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Reverse bubbling in geometric flows

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Dedicated to Rick Schoen in celebration of his 60th birthday.

Abstract

Reverse bubbling refers to singularities which develop in geometric flows as one takes a reverse limit $t \downarrow T$ rather than the traditional $t \uparrow T$. Flows with this type of singularity may have different or better properties, and can provide alternative ways of flowing past singularities. We survey the original reverse bubbling theory for the harmonic map flow, including recent developments, and otherwise focus mainly on the various analogous phenomena for Ricci flow.

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1 Introduction

In 1964 Eells and Sampson started the field of geometric flows by introducing the harmonic map flow [19]. This flow takes a map $u_0 : (\mathcal{M}, \gamma) \to (\mathcal{N}, g)$ between closed manifolds and tries to deform it to reduce the harmonic map energy

$$E(u) := \frac{1}{2} \int_{\mathcal{M}} |du|^2$$

as quickly as possible. More precisely, this flow is the L^2-gradient flow for E and deforms the map under a second order nonlinear parabolic equation.

The original insight of Eells and Sampson was that such a flow can be used to prove results in topology and differential geometry. The original application [19] was to prove that in the case that the target manifold $(\mathcal{N}, g)$ has weakly negative

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curvature, every map $u_0 : \mathcal{M} \to \mathcal{N}$ is homotopic to a *harmonic* map, that is, a critical point of the energy E; the flow performs the homotopy.

More generally, a geometric flow can take a wide variety of geometric objects, such as maps like above, or more generally connections, submanifolds, or Riemannian manifolds e.t.c. and deform them under an evolution equation in order to make them special, and hence understandable.

A particularly successful flow in this respect is Hamilton's Ricci flow which takes a Riemannian metric g on a manifold $\mathcal{M}$ and deforms it under the nonlinear PDE

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g), \tag{1}$$

or the equivalent *normalised* version

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric} + \frac{2}{n}\left(⨍ R\right) g, \tag{2}$$

which keeps the total volume fixed. Here Ric denotes the Ricci curvature and R the scalar curvature of g. (See [48] for more background on the fundamentals for this flow.) Since the original work of Hamilton [25] in 1982, a string of papers have demonstrated how compact Riemannian manifolds satisfying various curvature conditions will flow under normalised Ricci flow to round spheres, or one of their quotients. Highlights include the work of Böhm-Wilking [5] showing that compact manifolds with positive curvature operator flow to space forms, and the work of Brendle-Schoen [6] who showed the same for so-called quarter-pinched manifolds, thus solving the long-standing Differentiable Sphere Theorem Conjecture. (See [7] for a survey.)

So far, we have only discussed smooth flows which also ultimately converge smoothly. However, in many applications the flows develop singularities, and the subject becomes electrifying from an analysis/PDE point of view. These singularities often occur out of necessity. To illustrate this, recall that Eells and Wood [20] showed that there exist homotopy classes of maps which contain no harmonic maps; e.g. there is no harmonic map $T^2 \to S^2$ of degree 1. The harmonic map flow when confronted with such a map must find a way of changing homotopy class. As we shall see later, in the case of two-dimensional domains (the critical dimension in the PDE sense) the flow does this in a remarkable way. It makes the gradient of the map blow up, and concentrates energy. In Section 2 we will see that if one rescales the flow to get the energy under control, one can pass to a limit to obtain a so-called 'bubble'. Effectively, the flow generates bubbles to change the topology of the map, and can then continue, regaining its smoothness.

If one can understand these singularities well enough then topological and/or geometric applications can follow. One guiding problem which has prompted the development of the analysis of singularities in the harmonic map flow comes from a suggestion of Atiyah aimed at understanding the topology of whole spaces of maps in the spirit of the work of Segal [38]. It is well known that the homotopy classes of maps between 2-spheres are characterised by their degree, but one can then ask more refined questions about the topology of the space of all maps $\mathcal{M}_k$ of degree $k \in \{0, 1, \ldots\}$. Within $\mathcal{M}_k$ one has the space of *harmonic* maps of degree

k which in this case are precisely the rational maps Rat_k of degree k, i.e. viewing $S^2 = \mathbb{C} \cup \{\infty\}$, the maps are a quotient

$$\frac{\alpha_k z^k + \cdots + \alpha_0}{\beta_k z^k + \cdots + \beta_0}$$

of polynomials in z, or a quotient of polynomials in $\bar{z}$, with no common factor and $\alpha_k \beta_k \neq 0$. All these rational maps minimise energy within their homotopy class. A natural question then is to which extent the topology of $\mathcal{M}_k$ is the same as that of Rat_k. It turns out (Segal [38]) that the homotopy groups π_l of these spaces coincide for $l < k$. In fact, these groups need not agree in general; for example, in the case $k = 0$, the rational maps are precisely the constant maps, so $Rat_k \simeq S^2$, but $\mathcal{M}_0$ is more complicated. Indeed, $\mathcal{M}_0$ is not simply connected and one can find a smooth loop of smooth degree zero maps so that however we smoothly deform it, it will always contain a map of energy at least 8π, even though each individual map can clearly be deformed to a given constant map.

