Uniqueness and nonuniqueness for Ricci flow on surfaces: Reverse cusp singularities

Peter M. Topping

October 14, 2010

Abstract

We extend the notion of what it means for a complete Ricci flow to have a given initial metric, and consider the resulting well-posedness issues that arise in the 2D case. On one hand we construct examples of nonuniqueness by showing that surfaces with cusps can evolve either by keeping the cusps or by contracting them. On the other hand, by adding a noncollapsedness assumption for the initial metric, we establish a uniqueness result.

1 Introduction

A complete Ricci flow $(\mathcal{M}, g(t))$ is a smooth family of complete Riemannian metrics on a manifold $\mathcal{M}$, for t within some interval in $\mathbb{R}$, which satisfies Hamilton's nonlinear PDE

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}[g(t)]. \tag{1.1}$$

In the case that $\mathcal{M}$ is two-dimensional, the flow preserves the conformal class of the metric and can be written

$$\frac{\partial}{\partial t} g(t) = -2K g(t),$$

where K is the Gauss curvature of $g(t)$. In this case, we may take local isothermal coordinates x and y, and write the flow $g(t) = e^{2u}(dx^2 + dy^2)$ for some locally-defined scalar time-dependent function u which will then satisfy the local equation

$$\frac{\partial u}{\partial t} = e^{-2u}\Delta u = -K. \tag{1.2}$$

Returning to the case that $\mathcal{M}$ is of arbitrary dimension, Hamilton [7] and Shi [10] developed an existence theory for this equation when a complete bounded-curvature initial metric g_0 was specified. In other words, they found a complete bounded-curvature Ricci flow $g(t)$ for $t \in [0, T]$ (some $T > 0$) with $g(0) = g_0$. Hamilton [7] and Chen-Zhu [2] proved that this flow is unique within the class of complete bounded-curvature Ricci flows.

In this paper we consider existence and particularly uniqueness issues when we drop the restriction that the complete Ricci flows have bounded curvature, and generalise the notion of initial metric as follows.

Definition 1.1. We say that a complete Ricci flow $(\mathcal{M}, g(t))$ for $t \in (0, T]$ has a complete Riemannian manifold $(\mathcal{N}, g_0)$ as initial condition if there exists a smooth map $\varphi : \mathcal{N} \to \mathcal{M}$, diffeomorphic onto its image, such that

$$\varphi^*(g(t)) \to g_0$$

smoothly locally on $\mathcal{N}$ as $t \downarrow 0$.

In practice, we will be interested in the case that $(\mathcal{N}, g_0)$ has bounded curvature but $g(t)$ is allowed to have curvature with no uniform upper bound. In this way, $\mathcal{M}$ and $\mathcal{N}$ may not be diffeomorphic since parts of $\mathcal{M}$ may be shot out to infinity as $t \downarrow 0$ resulting in a change of topology in the limit.

This generalised notion of initial condition permits some new types of solution which do not fit into the classical framework. In particular, we show that a bounded-curvature Riemannian surface with a hyperbolic cusp need not be obliged to flow forwards in time retaining the cusp (as in Shi's solution) but can add in a point at infinity, removing the puncture in the surface, and let the cusp contract in a controlled way. More generally we have:

Theorem 1.2. *Suppose $\mathcal{M}$ is a compact Riemann surface and $\{p_1, \ldots, p_n\} \subset \mathcal{M}$ is a finite set of distinct points. If g_0 is a complete, bounded-curvature, smooth, conformal metric on $\mathcal{N} := \mathcal{M} \backslash \{p_1, \ldots, p_n\}$ with strictly negative curvature in a neighbourhood of each point p_i, then there exists a Ricci flow $g(t)$ on $\mathcal{M}$ for $t \in (0, T]$ (for some $T > 0$) having $(\mathcal{N}, g_0)$ as initial condition in the sense of Definition 1.1. We can take the map φ there to be the natural inclusion of $\mathcal{N}$ in $\mathcal{M}$.*

Moreover, the cusps contract logarithmically in the sense that for some $C < \infty$ and all $t \in (0, T]$ sufficiently small, we have

$$\frac{1}{C}(-\ln t) \leq diam(\mathcal{M}, g(t)) \leq C(-\ln t). \tag{1.3}$$

Furthermore, the curvature of $g(t)$ is bounded below uniformly as $t \downarrow 0$.

Thus a specific example of nonuniqueness would be when the Riemann surface $\mathcal{M}$ is a torus T^2, we remove one point to give $\mathcal{N}$, and let g_0 be the unique complete conformal hyperbolic metric on $\mathcal{N}$. One Ricci flow continuation would be the homothetically expanding one (which coincides with the solution constructed by Shi) but another continuation would see the cusp contract with the subsequent Ricci flow living on the whole torus $\mathcal{M}$.

One characteristic of these nonuniqueness examples is that the initial condition $(\mathcal{N}, g_0)$ does not have a lower bound for its injectivity radius, or equivalently that one can find unit balls of arbitrarily small area. In fact, we will see in a corollary to the following theorem that this is a necessary condition for nonuniqueness.

Theorem 1.3. *Suppose that $(\mathcal{N}, g_0)$ is a complete Riemannian surface with bounded curvature which is noncollapsed in the sense that for some $r_0 > 0$ we have*

$$\mathrm{Vol}_{g_0}(B_{g_0}(x, r_0)) \geq \varepsilon > 0 \tag{1.4}$$

for all $x \in \mathcal{N}$. If $(\mathcal{M}, g(t))$ is a complete Ricci flow for $t \in (0, T]$ (some $T > 0$) which has $(\mathcal{N}, g_0)$ as initial condition in the sense of Definition (1.1), then $(\mathcal{M}, g(t))$ has uniformly bounded curvature over some time interval $(0, \delta]$ (some $\delta \in (0, T]$). Moreover, the φ from Definition (1.1) must be a diffeomorphism (i.e. also surjective) and $g(t)$ can be extended smoothly down to $t = 0$ on the whole of $\mathcal{M}$ by setting $g(0) := \varphi_* g_0$.

The proof of this theorem uses the work of Chen [3], which in turn uses the work of Perelman [9]. It is possible to prove a variant of this result which is applicable to Ricci flows on higher-dimensional manifolds, albeit with slightly stronger hypotheses (see [16]).

