Abstracts of selected publications

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'Rigidity in the harmonic map heat flow.'
- appeared in 'The Journal of Differential Geometry,' volume 45 (1997) 593-610.

This article summarises some of the key results from part of my PhD thesis on the harmonic map flow from surfaces. My main result is the first asymptotic uniformity result to have been proved for the harmonic map flow in the presence of 'bubbling' (i.e. singularity formation) at infinite time. I prove that for flows between 2-spheres for which all the bubbles at infinite time share a common orientation with the limiting map of the flow, the convergence of the flow to its limit is strong, uniform in time, and at exponential rate in L^p (for any $p < \infty$) and even in C^k (for any k) away from the blow-up points. All theory prior to this work was only able to establish convergence of the flow at a sequence of times converging to infinity. I also give a perturbation result, saying that perturbing the initial map of a flow without finite time singularities gives a new flow close to the old one in the best sense possible. The main new ingredient in this work is a method for filtering out the singularities in the flow using complex analysis whilst retaining enough information to control the energy decay. Finally, I give an example to demonstrate that with weaker hypotheses, the convergence of the flow need not be uniform in time.

This work grew mainly out of a visit I made to Stanford University, California, in the spring of 1994.

'The Optimal Constant in Wente's L^∞ Estimate.'
- appeared in 'Commentarii Mathematici Helvetici,' volume 72 (1997) 316-328.

This article concerns the remarkable compensation properties of Jacobian determinants which have been discovered, developed and applied with great success by geometric and applied analysts over the past thirty years. A geometric approach is adopted in which an isoperimetric inequality is used to prove Wente's L^∞ estimate with the optimal constant, and which enables further improvements in the constant of up to a factor of two in certain cases. These improvements are also shown to hold for very general domains and subsequently applied to obtain new estimates on the size of compact immersed surfaces of constant mean curvature. The simplest and most easily stated result (which is described in context in the article for the benefit of nonexperts) may be described as follows. Suppose Ω is a bounded domain in $\mathbb{R}^2$ with regular boundary, and $u \in H^1(\Omega, \mathbb{R}^2)$. Then if φ is the unique solution in $W_0^{1,1}(\Omega, \mathbb{R})$ to

$$\begin{cases} -\Delta\varphi = \det(\nabla u) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

then we have the estimate

$$\|\varphi\|_{L^\infty(\Omega)} \leq \frac{1}{4\pi}\|\nabla u\|^2_{L^2(\Omega)}.$$

This work was a result of my year at the École Normale Supérieure at Cachan in Paris, where there is considerable expertise in these matters.

'Mean curvature flow and geometric inequalities.'
- appeared in 'Journal für die reine und angewandte Mathematik,' volume 503 (1998) 47-61.

This article explores the link between the mean curvature flow and inequalities relating geometric quantities such as area, volume, diameter and Willmore energy. In the first half, I observe how isoperimetric inequalities can be extracted from the theory of curve shortening on surfaces; in particular I give a new inequality which simultaneously generalises the inequalities of Alexandrov, Huber, Bol and others. The inequality relates the area A of a simply connected surface D to the length L of its boundary, in terms of the decreasing rearrangement $K^* : (0, A) \to \mathbb{R}$ of the Gauss curvature K, where K^* is defined to be the decreasing function satisfying

$$Area(\{x \in D \mid K(x) \geq s\}) = |\{y \in (0, A) \mid K^*(y) \geq s\}|,$$

for all $s \in \mathbb{R}$. The relation (proved with extra but unnecessary hypotheses in this article for simplicity) is then

$$4\pi A \leq L^2 + 2\int_0^A (A - x)K^*(x)dx. \qquad (0.1)$$

In the second half of the article, I find geometric inequalities which must be satisfied for the mean curvature flow in higher dimensions to evolve smoothly before disappearing at a point; in particular I give a lower bound for the area of a dumbbell in terms of its length which must be satisfied for its neck not to pinch off.

This work was carried out at the ETH in Zürich.

'The isoperimetric inequality on a surface.'
- appeared in 'Manuscripta Mathematica,' volume 100 (1999) 23-33.

Following the previous article in which I discovered the inequality (0.1), in this work I give a proof in the general case (without boundary convexity or total curvature conditions as required for a curve shortening proof) and extend the inequality to multiply connected domains for which I now estimate $4\pi A\chi$, where χ is the Euler characteristic of the surface, instead of $4\pi A$.

In order to prove the inequality, I approximate the surface by polyhedra and analyse the properties of the boundary equidistants on the surface and its approximations.