One can hope to prove results in the spirit of Segal's theorem by using the harmonic map flow. This flow tries to deform each element of $\mathcal{M}_k$ to an element of Rat_k, and given that there is no topological obstruction to this, naïvely one might be disappointed that it does not manage this without forming singularities. However, the flow is in fact trying to do much more, namely make a deformation retract of the entire space $\mathcal{M}_k$ onto Rat_k. The flow does manage to make such a deformation retract of the space of degree k maps of energy less than 8π onto Rat_k [44]. However, given that there are obstructions to this retraction in general, as described above, it must generate bubbles to change the topology. The dream then is by making a good enough analysis of the bubbling singularities, and finding the right way of flowing through singularities, one might be able to measure the discrepancy in the topology of these spaces $\mathcal{M}_k$ and Rat_k by accounting for the bubbles that are generated.

So, how can the harmonic map flow from two-dimensional domains flow beyond a singularity at a time T, say? Traditionally, this has been done by taking a weak limit of the flow maps $u(t)$ as t increases to the singular time, ignoring any energy that concentrated to form bubbles, and then restarting the flow. (See Section 2 for more details of this procedure.) In fact, as we shall see in Section 2, there can be alternative natural continuations of the flow with better topological properties. Roughly speaking, it can be possible to continue the flow by preserving the concentrated energy and allowing it to *deconcentrate* as we move beyond the singular time. We see here the central themes of this survey: First how a flow can be solved right through a singular time, and the resulting issues of uniqueness, but also, how one can have blow-up in reverse. Concerning this latter point, whereas traditional blow-up singularities occur as t increases to some singular time, we will focus on singularities which develop as t *decreases* to the singular time. We will be able to think of the flow at the singular time as initial data for the flow which is achieved in some sort of weak or local sense in this reverse limit as t decreases to the singular time, with some sort of blow-up occurring in this limit which we generally call 'reverse singularities' or 'reverse bubbling' after the terminology introduced in [46]. (We will see plenty of examples in what follows.)

It may be worth stressing that we are dealing with (essentially) parabolic equations in this survey, and in particular we are considering reverse bubbling when there is an *arrow of time*. Of course, deconcentrating singularities exist in time-reversible flows whenever concentrating singularities exist.

The other main flow for which we discuss singularities in reverse, uniqueness (and well-posedness) issues and issues concerning flowing through singularities, is the Ricci flow. As we shall see, there is a good classical well-posedness theory for Ricci flows starting at complete, bounded-curvature Riemannian manifolds. In Section 3.1 we will ask how to do Ricci flow starting at a noncompact manifold which may not even be complete or have bounded curvature. Considering this generality causes severe nonuniqueness issues since there is too much flexibility for flows to behave at will 'out at infinity'. We address this problem by imposing the constraint on flows on surfaces that they be complete for all later times. This *instantaneous completeness* can be viewed as a substitute for a boundary condition as we explain later. Whenever we start with an incomplete metric, the subsequent Ricci flow must develop blow-up as $t \downarrow 0$ and we have a reverse singularity at spatial infinity; the initial data is still achieved smoothly locally in this limit $t \downarrow 0$. At this stage we can describe the complete picture concerning existence of instantaneously complete Ricci flows, Theorem 3.3, and partial results towards the conjectured uniqueness within this class.

In Section 3.2 we survey a quite different way that Ricci flows on surfaces can develop reverse singularities: we allow the underlying manifold to jump to one of different topology as time lifts off from zero. To do this, we make an adjustment of Hamilton's way of posing Ricci flow. Instead of looking for a flow $g(t)$ with $g(0)$ prescribed, we look for a flow $g(t)$ with, essentially, the *Cheeger-Gromov limit* of $g(t)$ as $t \downarrow 0$ prescribed. This allows topology to fly in from spatial infinity as time lifts off from zero.

We will see concrete examples of this taking place. If we take a manifold with some hyperbolic cusps (complete and with bounded curvature) there is a standard way of flowing, according to Shi [39] (see Section 3) in which the manifold evolves with the cusps intact. We will see that an alternative is to add in a point at infinity, at the end of the cusp, and let the cusp contract as the flow starts. In other words, we will see a reverse singularity as $t \downarrow 0$ as one point in the surface is sent off to infinity to form a cusp in the limit. One can also consider these flows as giving examples of nonuniqueness, because we have an alternative to Shi's flow. On the other hand, we will also see a criterion which will rule out this nonunique behaviour: it cannot happen if the injectivity radius of the initial (bounded curvature) metric is uniformly positive.

Ricci flows on manifolds of dimension three and higher are harder to analyse. In general they must develop finite-time singularities, even for the normalised version (2). Indeed, as for the harmonic map flow, there are topological reasons why a general 3-manifold, say, cannot flow smoothly for all time under (2) and converge at infinite time. This time it is curvature that blows up at the singular time (analogous to the energy density blow-up for the harmonic map flow) and one can hope to make an analysis by rescaling and taking a limit.

The famous programme of Hamilton directed at proving Thurston's geometrization conjecture sought to analyse the singularities well enough to be able to flow through them. It was Perelman's triumph [31, 32] to extend this analysis far enough to be able to fully carry out Hamilton's surgery process. In fact, the continuation of the flow offered by surgery is somewhat ad hoc, and it is a tantalising possibility that one might be able to modify the argument to make the Ricci flow canonically flow through a singular time T, with as much regularity as is possible. One would have a reverse singularity as time moved on from the singular time, with curvature blow-up as $t \downarrow T$ in addition to the familiar curvature blow up as $t \uparrow T$.