Corollary 1.4. *With $(\mathcal{N}, g_0)$ as in the theorem above, if for $i = 1, 2$ we have complete Ricci flows $(\mathcal{M}_i, g_i(t))$ for $t \in (0, T_i]$ (some $T_i > 0$) with $(\mathcal{N}, g_0)$ as initial condition, then these two Ricci flows must agree over some nonempty time interval $t \in (0, \delta]$ in the sense that there exists a diffeomorphism $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ with $\psi^*(g_2(t)) = g_1(t)$ for all $t \in (0, \delta]$.*

Despite the nonuniqueness implied by Theorem 1.2, that construction throws up a quite different uniqueness issue: Does there exist more than one flow which does the same job of contracting the cusps? The next result shows that there does not.

Theorem 1.5. *In the situation of Theorem 1.2 (in which φ is the natural inclusion of $\mathcal{N}$ into $\mathcal{M}$) if $\tilde{g}(t)$ is a smooth Ricci flow on $\mathcal{M}$ for some time interval $t \in (0, \delta)$ ($\delta \in (0, T]$) such that $\tilde{g}(t) \to g_0$ smoothly locally on $\mathcal{N}$ as $t \downarrow 0$ and the Gauss curvature of $\tilde{g}(t)$ is uniformly bounded below, then $\tilde{g}(t)$ agrees with the flow $g(t)$ constructed in Theorem 1.2 for $t \in (0, \delta)$.*

Returning to Theorem 1.2, one can ask at what rate the curvature of $g(t)$ must blow up in the limit $t \downarrow 0$. General theory tells us that this sort of behaviour cannot occur if the curvature blows up no faster than C/t ([11], [3]). By analogy with the terminology of Hamilton for blow up rates [8], we might say then that we have a 'Type II(c) singularity' meaning that

$$\limsup_{t \downarrow 0} \left[t \sup_{\mathcal{M}} |\mathrm{Rm}(\cdot, t)| \right] = \infty.$$

In fact, a rough asymptotic analysis of a contracting cusp in the rotationally symmetric case, modelled by a hyperbolic cusp capped off by an appropriately scaled cigar soliton, suggests that the curvature blows up at a rate C/t^2.

Finally, we point out that Theorem 1.2 provides an answer to Perelman's question [9, §10.3] of whether the volume ratio hypothesis is necessary in his pseudolocality theorem: It is. More elementary examples can also be constructed ([13]).

The paper is organised as follows. In Section 2 we prove Theorem 1.3 and its Corollary 1.4. In Section 3 we derive a selection of estimates for metrics on punctured discs, and use them to construct useful barriers and prove useful estimates for Ricci flow, with the key tool being Lemma 3.3. This technology is then used to prove Theorem 1.2. Finally, in Section 4, we prove the uniqueness assertion of Theorem 1.5.

Acknowledgements: This work was supported by The Leverhulme Trust. Parts of this work were carried out when the author was visiting the Max Planck Albert Einstein Institute, Golm, and the Free University, Berlin, and he would like to thank Gerhard Huisken and Klaus Ecker for their hospitality. Thanks also to Gregor Giesen for discussions on the paper [3].

2 Noncollapsed initial metrics

In this section we prove Theorem 1.3 and its Corollary 1.4. The proof will extend slightly the uniqueness result in the work of Chen [3], which appeals strongly to the remarkable properties of the distance function on a Ricci flow discovered by Perelman [9] in order to prove the following curvature estimate.

Proposition 2.1. *(Chen [3, Proposition 3.9], cf. Perelman [9, §10.3].) Let $\mathcal{M}$ be a surface and $g(t)$ a smooth Ricci flow on $\mathcal{M}$ for $t \in [0, T]$. Suppose that $x_0 \in \mathcal{M}$ and*

$r_0 > 0$, and that $B_{g(t)}(x_0, r_0) \subset\subset \mathcal{M}$ for all $t \in [0, T]$. If

$$|R[g(0)]| \leq r_0^{-2} \text{ on } B_{g(0)}(x_0, r_0) \qquad \text{and} \qquad \mathrm{Vol}_{g(0)}(B_{g(0)}(x_0, r_0)) \geq v_0 r_0^2$$

for some $v_0 > 0$, then there exists $\delta > 0$ depending on v_0 such that

$$|R[g(t)]| \leq 2r_0^{-2} \text{ on } B_{g(t)}\left(x_0, \frac{r_0}{2}\right)$$

for all $t \in [0, T]$ with $t \leq \delta r_0^2$.

Proof. (Theorem 1.3.) First note that by the Bishop-Gromov comparison theorem, we may reduce r_0 to any smaller positive value and still have the noncollapsedness condition (1.4) for some new, possibly smaller, positive value of ε. In particular, by making such a reduction we may assume also that $|R[g_0]| \leq \frac{1}{2} r_0^{-2}$ thoughout $\mathcal{N}$.

We now set $v_0 = \frac{\varepsilon}{2} r_0^{-2}$ and attempt to apply Proposition 2.1 to $(\mathcal{M}, g(t))$. Since $(\mathcal{N}, g_0)$ is the initial condition for $(\mathcal{M}, g(t))$, we see that for all $x_0 \in \mathcal{M}$, and sufficiently small $t_0 > 0$ (depending on x_0) we have

$$|R[g(t_0)]| \leq r_0^{-2} \text{ on } B_{g(t_0)}(x_0, r_0) \qquad \text{and} \qquad \mathrm{Vol}_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \geq \frac{\varepsilon}{2} = v_0 r_0^2.$$

Keeping in mind that we may take $t_0 > 0$ arbitrarily small, Proposition 2.1 then implies that

$$|R[g(t)]| \leq 2r_0^{-2} \text{ on } B_{g(t)}\left(x_0, \frac{r_0}{2}\right)$$

for all $t \in (0, T]$ with $t \leq \delta r_0^2$. Since x_0 was arbitrary, we have established the required uniform curvature bound for $g(t)$.

The uniform curvature bound is then enough to force φ to be a diffeomorphism. Indeed, if we suppose that φ is not surjective, then we can pick $y \in \mathcal{M}$ outside its image. We can then take any smooth immersed curve $\gamma : [0, 1] \to \mathcal{M}$ so that $\gamma(0)$ lies within the image of φ and $\gamma(1) = y$. By truncating and reparametrising the curve, and adjusting y, we may assume that $\gamma(s)$ lies in the image of φ precisely for $s \in [0, 1)$ and $y = \gamma(1)$. Therefore there must exist a smooth curve $\sigma : [0, 1) \to \mathcal{N}$ such that $\varphi(\sigma(s)) = \gamma(s)$ for all $s \in [0, 1)$ and which converges to infinity in the sense that for every compact subset Ω of $\mathcal{N}$, we have $\sigma(s) \notin \Omega$ for $s \in [0, 1)$ sufficiently close to 1. In particular, the curve σ must have infinite length with respect to g_0. By Definition 1.1, for any $M > 0$, the length of γ with respect to $g(t)$ must then be at least M for $t > 0$ sufficiently small depending on M.

