The Optimal Constant in Wente's L^∞ Estimate

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Abstract

We explore some geometric aspects of compensation compactness associated to Jacobian determinants. We provide the optimal constant in Wente's inequality - the original motivation of this work - and go on to give various extensions to geometric situations. In fact we improve Wente's inequality somewhat, making it more appropriate for applications in which optimal results are required. This is demonstrated when we prove an optimal inequality for immersed surfaces of constant mean curvature in $\mathbb{R}^3$, contolling their diameter in terms of their area and curvature.

1 Introduction

Given a map $u \in H^1(\mathbb{R}^2, \mathbb{R}^2)$ it is clear that $\det(\nabla u) \in L^1(\mathbb{R}^2, \mathbb{R})$. However, over the past thirty years it has become clear that such quantities $\det(\nabla u)$ posess further 'regularity' properties. The earliest observations of this form seem to be due to Wente [10] whilst more modern work [5] has established that $\det(\nabla u)$ lies in the Hardy space $\mathcal{H}^1 \subset L^1$.

Experience has shown that the quantity $\det(\nabla u)$ arises in, or can be extracted from, a large number of partial differential equations from geometry and physics. At the heart of many of these situations has been the problem

$$\begin{cases} -\Delta\varphi = \det(\nabla u) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Moreover, it has been the crucial step in many situations to control φ in L^∞. In particular this provides control of $\nabla\varphi$ in L^2 and the continuity of φ via simple arguments. Whilst $\Delta\varphi \in L^1$ is not sufficient to control φ in L^∞, the slightly stronger statement $\Delta\varphi \in \mathcal{H}^1$ is indeed enough. More modern applications of these improved regularity phenomena have called for optimal constants in the estimates. A large number of references may be found in the forthcoming book of Frédéric Hélein [9].

Prior to this work it was known (see [11], [1] and [3]) that in the case that $\Omega = D$, the 2-disc (and consequently for any simply connected domain Ω by the conformal invariance of the problem) we have the estimate

$$\|\varphi\|_{L^\infty(\Omega)} \leq \frac{1}{4\pi}\|\nabla u\|^2_{L^2(\Omega)}$$

for solutions of (1). Examples of Baraket in [1] show that the constant $\frac{1}{4\pi}$ is the best we can hope for in such an estimate (whatever the domain Ω). For general Ω, Bethuel and Ghidaglia [2] (see also [4]) established that

$$\|\varphi\|_{L^\infty(\Omega)} \leq 13\|\nabla u\|^2_{L^2(\Omega)}. \tag{2}$$

In this work we prove such an estimate for general Ω, but with an optimal constant.

Theorem 1 *Suppose Ω is a bounded domain in $\mathbb{R}^2$ with regular boundary, and $u \in H^1(\Omega, \mathbb{R}^2)$. Then if φ is the unique solution in $W_0^{1,1}(\Omega, \mathbb{R})$ to (1), we have the estimate*

$$\|\varphi\|_{L^\infty(\Omega)} \leq \frac{1}{4\pi}\|\nabla u\|^2_{L^2(\Omega)}. \tag{3}$$

We refer to (3) as 'Wente's inequality.' It is clear from our proof that there is equality in (3) only when u is constant (on connected components of Ω).

Theorem 1 follows from a more general inequality in which equality is much easier to obtain. To state this, we must define the quantity

$$\omega(u) = \frac{1}{2}\left(\left|\frac{\partial u}{\partial x}\right|^2 - \left|\frac{\partial u}{\partial y}\right|^2 - 2i\left\langle\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle\right).$$

We note that both $2\omega(u)dz^2$ and $\frac{1}{2}\omega(u)dz^2$ (where $z = x + iy$) are often referred to as the 'Hopf differential,' the latter being the $(2,0)$ part of the pullback under u of the metric tensor on $\mathbb{R}^2$. A priori we have the pointwise estimate $|\omega(u)| \leq \frac{1}{2}|\nabla u|^2$, and so defining the global quantities

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2, \qquad \Theta(u) = \int_\Omega |\omega(u)|$$

(so E is the usual Dirichlet energy) we have the inequality

$$\Theta(u) \leq E(u).$$

We remark that $\Theta(u)$ measures, in some sense, the extent to which u is not conformal.