With current knowledge, both existence and also uniqueness issues are unclear as the flow moves on from the incomplete, degenerate metric one would have at the singular time. Note however that Angenent-Caputo-Knopf [1] have recently constructed examples of Ricci flows through some symmetric neck pinch singularities, and there is a theory under development in the special case of Kähler Ricci flows which seeks to build on a concrete example of a flow through a singularity constructed by Feldman-Ilmanen-Knopf [21]. See [42], [41] and the references therein.

The successful development of programmes like this will lead to substantial new applications of analysis and PDE to geometry. However, we also see this subject as being a vehicle for the application of geometry to analysis and PDE. Taking a geometric viewpoint has led to the discovery of new PDE phenomena and has both raised and solved issues of uniqueness in natural PDE which echo constructions made elsewhere in PDE such as those of Scheffer [37], Shnirelman [40] and De Lellis-Székelyhidi [14] for the Euler equation in the theory of hydrodynamics.

2 The harmonic map flow

As we mentioned in the introduction, the original situation in which the terminology 'reverse bubbling' arose was the harmonic map flow, and now we turn to that subject to see some more of the details, including advances that have been made in the last few years.

In order to express the analytic aspects of the harmonic map flow it is convenient to embed the target manifold $(\mathcal{N}, g)$ isometrically in some Euclidean space $\mathbb{R}^N$. The harmonic map flow, being gradient flow for the energy

$$E(u) := \frac{1}{2}\int_{\mathcal{M}} |\nabla u|^2$$

can then be seen to be governed by the equation

$$\frac{\partial u}{\partial t} = \tau(u(t)) := (\Delta u)^T \tag{3}$$

where the superscript T means the projection onto $T_{u(x)}\mathcal{N} \subset \mathbb{R}^N$ and τ is referred to as the tension field. One can compute that

$$\tau(u) = \Delta u + \gamma^{\alpha\beta} A(u)\left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right)$$

where $A(u)$ is the second fundamental form of the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^N$. In particular, we see that the nonlinearity of the PDE is quadratic in the gradient, and it makes sense to consider weak solutions $u \in W^{1,2}(\mathcal{M} \times [0,\infty), \mathcal{N})$ (that is, $u \in W^{1,2}(\mathcal{M} \times [0,\infty), \mathbb{R}^N)$ with $u(x) \in \mathcal{N} \hookrightarrow \mathbb{R}^N$ for almost all $x \in \mathcal{M}$). When we do that, we denote by $u(t)$ the trace of the map u on the submanifold $\mathcal{M} \times \{t\}$ of $\mathcal{M} \times [0,\infty)$.

At times t at which the flow is smooth, the energy of the flow decays according to

$$\frac{d}{dt}E(u(t)) = -\int_{\mathcal{M}} |\tau(u(t))|^2 \leq 0. \tag{4}$$

From now on in this section we will only consider the case that the domain is two-dimensional. The foundational result asserting the existence of a global solution is the following.

Theorem 2.1. *(Struwe [43].) Suppose* $\dim \mathcal{M} = 2$. *Given* $u_0 \in W^{1,2}(\mathcal{M}, \mathcal{N})$, *there exists a weak solution* $u \in W^{1,2}_{loc}(\mathcal{M} \times [0,\infty), \mathcal{N})$ *to* (3) *which is smooth in* $\mathcal{M} \times [0,\infty)$ *except possibly at finitely many singular points in* $\mathcal{M} \times (0,\infty)$, *and has the following properties:*

1. $u(0) = u_0$;
2. $E(u(t))$ *is a (weakly) decreasing function of* t *on* $[0,\infty)$;
3. *If the flow is smooth for* $t \in [0,T)$, *then it is the unique smooth solution over this time interval with the given initial data;*
4. *If* $(x,T) \in \mathcal{M} \times (0,\infty)$ *is a singular point, then energy concentrates in the sense that*
$$\lim_{\nu \downarrow 0} \limsup_{t \uparrow T} E(u(t), B_\nu(x)) \neq 0.$$

We call this flow the 'Struwe solution' or 'Struwe flow.'

To clarify, we are using here the notation $E(u,\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2$.

Theorem 2.1 allows the possibility of some finite-time singularities, but it was only later, with the work of Chang-Ding-Ye [8] that it was actually proved that such singularities could occur. The following basic description of what the singularities look like had already been given by Struwe: Energy concentrates near the singular points, and we can rescale the map as we approach the singular time in order to extract a bubble. The global solution of Theorem 2.1 is constructed by throwing away the concentrated energy at a singular time T by defining $u(T)$ to be the weak limit of $u(t)$ as $t \uparrow T$ and then continuing the flow with $u(T)$ as initial map.

Theorem 2.2. *(Slight variant of Struwe [43].) Suppose* $u \in W^{1,2}_{loc}(\mathcal{M} \times [0,\infty), \mathcal{N})$ *is a Struwe flow from Theorem 2.1, and* $T \in (0,\infty)$ *is a singular time. Then there exist times* $t_n \uparrow T$ *with the property that*

$$\|\tau(u(t_n))\|^2_{L^2(\mathcal{M})}(T - t_n) \to 0 \tag{5}$$

as $n \to \infty$*. Moreover, for every singular point* $(x_0, T) \in \mathcal{M} \times (0, \infty)$ *at time* T*, when we view* u *as a map in local isothermal coordinates on the domain with* x_0 *corresponding to the origin, there exist sequences* $a_n \to 0 \in \mathbb{R}^2$ *and* $\lambda_n \downarrow 0$ *with* $\lambda_n(T - t_n)^{-\frac{1}{2}} \to 0$ *as* $n \to \infty$*, and a nonconstant harmonic map* $\omega : S^2 \to \mathcal{N}$ *(which we view as a map* $\mathbb{R}^2 \cup \{\infty\} \to \mathcal{N}$*) such that*

$$u(a_n + \lambda_n y, t_n) \to \omega(y),$$

as functions of y*, in* $W^{2,2}_{loc}(\mathbb{R}^2, \mathcal{N})$ *as* $n \to \infty$*.*