However, the curve γ has some finite length with respect to each of the metrics $g(t)$, and by virtue of the uniform curvature bound, these lengths are uniformly bounded above by some number L, say (see for example [14, Lemma 5.3.2]). This is a contradiction, and we have concluded that φ must be a diffeomorphism.

The fact that $g(t)$ can be extended smoothly down to $t = 0$ then follows directly from Definition 1.1. □

Proof. (Corollary 1.4.) By Theorem 1.3, both of the Ricci flows $g_1(t)$ and $g_2(t)$ can be extended to $t = 0$ and then have uniformly bounded curvature over some nonempty time interval $[0, \delta]$. If we let $\varphi_1 : \mathcal{N} \to \mathcal{M}_1$ and $\varphi_2 : \mathcal{N} \to \mathcal{M}_2$ be the maps from Definition 1.1 corresponding to $g_1(t)$ and $g_2(t)$ respectively – which are diffeomorphisms in this case – then $g_0 = \varphi_1^*(g_1(0)) = \varphi_2^*(g_2(0))$, and so $\psi := \varphi_2 \circ (\varphi_1^{-1})$ is an isometry from $(\mathcal{M}_1, g_1(0))$ to $(\mathcal{M}_2, g_2(0))$. Thus $g_1(t)$ and $\psi^*(g_2(t))$ are both complete bounded-curvature Ricci flows, for $t \in [0, \delta]$, which agree at $t = 0$ and are thus identical by the uniqueness result of Chen-Zhu [2], or (more simply in this two-dimensional situation) by the uniqueness implied by [5, Theorem 4.2]. □

3 Flows contracting cusps

3.1 Metrics on the punctured disc

We will require some asymptotic information about metrics on the two-dimensional punctured disc $D\backslash\{0\}$ which are complete with negative curvature near the puncture.

We will be working on $D\backslash\{0\}$ either with respect to the standard complex coordinate $z = x + iy$, sometimes appealing to the corresponding standard polar coordinates (r, θ), or with respect to the cylindrical coordinates (s, θ), where $s = -\ln r$. Note that (s, θ) coordinates are conformally equivalent to the original (x, y) coordinates, and changing coordinates (x, y) to (s, θ) changes the conformal factor according to

$$|\mathrm{d}z|^2 = dx^2 + dy^2 = r^2(ds^2 + d\theta^2). \tag{3.1}$$

With this notation, the complete conformal hyperbolic metric on $D\backslash\{0\}$ can be written $e^{2v}(ds^2 + d\theta^2)$ where $v = -\ln s$ (for $s > 0$).

Lemma 3.1. *If $g_0 = e^{2a}|\mathrm{d}z|^2$ is any smooth conformal metric on the punctured disc $D\backslash\{0\}$ with Gauss curvature bounded above by -1 (with g_0 not necessarily complete) and $H = [r \ln r]^{-2}|\mathrm{d}z|^2$ is the complete conformal hyperbolic metric on $D\backslash\{0\}$, then $g_0 \leq H$, or equivalently*

$$a \leq -ln[r(-\ln r)]. \tag{3.2}$$

Moreover, if $g(t) = e^{2u(t)}|\mathrm{d}z|^2$ is any smooth Ricci flow on D ($t \in [0, T]$) with $g(0) \leq g_0$ then

$$u \leq -ln[r(-\ln r)] + \frac{1}{2}\ln(1 + 2t). \tag{3.3}$$

Proof. With respect to (s, θ) coordinates as introduced at the start of Section 3.1, the conformal factor

$$v_0 := -\ln s$$

gives rise to the complete hyperbolic metric on $(0, \infty) \times S^1$. Moreover, for $\delta > 0$ the conformal factor

$$v_\delta := -\ln\left[\frac{\sin(\delta(s - \delta))}{\delta}\right]$$

defines the complete hyperbolic metric over the range $s \in I_\delta := (\delta, \frac{\pi}{\delta} + \delta)$. It is elementary to see that this conformal factor must be pointwise at least as large as the conformal factor w of any other conformal metric on $\overline{I_\delta} \times S^1$ with Gauss curvature no higher than -1. Indeed, for sufficiently large $0 < M < \infty$, we must have $v_\delta + M > w$ (since the right-hand side is bounded and v_δ is bounded below) and then we can reduce $M > 0$ continuously without this condition failing until possibly at $M = 0$ since if it suddenly failed for $M > 0$ at some point p, then $v_\delta - w + M$ would be a weakly positive function with a zero at p but with strictly negative Laplacian at p:

$$\begin{aligned}
\Delta(v_\delta - w + M) &= -e^{2v_\delta}K[e^{2v_\delta}|\mathrm{d}z|^2] + e^{2w}K[e^{2w}|\mathrm{d}z|^2] \\
&\leq e^{2v_\delta} - e^{2w} = e^{2w}(e^{-2M} - 1) \\
&< 0
\end{aligned}$$

which is a contradiction.

Thus

$$w \leq v_\delta \to v_0 \qquad \text{as } \delta \downarrow 0,$$

and returning from (s, θ) to (x, y) coordinates, keeping in mind (3.1), we deduce the first part (3.2) of the lemma.

For the second part of the lemma, note that the function $v_\delta + \frac{1}{2}\ln(1+2t)$ is the conformal factor of a Ricci flow on $I_\delta \times S^1$ which starts at $t = 0$ above v_0, and hence above any conformal factor w as above. By the maximum principle, $v_\delta + \frac{1}{2}\ln(1+2t)$ must then lie above any conformal factor on D which represents a Ricci flow and which starts below w at $t = 0$. Letting $\delta \downarrow 0$ then yields (3.3). □

We now turn to the subtler issue of lower bounds for conformal factors of metrics g_0 as in Lemma 3.1.

Lemma 3.2. *Suppose $g_0 = e^{2a}|\mathrm{d}z|^2$ is a smooth conformal metric on the punctured disc $D\backslash\{0\}$ with Gauss curvature bounded within some interval $[-M, -1]$ and with g_0 complete at the origin. Denoting the complete conformal hyperbolic metric on $D\backslash\{0\}$ by $H = e^{2v}|\mathrm{d}z|^2$, where $v = -\ln[-r\ln r]$ as above, we have*

$$a - v \geq -C \tag{3.4}$$

for some $C < \infty$ (depending on g_0) and any $r \in (0, \frac{1}{2})$, and in particular, $a \to \infty$ as $r \downarrow 0$.