Our generalisation of Wente's inequality is then as follows.

Theorem 2 *Suppose Ω is a bounded domain in $\mathbb{R}^2$ with regular boundary, and $u \in H^1(\Omega, \mathbb{R}^2)$. Then if φ is the unique solution in $W_0^{1,1}(\Omega, \mathbb{R})$ to (1), we have the estimate*

$$\|\varphi\|_{L^\infty(\Omega)} \leq \frac{1}{4\pi}\left(E(u) + \Theta(u)\right). \tag{4}$$

The a priori inequality

$$E(u) + \Theta(u) \leq \|\nabla u\|^2_{L^2(\Omega)}$$

ensures that Theorem 2 indeed generalises Theorem 1.

We remark that equality in (4) is attained for a wide variety of maps u (in contrast to (3)). For example, when $\Omega = D$ and $u = id$ we find that $\det(\nabla u) = 1$, $\varphi = -\frac{1}{4}(x^2 + y^2 - 1)$, $E(u) = \pi$ and $\Theta(u) = 0$, and hence that both sides in (4) are equal to $\frac{1}{4}$.

In section 4 we will discuss generalisations of Theorem 2 to the case that Ω is a more general surface. This generalisation is then applied to the study of immersed surfaces of constant mean curvature in section 5. The extent to which the target $\mathbb{R}^2$ of the map u may be generalised to other surfaces is considered in section 6.

2 The isoperimetric inequality

Central to our proof will be an isoperimetric inequality. The simplest such inequality relates the area $A(\Omega)$ of a domain $\Omega \subset \mathbb{R}^2$ to the length $L(\partial\Omega)$ of its boundary, and is very well known.

Lemma 1 *Given a domain $\Omega \subset \mathbb{R}^2$ with regular boundary, we have the estimate*

$$4\pi A(\Omega) \leq L(\partial\Omega)^2.$$

We offer a new proof, inspired by work of Frédéric Hélein [8], which we believe to be shorter than any previously known proof. We use, in order of appearance, simple integration, Cauchy's theory (plus the fact that $dz \wedge d\bar{z} = \frac{2}{i} dx \wedge dy$ when $z = x + iy$), Fubini's Theorem, Stokes' Theorem, and simple estimation. We will denote the path corresponding to $\partial\Omega$, and keeping Ω on the left, by γ.

Proof.

$$4\pi A = \int_\Omega (2\pi i)\tfrac{2}{i} dxdy = \int_\Omega \left[\int_\gamma \frac{dw}{w-z}\right] dz \wedge d\bar{z} = \int_\gamma \left[\int_\Omega \frac{d\bar{z} \wedge dz}{z-w}\right] dw = \int_\gamma \left[\int_\gamma \frac{\bar{z}-\bar{w}}{z-w} dz\right] dw \leq L^2.$$

■

Note that in future we will use $\partial\Omega$ to represent both the boundary of Ω and the corresponding path traversed keeping Ω on the left.

In fact, we shall require a functional form of the isoperimetric inequality. Given a regular domain $\Sigma \subset \mathbb{R}^2$ and $u \in C^\infty(\overline{\Sigma}, \mathbb{R}^2)$ we may define

$$\mathcal{A}(\Sigma, u) = \int_\Sigma \det(\nabla u), \qquad \mathcal{L}(\partial\Sigma, u) = \int_{\partial\Sigma} |\nabla u.\tau|,$$

the area of $u(\Sigma)$ and the length of $u(\partial\Sigma)$ counted with algebraic and geometric multiplicity respectively. Here τ denotes a unit length vector tangent to the path on which we are integrating.

Lemma 2 *For any regular domain $\Sigma \subset \mathbb{R}^2$ and $u \in C^\infty(\overline{\Sigma}, \mathbb{R}^2)$ we have the inequality*

$$|\mathcal{A}(\Sigma, u)| \leq \frac{1}{4\pi}\mathcal{L}(\partial\Sigma, u)^2.$$

Of course, when u is the identity map, we recover the simplest form of the isoperimetric inequality.