The map ω arising as a blow-up limit near the singularity is the object we call a bubble. In fact, in principle, there may be many bubbles developing near each singular point (cf. [45]) i.e. we might be able to take different sequences a_n and λ_n which lead to convergence to a different bubble. It follows from the theorem that

$$\lim_{t \uparrow T} E(u(t)) - E(u(T)) \geq E(\omega). \tag{6}$$

However, if we capture *all* the bubbles at each singular point and replace the right-hand side with the sum of all their energies, then it is a result of Ding-Tian [16] that we would have equality in (6). For more details, and the most refined result (based on the work of Struwe [43], Ding-Tian [16], Qing-Tian [33] and Lin-Wang [30]) see [47].

Note that the behaviour of the flow at the singularity can still be quite bad. For example, the flow map $u(T)$ at time T might be discontinuous, despite being in $W^{1,2}(\mathcal{M}, \mathcal{N})$, and it is sometimes possible to change the sequence of times $t_n \uparrow T$ and end up with a different set of bubbles, even with different images. See [47] for details of these constructions.

At this point, we have not considered the question of uniqueness of the flow. In his original work, Struwe [43] demonstrated that within a class of relatively smooth solutions $u : \mathcal{M} \times [0, T] \to \mathcal{N}$ (and in particular too smooth to allow any singularities within the time interval $[0, T]$) the solution uniquely depends on the initial map $u(0)$.

This work was then improved, in particular by Freire who proved the following uniqueness theorem.

Theorem 2.3. *(Freire [22].) Suppose* $\dim \mathcal{M} = 2$*. If* $u \in W^{1,2}_{loc}(\mathcal{M} \times [0, T], \mathcal{N})$ *is a weak solution of the harmonic map flow such that* $u(0) = u_0 \in W^{1,2}(\mathcal{M}, \mathcal{N})$*, with the property that*

$$t \mapsto E(u(t)) \text{ is a weakly decreasing function of } t, \tag{7}$$

then u *coincides with the Struwe solution from Theorem 2.1 (with initial data* u_0*) over the time interval* $[0, T]$*.*

A natural question at this point was whether the hypothesis (7) could be dropped without compromising uniqueness. In contrast, in 1996 we made the following conjecture.

Conjecture 2.4. (From [44].) Define ε^* to be the least energy of a bubble:

$$\varepsilon^* := \inf\{E(v) \mid v : S^2 \to \mathcal{N} \text{ is nonconstant and harmonic } \} > 0.$$

Then it should be possible to weaken the hypothesis (7) that $E(u(t))$ is a weakly decreasing function of t to the hypothesis that

$$\limsup_{t \downarrow t_0} E(u(t)) < E(u(t_0)) + \varepsilon^*, \tag{8}$$

for all $t_0 \in [0, T)$, i.e. that the energy may increase provided that any upward jump must be less than the least energy of a bubble.

We remark that the least energy of a bubble is necessarily strictly positive (e.g. [44]) and in the popular case that the target is S^2, this least energy would be 4π, the energy of the identity map.

The motivation for this conjecture was that a type of 'reverse bubbling' might occur: In the bubbling of Theorem 2.2, energy gradually concentrates as we approach a singular time, generating a bubble, then we throw the bubble away and continue. In the following constructions, from [46] and [4], one sees the opposite behaviour. During a perfectly smooth flow, one can sometimes add in an infinitely concentrated bubble and continue the flow by allowing the energy to deconcentrate.

Theorem 2.5. *(Variant of [46] and [4].) Let* $id : S^2 \to S^2$ *be the identity map (which is harmonic). Then there exists a weak solution* $u \in W^{1,2}_{loc}(S^2 \times [0, \infty), S^2)$ *to the harmonic map heat equation* (3) *such that* $u(t) = id$ *for all* $t \in [0, 1]$*, but* $u(t) \neq id$ *for* $t > 1$.

The solution may be chosen so that $u(t) \in W^{1,2}(S^2, S^2)$ *for all* $t \geq 0$*, and even to be smooth in* $(S^2 \times [0, \infty))\backslash(p, 1)$*, where* p *is some point in* S^2*. Moreover, we have that*

$$\lim_{t \downarrow 1} E(u(t)) = E(u_0) + 4\pi = 8\pi.$$

Another phenomenon described in [46] is that one can sometimes continue a flow which has developed a bubble singularity at time T, not by throwing away the bubble and restarting the flow with the weak limit $u(T)$, but by 'twisting' the bubble, reattaching it, and continuing the flow allowing the bubble's infinitely concentrated energy to deconcentrate and redistribute itself. Such flows only just fail to satisfy the hypotheses of Freire's result Theorem 2.3 by having a single time T where the energy jumps down and then up again at the same instant.

Theorem 2.5 shows that Conjecture 2.4 is the best one can hope for. Recently, the conjecture has been essentially solved by Melanie Rupflin subject to the assumption that the energy as a function of time has bounded total variation.