To clarify, by *complete at the origin* we mean that g_0 restricted to, say, $\overline{D_{\frac{1}{2}}}\backslash\{0\}$ should be a complete manifold with boundary.

Proof. (cf. [6].) Choose any cut-off function $\varphi \in C_c^\infty(D_{\frac{3}{4}}, [0, 1])$ with $\varphi \equiv 1$ on $D_{\frac{1}{2}}$, and consider the metric $\Omega = e^{2\alpha}|\mathrm{d}z|^2$ defined by

$$\alpha = \varphi a + (1 - \varphi)v.$$

For $r \in (\frac{3}{4}, 1)$, we have $\Omega = H$, and so $K[\Omega] = -1$. For $r \in (0, \frac{1}{2})$, we have $\Omega = g_0$, and so $K[\Omega] \geq -M$. In the remaining *compact* region $\frac{1}{2} \leq r \leq \frac{3}{4}$, the curvature $K[\Omega]$ has *some* lower bound, and thus there exists $\beta \leq \infty$ such that

$$K[\Omega] \geq -\beta$$

throughout $D\backslash\{0\}$. Since Ω is clearly complete, we may apply Yau's Schwarz lemma (see [17] and [5, Theorem 2.3]) to deduce that $H \leq \beta e^{2\alpha}|\mathrm{d}z|^2$, or equivalently

$$v \leq \frac{1}{2}\ln\beta + \alpha.$$

Since $\alpha = a$ on $D_{\frac{1}{2}}$, the lemma is proved with $C = \frac{1}{2}\ln\beta$. □

3.2 Spherical upper barriers

In this section we consider Ricci flows on the disc which begin at a metric as considered in Lemma 3.1. The goal is to exploit the estimates from the previous section in order to construct an upper barrier which gives decay of the conformal factor like $1/t$.

Lemma 3.3. *If $g_0 = e^{2a}|\mathrm{d}z|^2$ is any smooth conformal metric on the punctured disc $D\backslash\{0\}$ with Gauss curvature bounded above by -1 (with g_0 not necessarily complete) and $g(t) = e^{2u(t)}|\mathrm{d}z|^2$ is any smooth Ricci flow on D ($t \in [0,T]$) with $g(0) \leq g_0$ then there exists $\beta < \infty$ universal such that*

$$u \leq \frac{\beta}{t} \tag{3.5}$$

for $r \leq \frac{1}{2}$ and $0 < t < \min\{1, T\}$.

The function $s : \mathbb{R}^2 \to \mathbb{R}$ defined in polar coordinates by

$$s(r) := \ln \frac{2}{1+r^2}$$

gives rise to the metric $e^{2s}|\mathrm{d}z|^2$ of the round (punctured) sphere. One may also dilate this conformal factor to $s(\frac{r}{\lambda})$, or add a constant, giving another spherical metric of a possibly different curvature. Under Ricci flow such a conformal factor evolves simply by shifting downwards - i.e. subtracting off a time-dependent constant. For example, for any $\lambda > 0$, one Ricci flow would be given by the conformal factor

$$(r,t) \mapsto s\left(\frac{r}{\lambda}\right) - \ln\lambda + \frac{1}{2}\ln(1-2t).$$

The idea in this section is to use these spherical metrics, appropriately restricted, as upper barriers for the Ricci flow $g(t)$ of the lemma. Moreover, we evolve them not just by Ricci flow (i.e. subtracting off a time-dependent constant) but also by dilating within the domain. Whereas a Ricci flow would make the radius of a sphere shrink like $\sqrt{C-t}$, our barriers will have a radius which is *increasing* like t.

One difficulty with this approach is that one must take care in any maximum principle argument about what is happening on the boundary of the domain on which one is working. This is where the estimates of the previous section first come in.

Proof. (Lemma 3.3.) Without loss of generality, we may assume that $T \leq 1$. Then by Lemma 3.1, for $t \in (0,T)$, we have the upper bound

$$u(r,t) < h(r) := -\ln[r(-\ln r)] + \frac{1}{2}\ln 3$$

for $r \in (0,1)$. On the other hand, we consider the function $S : D \times (0,T) \to \mathbb{R}$ defined by

$$S(r,t) := s\left(\frac{r}{\lambda}\right) - \ln[\lambda(-\ln\lambda)] + \frac{1}{2}\ln 3$$

where $\lambda = \lambda(t) := e^{-\frac{6}{t}}$ will be motivated in a moment. As mentioned above, $S(\cdot,t)$ represents the conformal factor of part of some sphere for each t. Note that

$$S(\lambda,t) = h(\lambda),$$

and so we can define a continuous function $U : D \times (0,T) \to \mathbb{R}$ by

$$U(r,t) = \begin{cases} S(r,t) & 0 \leq r < \lambda \\ h(r) & \lambda \leq r < 1 \end{cases}.$$

Claim: On the whole of $D \times (0,T) \to \mathbb{R}$ we have

$$u \leq U$$

Proof of Claim: For $0 \leq r \leq \lambda$, we have $S(r,t) \geq -\ln[\lambda(-\ln\lambda)] \to \infty$ as $t \downarrow 0$. Therefore for sufficiently small $t > 0$ (depending on the flow in question) we must have $u(r,t) < U(r,t)$ for all $r \in [0,1)$.

Now suppose at some first time $t_0 \in (0,T)$ the function $U(\cdot,t_0)$ fails to be a strict upper barrier for $u(\cdot,t_0)$. Then we can find $r_0 \in (0,\lambda(t_0))$ such that $U(r_0,t_0) = u(r_0,t_0)$ even though $U(\cdot,t_0) \geq u(\cdot,t_0)$. At (r_0,t_0) we then have

$$\frac{\partial(U-u)}{\partial t} \leq 0; \qquad \Delta(U-u) \geq 0,$$

but the Ricci flow equation (1.2) gives $\frac{\partial u}{\partial t} = e^{-2u}\Delta u$, and so

$$\frac{\partial U}{\partial t} \leq e^{-2U}\Delta U \tag{3.6}$$

at (r_0,t_0). On the other hand, keeping in mind that $\lambda = e^{-\frac{6}{t}}$, we have at (r_0,t_0) that

$$\begin{aligned}\frac{\partial U}{\partial t} = \frac{\partial S}{\partial t} &= \left[-\frac{r_0}{\lambda^2}s'\left(\frac{r_0}{\lambda}\right) - \frac{1}{\lambda} - \frac{1}{\lambda\ln\lambda}\right]\frac{d\lambda}{dt}\\ &\geq -\frac{1}{\lambda}\frac{d\lambda}{dt} = -\frac{6}{t^2}\end{aligned}$$

and

$$e^{-2U}\Delta U = e^{-2S}\Delta S = -\frac{1}{3}(\ln\lambda)^2 = -\frac{12}{t^2},$$

and so

$$\frac{\partial U}{\partial t} - e^{-2U}\Delta U \geq \frac{6}{t^2} > 0,$$

contradicting (3.6) and proving the claim.

