITY ESTIMATE FOR MAPS FROM S^2 TO S^2 VIA THE HARMONIC MAP FLOW

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Abstract

We show how a rigidity estimate introduced in recent work of Bernand-Mantel, Muratov and Simon can be derived using the harmonic map flow. The estimate controls how far a degree one map between 2-spheres can be from a Möbius map in terms of its energy, irrespective of whether or not the map has small tension.

1 Introduction

For maps $u \in W^{1,2}(S^2, S^2)$, consider the harmonic map energy

$$E(u) = \frac{1}{2}\int |Du|^2.$$

Following Schoen-Uhlenbeck [5] we know that such a map can be approximated in $W^{1,2}(S^2, S^2)$ by smooth maps u_i, whose degree will stabilise at some integer that gives a well-defined notion of degree of u. See also [8, Theorem 6.2]. For smooth maps w from a surface to any Riemannian manifold, a simple computation confirms that $E(w)$ is always at least the area of the image of w, with equality if w is weakly conformal. This implies that degree $k \in \mathbb{N}$ maps from S^2 to itself must have energy at least $4\pi k$, with equality implying that the map is a rational map. A special case of this is that if u is of degree one then $E(u) \geq 4\pi$ with equality if and only if u is a Möbius map.

The following theorem was recently proved by Bernand-Mantel, Muratov and Simon [1].

Theorem 1.1. *There exists a universal constant $C < \infty$ such that for each $u \in W^{1,2}(S^2, S^2)$ of degree one, there exists a Möbius map $v : S^2 \to S^2$ such that*

$$\int |D(u - v)|^2 \leq C[E(u) - 4\pi].$$

The purpose of this note is to show that the estimate above can be derived from the theory of the harmonic map flow developed in [10]. Independently, Hirsch and Zemas have given a different shorter proof [3].

2 The harmonic map flow from surfaces

The harmonic map flow [2] is the L^2-gradient flow for the harmonic map energy. If we view a flow $u : S^2 \times [0,T) \to S^2 \hookrightarrow \mathbb{R}^3$ as taking values in $\mathbb{R}^3$, then the equation in this case can be written

$$\frac{\partial u}{\partial t} = \tau(u) := (\Delta u)^T$$

where $(\Delta u)^T$ is the projection of Δu onto the tangent space of the target, i.e. $(\Delta u)^T = \Delta u + u|Du|^2$. The energy $E(t) := E(u(t))$ decays according to

$$\frac{dE}{dt} = -\|\tau(u)\|^2_{L^2(S^2)}. \tag{2.1}$$

In 1985 Struwe [7] initiated a theory for the harmonic map flow in the case that the domain is a surface, as it is here. He showed how one can start a flow with smooth or even $W^{1,2}$ initial data, giving a global weak solution that is smooth away from finitely many points in space-time, at which bubbling occurs. At each finite-time singularity, concentrated energy at the singular points is thrown away by taking a weak limit, and the flow restarted. Thus, the energy drops down at each singular time by at least the minimum energy of one bubble, i.e. of a nonconstant harmonic map from S^2 to the target. However, the map $t \mapsto u(t)$ is continuous into L^2, even across singular times. (Later [11], different continuations through certain singularities were constructed that did not require a drop in energy, but we will not need them here.)

Concerning the asymptotics at infinite time, there exists a sequence $t_i \to \infty$ such that maps $u(t_i)$ converge smoothly to a limiting harmonic map u_∞ away from finitely many bubble points. (See, for example, [8].) In general, even in the absence of bubbling, convergence of the form $u(t) \to u_\infty$ as $t \to \infty$ may fail, even in L^1, see [9, 10]. However, a theory was developed in [10, 12] concerning such uniform convergence for maps from S^2 to itself. A key lemma from that work, which will be useful to us now, gives a relationship between the tension field of a map u and its excess energy. It differed from previous estimates of 'Lojasiewicz-Simon' type [6] in that it could handle singular objects. In particular the map u below is not asked to be $W^{1,2}$ close to a harmonic map.