Theorem 2.6. *(Rupflin [35].) Suppose* $\dim \mathcal{M} = 2$*. Let* $u \in W^{1,2}(\mathcal{M} \times [0, T], \mathcal{N})$ *be any weak solution of* (3) *with* $u(0) = u_0 \in W^{1,2}(\mathcal{M}, \mathcal{N})$ *such that the function* $t \mapsto E(u(t))$ *has finite total variation and*

$$\limsup_{t \downarrow t_0} E(u(t)) < E(u(t_0)) + \varepsilon^*$$

for all $t_0 \in [0, T)$*. Then* u *coincides with the Struwe solution with initial data* u_0.

A subresult on the way to proving Theorem 2.6 is to prove that one has uniqueness within the smaller class in which one replaces ε^* by some tiny ε. In that case Rupflin can control the energy concentration sufficiently well to be able to deduce enough smoothness to be sure to be within Struwe's original uniqueness class.

Given a flow u as in Theorem 2.6, the bounded total variation hypothesis ensures that there can only be finitely many times at which the energy jumps by more than this tiny amount ε. Rupflin was then able to make an analysis assuming *smoothness* away from these times. It turns out that if there is any jump up in the energy at time T, it becomes possible to extract a bubble by taking a sequence of times $t_i \downarrow T$, and this must account for a jump up in the energy of at least ε^*. See [35] for further details.

Open problems: Rupflin's result gives a satisfying resolution to one aspect of the theory of reverse bubbling, but there are many intriguing open questions and issues within the theory, some of which we now highlight.

1. The construction in [46] shows that one can have *examples* of harmonic map flows through singularities which retain the energy of the bubble(s). Is there a general existence theorem of this type? Can one characterise when they can occur either in terms of the singularity that formed, or in terms of the initial data?

2. One could construct an example of a weak solution of the harmonic map flow for which there are multiple continuations beyond the singular time, all of which exhibit reverse bubbling, and all of which have
$$\lim_{t\uparrow T} E(u(t),\Omega) = \lim_{t\downarrow T} E(u(t),\Omega).$$
for all subsets Ω of the domain. Is there a natural selection criterion which can pick out one of these as the 'right' continuation?

3. Consider again the weak solution of the harmonic map flow we constructed in [46] where a bubble forms, is instantaneously twisted 180°, and then dissipates its energy via a reverse bubbling singularity. We conjecture that this flow (and possibly similar flows of this type) can be approximated by smooth solutions. In other words, if the original flow exists over a time interval $[0,T]$ with the (reverse) bubbling singularity at time $t_0 \in (0,T)$, it should be possible to find a sequence of smooth solutions over the whole time interval $[0,T]$ which converge to the singular flow (e.g. smoothly locally away from the singular point in space-time). Compelling recent numerical computations of Jan Bouwe van den Berg and J.F. Williams support this possibility [3]. Finding such approximations in greater generality may lead to an answer to the previous question. This whole issue is related to the question of stability of finite-time singularities: Under which perturbations of the initial map will such singularities remain?

3 Ricci flow

In the introduction we saw the definition of Ricci flow (1). We now return to that flow, and will focus mainly on the special case that the underlying manifold is two-dimensional. In that case the Ricci curvature of a metric g can be written $\mathrm{Ric}[g] = K[g]g$, where $K[g]$ is the Gauss curvature, and being a multiple of g, we see that any Ricci flow must preserve the conformal class of the initial metric.

If we choose a local complex coordinate $z = x + iy$ and write the metric locally as $g = e^{2u}|dz|^2$ (where $|dz|^2 := dx^2 + dy^2$) then we can write the Ricci flow locally as

$$\frac{\partial u}{\partial t} = e^{-2u}\Delta u \equiv -K[u], \tag{9}$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian in terms of the local coordinates, and where we abuse notation by abbreviating $K[e^{2u}|dz|^2]$ by $K[u]$. Equation (9) is now a strictly parabolic scalar equation which is often referred to as the logarithmic fast diffusion equation. We shall see that the nonlinearity of this equation, which stems from its geometric context, gives its solutions some remarkable properties.

In this section we will pick up the general ideas of the previous section and investigate more general notions of Ricci flow solution, corresponding issues of uniqueness and possible ways of flowing beyond singularities. But first we need to review the traditional well-posedness theory.

Given any metric g_0 on a *closed* manifold $\mathcal{M}$ of any dimension, there exists a Ricci flow $g(t)$ for $t \in [0, \varepsilon]$ (some $\varepsilon > 0$) with $g(0) = g_0$, as was shown by Hamilton [25] (simplified by DeTurck [15]). This solution is unique in that any other such flow on some time interval $[0, \varepsilon']$ with the same initial metric will agree with $g(t)$ while they both exist. Meanwhile, if $\mathcal{M}$ is not compact, but $(\mathcal{M}, g_0)$ is nevertheless complete, and of bounded curvature, then Shi [39] showed that a complete, bounded-curvature Ricci flow $g(t)$ exists on $\mathcal{M}$ over some nontrivial time interval $[0, T]$ with $g(0) = g_0$. This solution is then unique within that class of complete, bounded-curvature Ricci flows, as was proved by Chen-Zhu [10].

This picture looks more or less complete at first glance. However, there are numerous directions one can hope to develop the general well-posedness theory of Ricci flow. One direction we shall not pursue here is the following question:

What is the right notion of Ricci flow on a manifold with boundary?

Presumably there is a geometrically natural notion of boundary condition which gives well-posedness, possibly involving a constraint on the mean curvature of the boundary.