By inspection, the maximum of U for $r \leq \frac{1}{2}$ is achieved at the origin ($r = 0$):

$$\sup_{r\leq\frac{1}{2}} U = s(0) - \ln[\lambda(-\ln\lambda)] + \frac{1}{2}\ln 3 = \ln 2 + \frac{6}{t} - \ln(\frac{6}{t}) + \frac{1}{2}\ln 3 \leq \frac{C}{t}$$

for some universal C, since $t < 1$. By the claim above, we then have for $r \leq \frac{1}{2}$ and $t \in (0,T)$

$$u \leq \frac{C}{t}$$

as desired. □

3.3 Truncating cusps

In this section we clarify how to take a cusp-like metric on a punctured surface and smooth it out across the puncture in a controlled manner.

Lemma 3.4. *Suppose $g_0 = e^{2a}|\mathrm{d}z|^2$ is a smooth complete conformal metric on the punctured closed disc $\overline{D}\backslash\{0\}$ with Gauss curvature bounded within some interval $[-M,-1]$. Then there exists an increasing sequence of smooth conformal metrics $g_k = e^{2u_k}|\mathrm{d}z|^2$ on D such that*

(i) $g_k = g_0$ *on* $D\backslash D_{1/k}$;
(ii) $g_k \leq g_0$ *throughout* $D\backslash\{0\}$;
(iii) $\inf_{D_{1/k}} u_k \to \infty$ *as* $k \to \infty$;

(iv) the Gauss curvatures of g_k are uniformly bounded below independently of k.

Proof. Pick any smooth function $\psi : \mathbb{R} \to \mathbb{R}$ such that

(a) $\psi(s) = s$ for $s \leq -1$;
(b) $\psi(s) = 0$ for $s \geq 1$;
(c) $\psi'' \leq 0$.

The lemma is proved by taking a subsequence of the increasing sequence of metrics g_k defined by their conformal factors

$$u_k := \psi(a - k) + k.$$

One should view these metrics as a smoothed out version of the minimum of g_0 and the metric with constant conformal factor k. Indeed, it is clear that (ii) is satisfied. Note that u_k is identically equal to k in a neighbourhood of the origin because $a \to \infty$ at the origin, according to Lemma 3.2, and therefore the metrics extend smoothly across the origin. Assertion (iii) also follows because $a \to \infty$ at the origin.

In order to see that the Gauss curvature of the metrics g_k is uniformly controlled from below, we divide up into three cases:

(a) Where $a \geq k + 1$, we have $u_k \equiv k$, so $K(g_k) = 0$;
(b) Where $a \leq k - 1$, we have $u_k \equiv a$, so $K(g_k) = K(g_0) \geq -M$;
(c) Where $k - 1 < a < k + 1$, we have $u_k \geq a - 1$. We compute

$$\Delta u_k = \psi'(a - k)\Delta a + \psi''(a - k)|\nabla a|^2 \leq \psi'(a - k)\Delta a \leq \Delta a,$$

since $\psi'' \leq 0$, $\psi' \in [0, 1]$ and $\Delta a = -e^{2a}K(g_0) \geq 0$. Therefore

$$K(g_k) = -e^{-2u_k}\Delta u_k \geq -e^{-2u_k}\Delta a \geq -e^{-2(a-1)}\Delta a = e^2 K(g_0) \geq -e^2 M.$$

Finally, Assertion (i) can be guaranteed after passing to an appropriate subsequence. □

3.4 Proof of Theorem 1.2

We now combine the supporting results we have compiled in Section 3 into a proof of the existence of the Ricci flows claimed in Theorem 1.2.

For simplicity of notation, we restrict our discussion to the case that $n = 1$ – that is, there is a single puncture p on $\mathcal{M}$ – although the proof of the general case will then be an obvious extension.

We begin by taking isothermal coordinates x and y in a neighbourhood of p. By scaling and translating them, we may assume that p corresponds to $x = y = 0$, that the coordinates exist for $z = x + iy$ within a domain containing the closure of the unit disc $D \subset \mathbb{C}$, and that the supremum of the curvature of g_0 in $D\backslash\{0\}$ is strictly negative. By scaling the metric itself, we may then assume without loss of generality that the Gauss curvature of g_0 is less than -1 within $D\backslash\{0\}$.

By truncating g_0 within D using Lemma 3.4, we can find an increasing sequence of smooth conformal metrics g_k on $\mathcal{M}$ such that

(i) $g_k = g_0$ on $\mathcal{M}\backslash\{p\}$ outside the shrinking neighbourhood $D_{1/k}$ of p;
(ii) $g_k \leq g_0$ throughout $\mathcal{M}\backslash\{p\}$;

(iii) near p, the conformal factors u_k of g_k satisfy $\inf_{D_{1/k}} u_k \to \infty$ as $k \to \infty$;
(iv) the Gauss curvatures of g_k are uniformly bounded below independently of k.

We now flow each of the smooth metrics g_k under Ricci flow in order to give a time-dependent flow $g_k(t)$. Ricci flow theory in two dimensions due to Hamilton and Chow [4] tells us that the flows exist for all time if the genus of $\mathcal{M}$ is at least one, while if the genus is zero, then the existence time is equal to the area of $(\mathcal{M}, g_k)$ divided by 8π. In particular, the existence time is increasing with k since the areas of $(\mathcal{M}, g_k)$ are increasing with k, and we may pick some uniform $T > 0$ so that all of these flows exist for $t \in [0, T]$.

The maximum principle applied to conformal factors tells us that because the metrics $g_k(0)$ are increasing with k, so are the metrics $g_k(t)$ for each $t \in [0, T]$. Also, the maximum principle applied to curvatures, and condition (iv) above, tell us that the Gauss curvature of the flows $g_k(t)$ is uniformly bounded below independently of k and t.