Lemma 2.1 ([10, Lemma 1]). *There exist universal constants $\epsilon_0 > 0$ and $\kappa > 0$ such that if a smooth degree $k \in \mathbb{Z}$ map $u : S^2 \to S^2$ satisfies $E(u) - 4\pi|k| < \epsilon_0$, then*

$$E(u) - 4\pi|k| \leq \kappa^2 \|\tau(u)\|^2_{L^2}.$$

As is well known, such an estimate gives control on the gradient flow. Indeed, given a smooth solution $u : S^2 \times [0,T] \to S^2$ that is of degree $k \in \mathbb{Z}$, and satisfies $E(u) - 4\pi|k| < \epsilon_0$ at time $t = 0$ (and therefore also for later times) we can compute using (2.1) and then Lemma 2.1 that

$$-\frac{d}{dt}\left[E(u) - 4\pi|k|\right]^{\frac{1}{2}} = \frac{1}{2}\left[E(u) - 4\pi|k|\right]^{-\frac{1}{2}} \|\tau(u)\|^2_{L^2} \geq \frac{1}{2\kappa}\|\tau(u)\|_{L^2}$$

and integrating from $t = s \in [0, T)$ to $t = T$ gives

$$\begin{aligned} \int_s^T \|\tau(u)\|_{L^2} dt &\leq 2\kappa \left([E(u(s)) - 4\pi|k|]^{\frac{1}{2}} - [E(u(T)) - 4\pi|k|]^{\frac{1}{2}} \right) \\ &\leq 2\kappa \left[E(u(s)) - 4\pi|k| \right]^{\frac{1}{2}}. \end{aligned} \tag{2.2}$$

Since $\frac{\partial u}{\partial t} = \tau(u)$, the flow then cannot move far in L^2:

$$\|u(T) - u(s)\|_{L^2} \leq 2\kappa \left[E(u(s)) - 4\pi|k| \right]^{\frac{1}{2}}. \tag{2.3}$$

3 Proof of Theorem 1.1

Proof. First, if $E(u) = 4\pi$, then u must be a Möbius map and we can choose $v = u$, so from now on we may assume that $E(u) > 4\pi$. By approximation, using the definition of degree, it suffices to prove the result for $u \in C^\infty(S^2, S^2)$. Since the estimate is invariant under pre-composition by Möbius maps, it suffices to prove the theorem for u equal to a map u_0 with the property that $\int_{S^2} u_0 = 0 \in \mathbb{R}^3$. That this can be achieved follows from a topological argument: For $a \in B^3 \setminus \{0\}$, let $\varphi_a : S^2 \to S^2$ be the Möbius map that fixes $\pm\frac{a}{|a|}$, whose differential does not rotate the tangent spaces at $\pm\frac{a}{|a|}$, and which can be extended to a conformal map $\overline{B^3} \mapsto \overline{B^3}$ sending the origin to a. For the case $a = 0 \in B^3$ we set φ_0 to be the identity. When restricted back to $S^2 \hookrightarrow \mathbb{R}^3$, the map φ_a can be written explicitly as

$$\varphi_a(x) = (1 - |a|^2)\frac{(x + a)}{|x + a|^2} + a,$$

although the extension mentioned above would be this map composed with the inversion $x \mapsto \frac{x}{|x|^2}$ that fixes S^2. Thus as a approaches some $a_0 \in S^2 = \partial B^3$, the composition $u \circ \varphi_a$ converges to $u(a_0)$ away from $-a_0$, so $\frac{1}{4\pi}\int u \circ \varphi_a \to u(a_0)$. Thus the map

$$a \mapsto \frac{1}{4\pi}\int u \circ \varphi_a$$

extends to a continuous map Φ from $\overline{B^3}$ to itself that agrees with the degree one map u on the boundary S^2. A topological argument then tells us that Φ is surjective, since otherwise Φ could be homotoped to a continuous map $\tilde{\Phi}$ from $\overline{B^3}$ to S^2 that restricts to $u : S^2 \to S^2$. But then $\tilde{\Phi}$ would provide a homotopy from the degree one map u to a constant map, a contradiction. In particular, there exists $a \in B^3$ such that $\Phi(a) = 0 \in B^3$. We can then set $u_0 := u \circ \varphi_a$ to achieve our objective. For a related argument in which one *post*-composes with Möbius maps see Li-Yau [4].