The directions that we will pursue involve starting flows with incomplete, unbounded-curvature metrics, and considering flows in which the topology of the underlying manifold can jump. While one would like results in general dimensions for applications (e.g. for flowing through singularities) we will mainly be considering Ricci flows on surfaces. In this case, certain aspects simplify (for example, the Chen-Zhu uniqueness of Ricci flows becomes simple to prove) and in the compact case there is a refined existence result, and an understanding that the Ricci flow has a uniformising effect:

Theorem 3.1. (Hamilton [26], Chow [11].) *Let* $(\mathcal{M}^2, g_0)$ *be a closed Riemannian surface. Then there exists a unique Ricci flow* $g(t)$ *for* $t \in [0,T)$ *with* $g(0) = g_0$*, where the maximal time* T *is given by*

$$T = \begin{cases} \frac{1}{4\pi\chi(\mathcal{M})} \operatorname{Vol}_{g_0} \mathcal{M} & \text{if } \chi(\mathcal{M}) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, the rescaled solution

$$\begin{cases} \frac{1}{2(T-t)}\, g(t) & \text{if } \chi(\mathcal{M}) > 0 \\ g(t) & \text{if } \chi(\mathcal{M}) = 0 \\ \frac{1}{2t}\, g(t) & \text{if } \chi(\mathcal{M}) < 0 \end{cases}$$

converges smoothly to a conformal metric of constant Gaussian curvature $1, 0, -1$ *respectively as* $t \to T$.

Here $\chi(\mathcal{M})$ denotes the Euler characteristic of $\mathcal{M}$.

While this form of the theorem suits us best here, one could also phrase it in terms of the normalised flow (2). We would then have existence of the flow for all time, and convergence to a constant curvature metric as $t \to \infty$.

3.1 Instantaneously complete Ricci flows

We have mentioned that the traditional well-posedness theory allows us to flow a Riemannian manifold under Ricci flow starting with a *complete* metric, of *bounded curvature*, giving a unique smooth complete bounded-curvature flow. We now consider the problem of Ricci flow starting with a manifold which may be *incomplete* and even of unbounded curvature. At first sight this looks hopeless, and it is not hard to see that there must be a high degree of nonuniqueness in general:

Example 3.2. Consider Ricci flow starting at a flat unit two-dimensional disc. This amounts to solving the PDE (9) for a time-dependent function on the disc with zero initial data. Standard theory of strictly parabolic equations tells us that we have infinitely many solutions, and even amongst solutions which are, say, smooth up to the boundary, we are free to smoothly prescribe u on this boundary at each time (which would then give uniqueness).

However, there is a more geometric way of posing the problem than prescribing the conformal factor on the boundary, and one that applies for a completely general surface, even where there is no such notion of 'boundary'. The trick is to give up the notion of boundary condition and replace it with the condition of *instantaneous completeness*:

Theorem 3.3. *([49] and, joint with Giesen, [24].) Let* $(\mathcal{M}^2, g_0)$ *be a smooth Riemannian surface which need not be complete, and could have unbounded curvature. Depending on the conformal type, we define* $T \in (0, \infty]$ *by*

$$T := \begin{cases} \frac{1}{8\pi} \operatorname{Vol}_{g_0} \mathcal{M} & \text{if } (\mathcal{M}, g_0) \cong \mathcal{S}^2, \\ \frac{1}{4\pi} \operatorname{Vol}_{g_0} \mathcal{M} & \text{if } (\mathcal{M}, g_0) \cong \mathbb{C} \text{ or } (\mathcal{M}, g_0) \cong \mathbb{R}P^2, \\ \infty & \text{otherwise.} \end{cases}$$

Then there exists a unique smooth Ricci flow $\big(g(t)\big)_{t\in[0,T)}$ *such that*

1. $g(0) = g_0$;
2. $g(t)$ *is instantaneously complete - i.e. complete for all* $t \in (0,T)$;
3. $g(t)$ *is maximally stretched - i.e. if* $\tilde{g}(t)$ *is any smooth Ricci flow on* $\mathcal{M}$ *for* $t \in [0,\varepsilon)$ *with* $\tilde{g}(0) \leq g(0)$, *then* $\tilde{g}(t) \leq g(t)$ *for all* $t \in [0, \min\{\varepsilon, T\})$.

If $T < \infty$, *then we have*

$$\mathrm{Vol}_{g(t)}\,\mathcal{M} = \left\{ \begin{array}{ll} 8\pi(T-t) & \textit{if } (\mathcal{M}, g_0) \cong \mathcal{S}^2, \\ 4\pi(T-t) & \textit{otherwise}, \end{array} \right\} \quad \longrightarrow \quad 0 \quad \textit{as } t \nearrow T,$$

and in particular, T *is the maximal existence time. Alternatively, if* $\mathcal{M}$ *supports a complete hyperbolic metric* H *conformally equivalent to* g_0 *(in which case* $T = \infty$*) then we have convergence of the rescaled solution*

$$\frac{1}{2t} g(t) \longrightarrow H \quad \textit{smoothly locally as } t \to \infty.$$

If additionally there exists a constant $M > 0$ *such that* $g_0 \leq MH$ *then the convergence is global: For any* $k \in \mathbb{N}_0$ *and* $\eta \in (0,1)$ *there exists a constant* $C = C(k, \eta, M) > 0$ *such that for all* $t \geq 1$

$$\left\| \frac{1}{2t} g(t) - H \right\|_{C^k(\mathcal{M},H)} \leq \frac{C}{t^{1-\eta}} \quad \overset{t\to\infty}{\longrightarrow} \quad 0.$$

In fact, in this case, for all $t > 0$ *we have* $\left\|\frac{1}{2t} g(t) - H\right\|_{C^0(\mathcal{M},H)} \leq \frac{C}{t}$ *and even*

$$0 \leq \frac{1}{2t} g(t) - H \leq \frac{M}{2t} H.$$

In particular, we stress that there is long-time existence in virtually all cases. The theorem applies for *any* metric on *any* underlying surface.