We also want to consider Shi's complete bounded-curvature Ricci flow $g_s(t)$ on $\mathcal{M}\backslash\{p\}$ with $g_s(0) = g_0$. Within $D\backslash\{0\}$, the conformal factor of g_0 is converging to infinity at the puncture (this is implicit above and follows from Lemma 3.2). Therefore, working directly from the Ricci flow equation (1.2) and using the fact that the curvature is bounded above, we see that the conformal factor of each of the metrics $g_s(t)$ must also converge to infinity at the puncture. This allows us to apply the maximum principle to compare each $g_k(t)$ with $g_s(t)$, and we conclude that for each $t \in [0, T]$,

$$g_k(t) \leq g_{k+1}(t) \leq g_s(t) \tag{3.7}$$

throughout $\mathcal{M}\backslash\{p\}$.

This estimate gives good control on the approximating flows $g_k(t)$ away from p, all the way down to $t = 0$. In particular, it allows us to define a flow

$$G(t) = \lim_{k\to\infty} g_k(t) \tag{3.8}$$

on $\mathcal{M}\backslash\{p\}$ for $t \in [0, T]$, and when we take any conformal chart not containing p, we see that the conformal factors of $g_k(t)$ are locally uniformly bounded, independently of k (since $g_k(t)$ is sandwiched between $g_1(t)$ and $g_s(t)$) and so we may apply standard parabolic regularity theory to get local uniform bounds on their derivatives. Therefore, we deduce that $G(t)$ is a smooth Ricci flow on $\mathcal{M}\backslash\{p\}$ for $t \in [0, T]$.

We propose that $G(t)$ extends to be the flow whose existence is asserted in the theorem. Certainly $G(t) \to g_0$ locally on $\mathcal{M}\backslash\{p\}$ as $t \downarrow 0$. However, at this point it is unclear whether for $t > 0$, $G(t)$ extends smoothly across p. Indeed, the truncated cusps within the metrics $g_k(t)$ might take longer and longer to contract as $k \to \infty$, and $G(t)$ might then coincide with Shi's flow $g_s(t)$, for example. In order to show that the truncated cusps within the metrics $g_k(t)$ contract at a rate which is independent of k, we apply Lemma 3.3 to each flow $g_k(t)$ restricted to D.

This gives us uniform control from above on the metrics $g_k(t)$ throughout $\mathcal{M}$, on any closed time interval within $(0, T]$, independently of k. Therefore for $t > 0$, we can extend the definition (3.8) to the whole of $\mathcal{M}$. Moreover, this uniform control allows us to apply the same standard parabolic regularity theory as above to get local uniform bounds on the derivatives of $g_k(t)$ across p, and we conclude that $G(t)$ is a smooth Ricci flow on the whole of $\mathcal{M}$ for $t > 0$.

We now see that the Ricci flow $(\mathcal{M}, G(t))$ for $t \in (0, T]$ has $(\mathcal{M}\backslash\{p\}, g_0)$ as initial metric in the sense of Definition 1.1, with the map φ of the definition equal to the obvious inclusion.

This completes the proof of the first part of the theorem. It remains to argue that the diameter of $(\mathcal{M}, G(t))$ decays precisely logarithmically in t as in (1.3). Since the flow is only singular at p, it suffices to argue that the distance from p ($z = 0$ in the chart considered above) to any other fixed point in $\mathcal{M}$ (say the point $z = 1/2$) decays in this logarithmic fashion.

Upper bound: Because the flow $G(t)$ above arose as a limit of flows $g_k(t)$, it suffices to prove a logarithmic upper bound for the distance from $z = 0$ to $z = 1/2$ in the flows $(\mathcal{M}, g_k(t))$ provided that it is independent of k. Taking β from Lemma 3.3, and exploiting that result together with Lemma 3.1 we compute

$$\begin{aligned} dist_{g_k(t)}(z=0,z=1/2) &\leq \int_0^{\frac{1}{2}} e^{u_k(s,t)}ds \\ &= \int_0^{e^{-\beta/t}} e^{u_k(s,t)}ds + \int_{e^{-\beta/t}}^{\frac{1}{2}} e^{u_k(s,t)}ds \\ &\leq \int_0^{e^{-\beta/t}} e^{\beta/t}ds + \int_{e^{-\beta/t}}^{\frac{1}{2}} \frac{\sqrt{3}}{s(-\ln s)}ds \\ &= 1+\sqrt{3}\Big[-\ln(-\ln s)\Big]_{e^{-\beta/t}}^{\frac{1}{2}} \end{aligned} \tag{3.9}$$

for $t \in (0, \min\{1, T\})$ sufficiently small. Therefore, for sufficiently small $t \in (0, \min\{1, T\})$ we have

$$dist_{g_k(t)}(z=0,z=1/2) \leq -2\ln t$$

as desired.

Lower bound: For points near the origin, we will derive control on the curvature for a certain time depending on the proximity to the origin. This will then show that we cannot deviate from the original metric too much for a controlled time (again, depending on the proximity to the origin) and will lead to a lower bound for the diameter.

Claim: There exists $C < \infty$ such that for $r > 0$ sufficiently small, on ∂D_r, the conformal factor of the Ricci flow is bounded below by $-C - \ln[r(-\ln r)]$ for a time $\frac{1}{C(\ln r)^2}$.

Throughout the argument, C will denote a positive constant which may get larger each time it is used.

Before proving the claim, we show how it would imply the desired lower bound on the diameter. Suppose that the claim holds for $r \in (0, \overline{r})$. Then for $t > 0$ sufficiently small, we would have the conformal factor at time t bounded below by $-C - \ln[r(-\ln r)]$ for $r \in (\underline{r}, \overline{r})$, where $\underline{r} := e^{-\frac{1}{C\sqrt{t}}}$. Then the distance between $\partial D_{\underline{r}}$ and $\partial D_{\overline{r}}$ with respect to the time t metric must be at least

$$\int_{\underline{r}}^{\overline{r}} e^{-C-\ln[r(-\ln r)]}dr \geq \frac{1}{C}\Big[-\ln(-\ln r)\Big]_{\underline{r}}^{\overline{r}} \geq \frac{1}{C}\ln\Big(\frac{1}{C\sqrt{t}}\Big) \geq -\frac{1}{C}\ln t$$

for sufficiently small t, as desired.

Proof of claim: Note that the lower bound of the claim holds at $t = 0$ by Lemma 3.2. By inspection of the Ricci flow equation (1.2), the claim will follow if we can bound the Gauss curvature on ∂D_r by $C(\ln r)^2$ for a time $\frac{1}{C(\ln r)^2}$ (for sufficiently small $r > 0$).