‘An example of a nontrivial bubble tree in the harmonic map heat flow.’ - appeared in ‘Harmonic Morphisms, Harmonic Maps and Related Topics.’ J. C. Wood et al ed., CRC Press (1999).

Here we give the first example of the formation of a nontrivial, nested bubble tree in the harmonic map heat flow. In other words, we give a flow in which more than one bubble develop simultaneously at the same point. The bubbles occur at infinite time and develop at different scales.

The method is to construct an appropriate warped target manifold $\mathcal{N}$ and an initial map from the 2-disc to $\mathcal{N}$ in a homotopy class containing no harmonic maps, such that *(i)* the subsequent heat flow cannot blow up in finite time, *(ii)* bubbles may only develop at the centre of the disc, and *(iii)* no one bubble can move the flow into a homotopy class containing a harmonic map. Two bubbles must then develop at the same place at infinite time.

‘Pressure estimates in two dimensional incompressible fluid flow.’ - appeared in ‘Physica D,’ volume 137 (2000) 143-156.

We derive sharp pressure estimates for two dimensional incompressible fluid flow, in terms of natural quantities such as enstrophy, energy and angular momentum. We cover both the Euler and Navier-Stokes equations, and both periodic planar flows and spherical flows.

Such estimates do not follow from classical theory - instead we must invoke theory and ideas we developed in our previous work on Wente’s inequality. For our treatment of spherical flows, we must develop this compensation theory further, to handle Jacobian determinants of *vector fields* on surfaces rather than simply determinants of maps into surfaces supporting isoperimetric inequalities as in our previous work.

This work was carried out at the IHES, Bures-sur-Yvette, France.

'Towards the Willmore conjecture.'
- appeared in 'Calculus of Variations and P.D.E.' volume 11 (2000) 361-393.

We develop a variety of approaches, mainly using integral geometry, to proving that the integral of the square of the mean curvature of a torus immersed in $\mathbb{R}^3$ must always take a value no less than $2\pi^2$. Our partial results, phrased mainly within the S^3-formulation of the problem, are typically strongest when the Gauss curvature can be controlled in terms of extrinsic curvatures or when the torus enjoys further properties related to its distribution within the ambient space. Corollaries include a recent result of Ros confirming the Willmore conjecture for surfaces invariant under the antipodal map, and a strengthening of the expected results for flat tori.

The value $2\pi^2$ arises in this work in a number of different ways - as the volume (or renormalised volume) of S^3, $SO(3)$ or $G_{2,4}$, and in terms of the length of shortest nontrivial loops in subgroups of $SO(4)$.

This project was started at the ETH - Zürich, and completed at the IHES, Bures-sur-Yvette, France.

'Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow.' - to appear in 'Annals of Math'.

Given a map u between 2-spheres for which the tension $\tau(u)$ (the negation of the gradient of the harmonic energy $E(u)$) is small in L^2, to what extent is u close to a harmonic map? It is not hard to prove the existence of a harmonic map L^2-close to u; however in general there will be no such map close to u in the natural energy norm $W^{1,2}$. Indeed u may resemble a harmonic 'body' map with bubbles attached. Nevertheless, by exploiting this structure, it is possible to show that u will have energy close to an integer multiple of 4π, when the tension is small.

One of the goals of this paper is to control just how close $E(u)$ must be to $4\pi k$, in terms of the tension. Aside from the intrinsic interest of such a nondegeneracy estimate, control of this form turns out to be the key to an understanding of the asymptotic properties of the harmonic map heat flow (L^2 gradient flow on E) of Eells and Sampson.

With fairly general hypotheses on the configuration of the bubble tree to which u must be close (including the case where there is no more than one bubble at each point, provided the body map has nonzero energy density at those points) we are able to establish an estimate

$$|E(u) - 4\pi k| \leq C\|\tau(u)\|^2_{L^2(S^2)},$$

controlling the degree of quantization of energy in terms of the tension. This then allows us to completely solve the question of asymptotic uniformity of the harmonic map flow in the situation considered; we find uniform exponential convergence in time and uniqueness of the positions of bubbles.

A further goal of this paper, which turns out to be a key ingredient in the theory described above, is a sharp estimate for the size of any bubbles which develop with opposite orientation to the body map, *in terms of the tension*, which we establish using an analysis of the Hopf differential and theory of the Hardy-Littlewood maximal function. The estimate asserts a repulsive effect between holomorphic and antiholomorphic bubbles and could never hold for bubbles of like orientation. (Indeed in general, bubbling may occur within sequences of harmonic maps.) From here, we proceed with a careful analysis of energy decay along necks, inspired by recent work of Qing-Tian and others, and a programme of 'analytic surgery,' which enables us to quantize the energy on each component of a bubble tree.