Proof. Let us see u as a map from $\Sigma \subset \mathbb{C}$ to $\mathbb{C}$. Then

$$\det(\nabla u) = |u_z|^2 - |u_{\bar{z}}|^2 = (\bar{u}u_z)_{\bar{z}} - (\bar{u}u_{\bar{z}})_z,$$

and so

$$\mathcal{A}(\Sigma, u) = \int_\Sigma [(\bar{u}u_z)_{\bar{z}} - (\bar{u}u_{\bar{z}})_z] \frac{1}{2i} d\bar{z} \wedge dz = \frac{1}{2i}\int_{\partial\Sigma} \bar{u}u_z dz + \bar{u}u_{\bar{z}} d\bar{z} = \frac{1}{2i}\int_{\partial\Sigma} \bar{u} du.$$

Writing $\gamma = u \circ \partial\Sigma$ and reinterpreting du as a form on the target $\mathbb{C}$, we have

$$\mathcal{A}(\Sigma, u) = \frac{1}{2i}\int_\gamma \bar{u} du.$$

Let us take any simply connected regular domain Λ in the target $\mathbb{C}$ which encloses $u(\partial\Sigma)$. Then

$$\mathcal{A}(\Sigma, u) = \frac{1}{2i}\int_\gamma \bar{u}\left(\frac{1}{2\pi i}\int_{\partial\Lambda} \frac{dv}{v-u}\right) du = \frac{1}{4\pi}\int_\gamma\int_{\partial\Lambda} \frac{\bar{u}}{u-v} dv\, du = \frac{1}{4\pi}\int_\gamma\int_{\partial\Lambda} \frac{\bar{u}-\bar{v}}{u-v} dv\, du,$$

where the last equality holds because γ cannot wind around $v \in \partial\Lambda$ and so

$$\int_\gamma \frac{1}{u-v} du = 0.$$

Simple estimation now gives us

$$|\mathcal{A}(\Sigma, u)| \leq L(\partial\Lambda)\mathcal{L}(\partial\Sigma, u).$$

The lemma follows upon shrinking Λ around $u(\partial\Sigma)$. ∎

3 The proof of Theorem 2

Let us consider the Green function $G = G^a$ related to Ω as in Theorem 2, and $a \in \Omega$ - in other words the solution $G : \overline{\Omega} \to \mathbb{R}$ to

$$\begin{cases} -\Delta G = \delta_a & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Recall that

$$G(x) = -\frac{1}{2\pi} \log|x-a| + h(x), \tag{6}$$

where $h = h^a \in C^\infty(\overline{\Omega}, \mathbb{R})$ is harmonic, and hence that G is smooth away from a. The maximum principle tells us that G is positive on Ω.

Let $\mathcal{S} \subset \Omega$ denote the set of critical points of G. Clearly $\mathcal{S}$ is a set of isolated points, as it also represents the zero set of the holomorphic function G_z. In particular $\mathcal{S}$ is countable which suffices for our purposes, though as we shall now argue, the critical points of G are isolated in $\overline{\Omega}$ and therefore $\mathcal{S}$ is a finite set of points. If this were not the case, we could pick an accumulation point $x \in \partial\Omega$ of critical points of G. Then by straightening out the boundary $\partial\Omega$ via a conformal reparameterisation of Ω and reflecting G across the boundary locally, the extension of G would have an accumulation point at x which is impossible as G_z is now holomorphic on a neighbourhood of x.

We make the further definitions $\mathcal{R}_+ = (0,\infty)\backslash G(S)$ and $\Omega' = \{x \in \Omega \mid G(x) \in \mathcal{R}_+\} = \Omega\backslash G^{-1}(G(S))$. Moreover, we label the level sets of G by $V(\gamma) = G^{-1}(\gamma)$ and define $W(\gamma) = G^{-1}((\gamma, \infty])$, so that $\partial W(\gamma) = V(\gamma)$. Note that the level sets of G were also considered in Bethuel and Ghidaglia's proof of (2). By a simple Implicit Function Theorem argument, for any γ, $V(\gamma)$ is locally a smooth curve away from the set $\mathcal{S}$, and hence for $\gamma \in \mathcal{R}_+$, $V(\gamma)$ is a union of smooth closed paths. A further consequence is that Ω and Ω' differ only by a set of measure zero.