Consider the hyperbolic metric H on $D\backslash\{0\}$ defined in terms of its conformal factor $h(z) = -\ln(-|z|\ln|z|)$. As $z \in D\backslash\{0\}$ approaches the origin, the injectivity radius at

z with respect to H is asymptotically $\frac{\pi}{-\ln|z|}$. Therefore, for z sufficiently close to the origin, we have the volume ratio bound

$$\frac{\mathrm{Vol}_H(B_H(z,s))}{\pi s^2} \geq 1$$

for $s \in (0, \frac{\pi}{-2\ln|z|})$, say.

By virtue of Lemmata 3.1 and 3.2, we see that g_0 is equivalent to H on $D_{\frac{1}{2}}\backslash\{0\}$, say, and thus for z sufficiently close to the origin and $r_0 = \frac{1}{-\ln|z|}$, we have

$$\frac{\mathrm{Vol}_{g_0}(B_{g_0}(z,r_0))}{r_0^2} \geq \frac{1}{C}.$$

By applying Proposition 2.1 to the flow $G(t)$ (or strictly speaking to $G(t+\varepsilon)$ for arbitrarily small $\varepsilon > 0$) on D with $x_0 \in D\backslash\{0\}$ sufficiently close to 0 and $r_0 = \frac{1}{-\ln|x_0|}$, we deduce the Gauss curvature control

$$|K|(x_0) \leq C(\ln|x_0|)^2 \text{ for } t \leq \frac{1}{C(\ln|x_0|)^2}$$

as required to complete the proof.

We remark that a by-product of the argument we have just given is that the supremum of the conformal factor at small time $t > 0$ is bounded below by $\frac{1}{C\sqrt{t}}$. This can be compared to the upper bound $\frac{C}{t}$ implied by Lemma 3.3.

4 Alternative uniqueness issues

The example of a contracting cusp that we have constructed in Section 3 demonstrates that Ricci flows are nonunique when posed as in Definition 1.1. However, one can also ask whether our newly constructed flows are unique amongst all flows which contract their cusps, and in Theorem 1.5 we asserted that they are. This section is devoted to proving that assertion.

The essential difficulty is that *a priori* we know little about the behaviour of any competitor flow near the punctures, for small time.

Recall that $\mathcal{M}$ is a compact Riemann surface and $\{p_1, \ldots, p_n\} \subset \mathcal{M}$ is a finite set of distinct points. We have a complete bounded-curvature smooth conformal metric g_0 on $\mathcal{N} := \mathcal{M}\backslash\{p_1, \ldots, p_n\}$ with strictly negative curvature in a neighbourhood of each point p_i, and (from Theorem 1.2) a complete Ricci flow $g(t)$ on $\mathcal{M}$ for $t \in (0, T]$ (for some $T > 0$) with curvature uniformly bounded below, and such that

$$g(t) \to g_0$$

smoothly locally on $\mathcal{N}$ as $t \downarrow 0$.

Claim 1: With $\tilde{g}(t)$ any Ricci flow as in Theorem 1.5, (that is, defined on $\mathcal{M}$ for $t \in (0, \delta)$, with curvature bounded below and satisfying $\tilde{g}(t) \to g_0$ smoothly locally on $\mathcal{N}$ as $t \downarrow 0$) if $\sigma(t)$ is any Ricci flow on $\mathcal{M}$ for $t \in [0, \delta)$ such that $\sigma(0) < g_0$ on $\mathcal{N}$, then $\sigma(t) \leq \tilde{g}(t)$ on $\mathcal{M}$ for $t \in (0, \delta)$.

We will use Claim 1 to prove:

Claim 2: Given two such flows $\tilde{g}_1(t)$ and $\tilde{g}_2(t)$ (that is, defined on $\mathcal{M}$ for $t \in (0, \delta)$, with curvature bounded below and converging to g_0 smoothly locally on $\mathcal{N}$ as $t \downarrow 0$) we must have $\tilde{g}_1(t) \leq \tilde{g}_2(t)$ for $t \in (0, \delta)$.

Once we have established Claim 2, by switching $\tilde{g}_1(t)$ and $\tilde{g}_2(t)$ we will have $\tilde{g}_1(t) = \tilde{g}_2(t)$, and by applying this in the case $\tilde{g}_1(t) = \tilde{g}(t)$ and $\tilde{g}_2(t) = g(t)$, we will have finished the proof of Theorem 1.5.

To prove Claim 2 from Claim 1, we will consider a scaling of the flow $\tilde{g}_1(t)$ starting at some early time $t_0 > 0$. By the lower curvature bound assumption $K[\tilde{g}_1(t)] \geq -\beta \leq 0$ say, we have $e^{-\beta t_0}\tilde{g}_1(t_0) \leq g_0$ and better still, $e^{-2\beta t_0}\tilde{g}_1(t_0) < g_0$. Therefore, the Ricci flow $\sigma(t) := e^{-2\beta t_0}\tilde{g}_1(e^{2\beta t_0}t + t_0)$ considered for $t \in [0, (\delta - t_0)e^{-2\beta t_0})$ satisfies the hypotheses of Claim 1 (with $\tilde{g}(t)$ there equal to $\tilde{g}_2(t)$ here) and so

$$e^{-2\beta t_0}\tilde{g}_1(e^{2\beta t_0}t + t_0) \leq \tilde{g}_2(t)$$

for $t \in [0, (\delta - t_0)e^{-2\beta t_0})$. Taking the limit $t_0 \downarrow 0$ then finishes the proof of Claim 2.

It remains to prove Claim 1, and for that we need some *a priori* control on solutions. The key ingredient is:

Claim 3: Take any flow $\tilde{g}(t)$ as above (that is, defined on $\mathcal{M}$ for $t \in (0, \delta)$, with curvature bounded below and satisfying $\tilde{g}(t) \to g_0$ smoothly locally on $\mathcal{N}$ as $t \downarrow 0$). Choose a local complex coordinate z about one of the punctures p_i, and write $\tilde{g}(t)$ locally as $e^{2u}|\mathrm{d}z|^2$. Then for any $M < \infty$, we have

$$u \geq M$$

in some neighbourhood of p_i, for sufficiently small $t > 0$.