Let now ϵ_0 be as in the key lemma 2.1. For later use, if necessary we reduce $\epsilon_0 > 0$ so that

$$(1 + 4\kappa^2)\epsilon_0 \leq \pi. \tag{3.1}$$

We may assume that our map u_0 satisfies $E(u_0) - 4\pi < \epsilon_0$ since otherwise the theorem is vacuously true.

To prove the theorem, run the harmonic map flow starting with u_0. By (2.3), the flow is constrained in how far it can move in L^2. Because of the balancing $\int_{S^2} u_0 = 0$, this implies that $\int_{S^2} u(t)$ remains close to the origin, which precludes bubbling both at finite and infinite time. More precisely, because

$$\frac{d}{dt}\int u = \int \tau(u)$$

we have

$$\left|\frac{d}{dt}\int u\right| \leq 2\sqrt{\pi}\|\tau(u)\|_{L^2},$$

and therefore, integrating from 0 to t using (2.2) we have

$$\left|\fint u(t)\right| \leq \frac{\kappa}{\sqrt{\pi}}\left[E(u_0) - 4\pi\right]^{\frac{1}{2}}. \tag{3.2}$$

Suppose we develop a singularity at a finite or infinite time $T \in (0, \infty]$. If we pick $t_i \uparrow T$ such that $u(t_i) \rightharpoonup w$ weakly in $W^{1,2}$, then we must have $E(w) \leq E(u_0) - 4\pi$, since the singularity loses at least 4π of energy (that being the least possible energy of a nonconstant harmonic map from S^2 to itself) but also because

$$\fint u(t_i) \to \overline{w} := \fint w$$

we have

$$|\overline{w}| \leq \frac{\kappa}{\sqrt{\pi}}\left[E(u_0) - 4\pi\right]^{\frac{1}{2}}.$$

But the Poincaré inequality tells us that

$$\int_{S^2} |w - \overline{w}|^2 \leq E(w),$$

and so integrating the inequality

$$1 = |w|^2 \leq 2|w - \overline{w}|^2 + 2|\overline{w}|^2,$$

we obtain

$$\begin{aligned} 4\pi &\leq 2\int |w - \overline{w}|^2 + 8\pi|\overline{w}|^2 \\ &\leq 2E(w) + 8\kappa^2\left[E(u_0) - 4\pi\right] \\ &\leq (2 + 8\kappa^2)\left[E(u_0) - 4\pi\right] \\ &\leq 2\pi, \end{aligned} \tag{3.3}$$

by (3.1), giving a contradiction.

We deduce that the flow exists for all time and converges smoothly to a Möbius map v. Here we only need convergence at some sequence of times $t_i \to \infty$, although we have convergence as $t \to \infty$ by [10].

We can also pass the inequality (3.2) to the limit $t \to \infty$ to give

$$|\overline{v}| \leq \frac{\kappa}{\sqrt{\pi}}\left[E(u_0) - 4\pi\right]^{\frac{1}{2}} \leq \kappa\left(\frac{\epsilon_0}{\pi}\right)^{\frac{1}{2}} \leq \frac{1}{2},$$

by (3.1). This estimate prevents v from being too concentrated, and we can deduce that

$$|Dv| \leq c_0, \tag{3.4}$$

for some universal c_0. This follows by noticing that every Möbius map, in particular v, can be written as one of the maps φ_a followed by a rotation of S^2. The derivative of φ_a can only get large as a approaches ∂B^3, which is ruled out by the inequality $|\overline{v}| \leq \frac{1}{2}$. An alternative is to argue by contradiction: If such an estimate (3.4) were not true, then we would take a sequence of Möbius maps v_i with $|\overline{v_i}| \leq \frac{1}{2}$ but with $\sup |Dv_i| \to \infty$. After a bubbling analysis (passing to a subsequence) the maps v_i would converge weakly in $W^{1,2}$ to a constant map $v_\infty : S^2 \to S^2$ (and so $|\overline{v_\infty}| = 1$) with $|\overline{v_\infty}| \leq \frac{1}{2}$, a contradiction.