This result was originally proved in [49] under the assumption that the Gauss curvature of g_0 was bounded above by some $K_0 \in \mathbb{R}$, and for a time of existence of at least $\frac{1}{2K_0}$ if $K_0 > 0$, and with existence for all time if $K_0 \leq 0$, and without the long-time asymptotic description. There are also a number of other Ricci flow results generalised by Theorem 3.3: see [24] for an overview.

In order to see what the solutions of Theorem 3.3 look like as they make an incomplete manifold complete instantaneously, let us return to Example 3.2 and see what the new flow does to the flat unit disc. After a short time, most of the flow (being smooth down to $t = 0$) looks much like the flat disc at which it started. However, the metric now blows up at the boundary when compared to the initial metric. Indeed, at a small time $t > 0$, the metric near the boundary will look like the complete metric on the disc which has constant curvature $-\frac{1}{2t}$.

Example 3.4. A further illustrative example would be when the initial manifold $(\mathcal{M}, g_0)$ is the punctured Euclidean 2-plane. In this case, our solution immediately develops a hyperbolic cusp (of curvature $-\frac{1}{2t}$) near the puncture in order to make the metric complete.

In fact, this particular example is rather special since it can be written as a *Ricci soliton* (see e.g. [48] for a definition) but is representative of what Theorem 3.3 does to a puncture on a more general manifold. Of course, Theorem 3.3 applies in far greater generality.

This result essentially finishes the issue of existence of instantaneously complete solutions, at least on surfaces. It also asserts uniqueness within the class of maximally stretched Ricci flows. However, we have said nothing yet about uniqueness without this maximal condition.

Conjecture 3.5. ([49].) The solution of Theorem 3.3 is *unique* within the class of instantaneously complete Ricci flows $g(t)$ with $g(0) = g_0$.

In other words, the completeness hypothesis kills any flexibility at infinity, and makes the problem rigid.

At the time of writing, this conjecture has been partially resolved, but some special cases remain open. It is easiest to classify the known results depending on the conformal type of the universal cover of $(\mathcal{M}, g_0)$.

- If the conformal type is S^2, then $\mathcal{M}$ must be compact, so the uniqueness is already included in the traditional theory.
- If the conformal type is $\mathbb{C}$, then with Giesen we completely solve the conjecture in [24].
- If the conformal type is the disc D, then partial results can be found in [23] and [24].

We have already seen how if one works with respect to local isothermal coordinates, then the equation for the conformal factor of a Ricci flow is (9). In particular, if $\mathcal{M}$ is conformally equivalent to $\mathbb{C}$, then (9) will hold on the whole plane. There is a large literature on this so-called logarithmic fast-diffusion equation which contains some special cases of the results mentioned above. In particular, the reader is directed to [12], [34] and [13] and the references therein.

3.2 Topology-jumping Ricci flows

We have discussed how it is desirable to consider Ricci flows for which the underlying manifold can jump topology, in order to flow past singularities in three dimensions, for example. This must involve considering flows with unbounded curvature. We now describe some progress one can make along these lines by generalising Hamilton's way of posing the Ricci flow, extending the notion of initial metric as follows.

Definition 3.6. We say that a complete Ricci flow $(\mathcal{M}, g(t))$ for $t \in (0, T]$ has a complete Riemannian manifold $(\mathcal{N}, g_0)$ as initial condition if there exists a smooth map $\varphi : \mathcal{N} \to \mathcal{M}$, diffeomorphic onto its image, such that

$$\varphi^*(g(t)) \to g_0$$

smoothly locally on $\mathcal{N}$ as $t \downarrow 0$.

Note that even if $(\mathcal{N}, g_0)$ has bounded curvature, the subsequent flow $g(t)$ might have unbounded curvature, and this allows $\mathcal{M}$ and $\mathcal{N}$ to be different since parts of $\mathcal{M}$ may be shot out to infinity as $t \downarrow 0$ resulting in a change of topology in the limit. The next theorem shows this really can happen, even for Ricci flow on surfaces. The Ricci flow can take a hyperbolic cusp, say, and add in the point at infinity, subsequently contracting the cusp.

Theorem 3.7. *([50].) Suppose $\mathcal{M}$ is a compact Riemann surface and $\{p_1, \ldots, p_n\} \subset \mathcal{M}$ is a finite set of distinct points. If g_0 is a complete, bounded-curvature, smooth, conformal metric on $\mathcal{N} := \mathcal{M}\backslash\{p_1, \ldots, p_n\}$ with strictly negative curvature in a neighbourhood of each point p_i, then there exists a Ricci flow $g(t)$ on $\mathcal{M}$ for $t \in (0, T]$ (for some $T > 0$) having $(\mathcal{N}, g_0)$ as initial condition in the sense of Definition 3.6. We can take the map φ there to be the natural inclusion of $\mathcal{N}$ in $\mathcal{M}$.*

Moreover, the cusps contract logarithmically in the sense that for some $C < \infty$ and all $t \in (0, T]$ sufficiently small, we have

$$\frac{1}{C}(-\ln t) \leq diam(\mathcal{M}, g(t)) \leq C(-\ln t). \tag{10}$$

Furthermore, the curvature of $g(t)$ is bounded below uniformly as $t \downarrow 0$.