To prove Claim 1 from Claim 3, look at a neighbourhood of a point p_i as in Claim 3. Denote the conformal factor of $\sigma(0)$ in this local chart by s (i.e. $\sigma(0) = e^{2s}|\mathrm{d}z|^2$) and define $M < \infty$ to be the supremum of s over some neighbourhood of p_i. By Claim 3, we may shrink this neighbourhood and be sure that $u \geq M \geq s$ for sufficiently small $t > 0$. Repeating for all the other punctures p_i, we can find an open set $\Omega \subset \mathcal{M}$ containing each point p_i so that $\tilde{g}(t) \geq \sigma(0)$ on Ω for $t \in (0, t_0)$ (for some $t_0 \in (0, \delta)$). By compactness of $\mathcal{M}\backslash\Omega$ and the fact that $\sigma(0) < g_0$, we may reduce $t_0 > 0$ further and be sure that $\tilde{g}(t) \geq \sigma(0)$ throughout the whole of $\mathcal{M}$ for $t \in (0, t_0)$. The comparison principle then tells us that for any $\tilde{t} \in (0, t_0)$, we have $\tilde{g}(t + \tilde{t}) \geq \sigma(t)$ for $t \in (0, \delta - \tilde{t})$. By taking the limit $\tilde{t} \downarrow 0$, Claim 1 is proved.

It now remains to prove Claim 3, and this in turn relies on a new claim:

Claim 4: In the setting of Claim 3, if Ω is a neighbourhood of p_i compactly contained in the neighbourhood where z is defined, then there exists *some* $m \in \mathbb{R}$ such that

$$u \geq m$$

within Ω for $t \in (0, \delta/2]$.

To prove Claim 4, we use the fact that the Gauss curvature is uniformly bounded below by $-\beta$, say. Then by the Ricci flow equation (1.2), for $z \in \Omega$, and $t \in (0, \delta/2]$,

$$u(z,t) \geq u(z, \delta/2) - \beta(\delta/2 - t) \geq \inf_{\Omega} u(\cdot, \delta/2) - \beta\delta/2 =: m,$$

and the claim is proved.

Finally, we prove Claim 3 from Claim 4. With respect to the local complex coordinate from Claim 3, we may write g_0 as $e^{2u_0}|\mathrm{d}z|^2$. Recalling that the curvature of g_0 is uniformly strictly negative in some neighbourhood of p_i, we see that without loss of generality, we may assume that the coordinate z is defined for $z \in D$, and that for $z \in D\backslash\{0\}$ the curvature of g_0 lies in some interval $[-C,-1]$. By Lemma 3.2, we then see that the conformal factor u_0 of g_0 is at least $M+2$ for z in some small disc D_ε within the conformal chart (away from the origin).

Consider

$$(M-u)_+ := \begin{cases} M-u & \text{if } M-u>0 \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that $u \to u_0$ smoothly locally on $\overline{D_\varepsilon}\backslash\{0\}$, together with Claim 4, we see that $\|(M-u)_+\|_{L^1(D_\varepsilon)} \to 0$ as $t \downarrow 0$. Moreover, because $u_0 \geq M+2$ on ∂D_ε, we see that $u \geq M+1$ on ∂D_ε over some nonempty time interval $(0,t_0)$. Pick any smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that

(a) $\phi(s) = s$ for $s \geq 1$;
(b) $\phi(s) = 0$ for $s \leq -1$;
(c) $\phi'' \geq 0$,

and note that $\phi' \geq 0$. We then have $\phi(M-u) = 0$ on ∂D_ε over the time interval $(0,t_0)$ and we may compute for $t \in (0,t_0)$,

$$\begin{aligned} \frac{d}{dt}\int_{D_\varepsilon} \phi(M-u) &= -\int \phi'(M-u)u_t = -\int \phi'(M-u)e^{-2u}\Delta u \\ &= -\int e^{-2u}|\nabla u|^2 \left(\phi''(M-u) + 2\phi'(M-u)\right) \\ &\leq 0. \end{aligned} \tag{4.1}$$

By allowing ϕ to decrease uniformly to the function $s \mapsto s_+$, we then see that

$$(M-u)_+ \equiv 0$$

on D_ε for $t \in (0,t_0)$, completing the proof.

References

[1] M. Bertsch, R. Dal Passo and R. van der Hout, *Nonuniqueness for the heat flow of harmonic maps on the disk.* Arch. Ration. Mech. Anal. **161** (2002) 93–112.

[2] B.-L. Chen and X.-P. Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds.* J. Differential Geom. **74** (2006) 119–154.

[3] B.-L. Chen, *Strong uniqueness of the Ricci flow.* J. Differential Geom. **82** (2009) 363–382.

[4] B. Chow and D. Knopf, The Ricci Flow: An Introduction, AMS, Mathematical surveys and monographs **110**, (2004).

[5] G. Giesen and P. M. Topping, *Ricci flow of negatively curved incomplete surfaces.* Calc. Var. **38** (2010), 357–367.

[6] G. Giesen and P. M. Topping, *Existence of Ricci flows of incomplete surfaces.* http://arxiv.org/abs/1007.3146

[7] R. S. Hamilton, *Three-manifolds with positive Ricci curvature.* J. Differential Geom. **17** (1982) 255–306.

[8] R. S. Hamilton, *The formation of singularities in the Ricci flow.* Surveys in differential geometry, Vol. II (Cambridge, MA, 1993) 7–136, Internat. Press, Cambridge, MA, 1995.

[9] G. Perelman *The entropy formula for the Ricci flow and its geometric applications.* http://arXiv.org/abs/math/0211159v1 (2002).

[10] W.-X. Shi, *Deforming the metric on complete Riemannian manifolds.* J. Differential Geom. **30** (1989), 223–301.

[11] M. Simon, *Local results for flows whose speed or height is bounded by $\frac{c}{t}$.* I.M.R.N. (2008) doi:10.1093/imrn/rnn097

[12] P. M. Topping, *Reverse bubbling and nonuniqueness in the harmonic map flow.* I.M.R.N. **10** (2002) 505–520.

[13] P. M. Topping, Talks at "Workshop on Geometric Analysis," Sendai (2005).

[14] P. M. Topping, *Lectures on the Ricci flow.* L.M.S. Lecture notes series **325** C.U.P. (2006) http://www.warwick.ac.uk/~maseq/RFnotes.html

[15] P. M. Topping, *Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics.* J. Eur. Math. Soc. (JEMS) **12** (2010) 1429–1451.

[16] P. M. Topping, *Reverse bubbling in geometric flows.* Preprint (2010).

[17] S.-T. Yau, *Remarks on conformal transformations.* J. Differential Geometry **8** (1973), 369–381.

mathematics institute, university of warwick, coventry, CV4 7AL, uk
http://www.warwick.ac.uk/~maseq