Returning to (2.3), we find that $\|u_0 - v\|_{L^2}^2 \leq 4\kappa^2[E(u_0) - 4\pi]$. But we can also compute

$$\int |D(u_0 - v)|^2 = \int |Du_0|^2 + \int |Dv|^2 - 2\int \langle Du_0, Dv\rangle$$

and because $-\Delta v = v|Dv|^2$, we can handle the final term using

$$\begin{aligned} -2\int \langle Du_0, Dv\rangle &= -2\int u_0(-\Delta v) = -2\int u_0 v|Dv|^2 \\ &= \int |u_0 - v|^2|Dv|^2 - 2\int |Dv|^2, \end{aligned} \tag{3.5}$$

where we have used that $|u_0| = |v| = 1$. Combining, we obtain

$$\begin{aligned} \int |D(u_0 - v)|^2 &= \int |Du_0|^2 - \int |Dv|^2 + \int |u_0 - v|^2|Dv|^2 \\ &\leq 2[E(u_0) - 4\pi] + c_0^2\|u_0 - v\|_{L^2}^2 \\ &\leq C[E(u_0) - 4\pi], \end{aligned} \tag{3.6}$$

for universal C. □

Remark 3.1. At the start of the argument we balanced our map to have 'centre of mass' at the origin. Without this step we could expect the harmonic map flow to generate a finite-time singularity. Indeed in [9, Theorem 5.5] we showed that for arbitrarily small $\epsilon_0 > 0$, there exists a smooth degree one map $u_0 : S^2 \to S^2$ with $E(u_0) < 4\pi + \epsilon_0$ such that the subsequent harmonic map flow must develop a singularity in finite time.

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References

[1] A. Bernand-Mantel, C. B. Muratov and T. M. Simon, *A quantitative description of skyrmions in ultrathin ferromagnetic films and rigidity of degree ±1 harmonic maps from $\mathbb{R}^2$ to S^2*. Arch. Rational Mech. Anal. **239** (2021) 219–299.
https://arxiv.org/abs/1912.09854

[2] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86** (1964) 109–160.

[3] J. Hirsch and K. Zemas, *A note on a rigidity estimate for degree* ± 1 *conformal maps on* S^2*.* Preprint (2021). https://arxiv.org/abs/2103.05390

[4] P. Li and S.-T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces.* Inventiones math. **69** (1982) 269–291.

[5] R. Schoen and K. Uhlenbeck, *Boundary regularity and the Dirichlet problem for harmonic maps.* J. Differential Geometry, **18** (1983) 253–268.

[6] L. Simon *Asymptotics for a Class of Non-Linear Evolution Equations, with Applications to Geometric Problems.* Annals of Math. **118** (1983) 525–571.

[7] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces.* Comment. Math. Helv. **60** (1985) 558–581.

[8] M. Struwe, 'Variational methods.' Fourth edition. Springer 2008.

[9] P. M. Topping, *The harmonic map heat flow from surfaces.* PhD thesis (1996). https://wrap.warwick.ac.uk/50788

[10] P. M. Topping, *Rigidity in the harmonic map heat flow.* J. Differential Geometry, **45** (1997) 593–610

[11] P. M. Topping, *Reverse bubbling and nonuniqueness in the harmonic map flow.* I.M.R.N. **10** (2002) 505–520.

[12] P. M. Topping, *Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow.* Annals of Mathematics, **159** (2004) 465–534.

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