This work was carried out mainly at MSRI, Berkeley, USA.

'Reverse bubbling and nonuniqueness in the harmonic map flow.'
- appeared in 'International Mathematics Research Notices,' volume 10 (2002) 505-520.

In 1985, Struwe developed a theory of the harmonic map heat flow for two dimensional domain manifolds. He constructed global weak solutions which are smooth except at finitely many points in space-time where bubbling occurs. Since then, much progress has been made in understanding finer properties of these flows with a view to applications, and a series of works have established uniqueness of these solutions in progressively greater generality culminating in a 1995 paper of Freire where it was only necessary to assume that the energy was nonincreasing.

In this paper, I construct some new solutions, also with smoothness away from a finite set of points, but now with a very different type of 'reverse' bubbling occuring at the singular points. These solutions not only settle the general uniqueness problem in the negative, but enjoy some new properties which may make them more useful for certain applications. For example, they may stay in the same homotopy class as a singularity is encountered, when the Struwe-solution would not.

'An approach to the Willmore conjecture.'
- to appear in 'The global theory of minimal surfaces,' (Berkeley, CA, 2001).

We highlight, and expand upon, one of the methods developed in our earlier work on the Willmore conjecture. In particular, we find a strict generalisation of the result for antipodally invariant tori, which may now be strong enough to settle the general conjecture.

‘Winding behaviour of finite-time singularities of the harmonic map heat flow.’ - (2002) to appear in Math. Zeit.

This article settles a number of questions about the properties of *finite*-time bubbling in the harmonic map flow on two dimensional domains. Here one has much weaker control on the behaviour of the tension near the singularity than for infinite-time bubbling. I demonstrate that a type of nonuniqueness of bubbles can occur at finite time, where bubble images wind around in the target with unbounded speed as the singular time is approached. This behaviour leads to an unexpected failure of convergence when the flow is (locally) lifted to the universal cover of the target manifold.

I also settle the ‘discontinuity of $u(T)$’ problem raised by Qing-Tian and Lin-Wang, showing that the weak limit of the flow at the singular time can be discontinuous. Finally, I determine exactly the (polynomial) rate of blow-up in one particular example.

‘Bubbling of almost-harmonic maps between 2-spheres at points of zero energy density.’ - appeared in ‘Progress in nonlinear differential equations and their applications,’ volume 59 (2004) 33-42.

This article considers both bubbling in the harmonic map flow at infinite time, and in sequences of almost-harmonic maps. I show how the position in the domain at which a bubble occurs may have a strong bearing on its properties. In my previous paper ‘Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow’ I established exponential convergence of the harmonic map flow as $t \to \infty$, and quantization and repulsion estimates, under a hypothesis on the energy density of the body map at certain bubble points. Here, that hypothesis is justified by proving that all these properties may fail otherwise.

‘Improved regularity of harmonic map flows with Hölder continuous energy.’ - to appear in Calc. Var.

As we showed in our ‘Winding behaviour...’ paper above, if $u(t)$ is a smooth harmonic map flow on a two dimensional domain over a maximal time interval $[0, T)$, then the weak limit $u(T)$ of $u(t)$ as $t \uparrow T$ is liable to be discontinous. In this paper, we show that the regularity of the map $u(T)$ is controlled by the regularity of the energy as a scalar function of time. In particular, if the energy is Hölder continuous, then the map $u(T)$ must be Hölder continuous also.

‘Diameter control under Ricci flow.’
- Preprint (2004).

It is currently unknown as to whether the diameter of a closed manifold evolving under Ricci flow can blow up in finite time. In this work we prove some estimates related to this question, controlling the diameter in terms of a natural scalar curvature integral. For three dimensional manifolds, we essentially control the diameter in terms of the total scalar curvature.

The proof builds on recent work of Perelman, combining some elements of his ‘no local collapsing’ result with a new geometric maximal function.

‘The harmonic map heat flow from surfaces.’ PhD thesis, University of Warwick (1996).

My thesis contains a comprehensive introduction to the harmonic map flow, a detailed exposition of the uniformity and stability work contained in my paper ‘Rigidity in the harmonic map heat flow,’ and several other results which are unavailable elsewhere in print. For example, I prove a series of related results concerning the amount of energy which is required in practice for singularities to develop. The results are shown to be sharp with a plethora of examples. We end by demonstrating how a flow with bubbles at a sequence of times $t_i \to \infty$ need not have any bubbles whatsoever at a different sequence of times $s_i \to \infty$.

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