These flows thus have a reverse singularity at $t = 0$. They also give another illustration of how singular behaviour can lead to nonuniqueness: The manifold $(\mathcal{N}, g_0)$ could be flowed using Shi's theorem to give a complete bounded-curvature flow on $\mathcal{N}$, or the cusps could be contracted as above.

There is something very particular about these examples which permits the nonuniqueness, namely the fact that the injectivity radius decays to zero as we move out to infinity along a cusp. In fact, if this cannot happen, then we have uniqueness even for higher dimensional Ricci flow, as we now describe.

For n-dimensional Ricci flows, there is a remarkable 'pseudolocality' theorem of Perelman [31] which says roughly that if we have a complete, bounded-curvature Ricci flow $g(t)$ and, at time $t = 0$ there is some ball $B_{g(0)}(x, r)$ which has bounded curvature ($\lesssim r^{-2}$) and volume bounded below ($\gtrsim r^n$), then the Ricci flow cannot make the curvature at x too large (compared to r^{-2}) for a time of order r^2.

By carefully applying this result along the cusps in the manifold $(\mathcal{N}, g_0)$ from Theorem 3.7, one can see that the cusp cannot move for a short time, although as we venture further along the cusp, we are obliged to consider smaller and smaller balls in order to achieve the lower volume bound, and we obtain control for a shorter and shorter time. Nevertheless, an argument along these lines is enough to obtain the lower bound on the diameter in (10).

If instead we start a Ricci flow from a manifold with a positive lower bound on its injectivity radius, then we can apply the pseudolocality theorem throughout the manifold on balls of fixed size, to deduce that the curvature remains bounded as time lifts off from $t = 0$. This prevents any topology from instantaneously flying in from infinity, and yields the following uniqueness result (see [50] for the details of a similar result, stronger but in two dimensions only).

Theorem 3.8. *(Variant of [50].) Suppose that* $(\mathcal{N}, g_0)$ *is a complete* n*-dimensional Riemannian manifold with bounded curvature such that the injectivity radius is bounded below by a positive number:*

$$\mathrm{inj}_{g_0}(x) \geq \varepsilon > 0$$

for all $x \in \mathcal{N}$*. Then there is a unique Ricci flow continuation in the sense that if for* $i = 1, 2$ *we have complete Ricci flows* $(\mathcal{M}_i, g_i(t))$ *for* $t \in (0, T_i]$ *(some* $T_i > 0$*) both with* $(\mathcal{N}, g_0)$ *as initial condition, and with bounded curvature for* t *in compact subsets of* $(0, T_i]$*, then these two Ricci flows must agree while they both exist. More precisely, there exists a diffeomorphism* $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ *such that* $\psi^*(g_2(t)) = g_1(t)$ *for all* $t \in (0, \min\{T_1, T_2\}]$*.*

In the case of surfaces, a similar result can be proved even without the assumption that the flows $g_i(t)$ have bounded curvature away from $t = 0$ (see [50]).

Returning to the nonuniqueness examples of Theorem 3.7, we remark that it is *a priori* unclear whether there are many different ways the Ricci flow can contract the cusp. In fact, there are not:

Theorem 3.9. *([50].) In the situation of Theorem 3.7, if* $\tilde{g}(t)$ *is a smooth Ricci flow on* $\mathcal{M}$ *for some time interval* $t \in (0, \delta)$ $(\delta \in (0, T])$ *such that* $\tilde{g}(t) \to g_0$ *smoothly locally on* $\mathcal{N}$ *as* $t \downarrow 0$ *and the Gauss curvature of* $\tilde{g}(t)$ *is uniformly bounded below, then* $\tilde{g}(t)$ *agrees with the flow* $g(t)$ *constructed in Theorem 3.7 for* $t \in (0, \delta)$*.*

4 Addendum - mean curvature flow

One other context in which the phenomena of the present survey can be seen is in mean curvature flow. That flow deforms a submanifold, most often a hypersurface in Euclidean space, by moving it with a velocity equal to the mean curvature vector (see [18] for an introduction and description of the notions we discuss below).

As sketched by Angenent-Ilmanen-Chopp in [2] and [29] there are believed to exist examples of solutions in which the curvature blows up, that can then continue with a reverse singularity, and in a possibly nonunique way. Indeed, there should exist a so-called self-shrinker in $\mathbb{R}^3$ (i.e. a hypersurface which moves under mean curvature flow simply by homothetic contraction towards the origin) which is asymptotic to a double cone. The flow shrinks homothetically, and in finite time converges to this asymptotic double cone. From there, the two parts of the cone can separate, giving a self-expanding two-sheeted continuation [17]. Alternatively, in certain situations, the double cone can desingularise itself by becoming a self-expanding one-sheeted evolving surface with the topology of a hyperboloid. In fact, generally we have nonuniqueness even amongst one-sheeted self-expanding flows.

We see all the themes of this article here: flows through singularities, reverse singularities, and nonuniqueness issues flowing out of the singularity. See [29] for more of this story. We also observe that this theory has implications for the study of the dynamics of charged fluid droplets and their tendency to coalesce or repel each other, thus explaining phenomena observed in storm cloud formation, ink-jet printing and numerous other applications. See parts of [28] for further details.

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