We will need to appeal several times to a coarea formula. Taking integration to be with respect to Lebesgue measure on $\mathbb{R}^2$ (or an appropriate induced measure) when not stated explicity, we give the following specialisation of [6, Theorem 3.2.12].

Proposition 1 *Let $\Sigma \subset \mathbb{R}^2$ be a regular domain. Then for any $g \in L^1(\Sigma, \mathbb{R})$ and $s \in C^\infty(\Sigma, \mathbb{R})$ with $\nabla s \neq 0$ (so that $s^{-1}(\gamma)$ is a smooth curve for every γ) we have*

$$\int_\Sigma g|\nabla s| = \int_{\mathbb{R}} \left(\int_{s^{-1}(\gamma)} g \right) d\gamma.$$

From the properties of the Green function G discussed above, and in particular the smallness of the set $\mathcal{S}$ of critical points, we can effectively ignore the critical points of G by removing small balls around them - putting $g = f/|\nabla G|$, $s = G$ and $\Sigma = W(\eta)$ for some $\eta \geq 0$, we have the following corollary.

Corollary 1 *Let $G : \Omega \to \mathbb{R}$ be the Green function as defined in* (5). *Then for any $\eta > 0$ and $f \in L^1(\Sigma, \mathbb{R})$ we have*

$$\int_{W(\eta)} f = \int_\eta^\infty \left(\int_{V(\gamma)} \frac{f}{|\nabla G|} \right) d\gamma.$$

Finally we prepare an a priori estimate on the directional energy density of a map.

Lemma 3 *For $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ and $\tau \in S^1 \hookrightarrow \mathbb{R}^2$, we have the pointwise estimate*

$$|\nabla u.\tau|^2 \leq \frac{1}{2}|\nabla u|^2 + |\omega(u)|.$$

Proof. Defining $\tau_\theta = (\cos\theta, \sin\theta)$, a calculation reveals that

$$\begin{aligned} |\nabla u.\tau_\theta|^2 &= \left|\frac{\partial u}{\partial x}\right|^2 \cos^2\theta + \left|\frac{\partial u}{\partial y}\right|^2 \sin^2\theta + \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle \sin 2\theta & (7) \\ &= \frac{1}{2}|\nabla u|^2 + \frac{1}{2}\left\langle \tau_{2\theta}, \left(\left|\frac{\partial u}{\partial x}\right|^2 - \left|\frac{\partial u}{\partial y}\right|^2, 2\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle \right) \right\rangle, & (8) \end{aligned}$$

at which point the lemma follows via the Cauchy-Schwarz inequality. ■

The more geometrically inclined reader may wish to carry through the following proof of Theorem 2 with the ordinary metric on $\Omega\backslash\mathcal{S}$ scaled by a conformal factor of $1/|\nabla G|^2$. This removes the factors of $|\nabla G|$ in the calculations, and does not alter the value of $\|\nabla u\|^2_{L^2(\Omega)}$.

Proof. (Theorem 2.)

Let us assume that $u \in C^\infty(\overline{\Omega}, \mathbb{R}^2) \cap H^1(\Omega, \mathbb{R}^2)$. For general u we may reduce to this case by taking a smooth approximating sequence $\{u_n\}$ with $u_n \to u$ in $H^1(\Omega, \mathbb{R}^2)$, and analysing the limiting behaviour of the solutions to (1).

Our objective is to control $|\varphi(a)|$. We begin with Green's representation

$$\varphi(a) = \int_\Omega \det(\nabla u) G,$$

to which we apply the coarea formula Corollary 1 with $f = \det(\nabla u)G$ and $\eta = 0$ to get

$$\varphi(a) = \int_0^\infty \left(\int_{V(\gamma)} \frac{\det(\nabla u)G}{|\nabla G|} \right) d\gamma.$$

A second application of the coarea formula with $f = \det(\nabla u)$ and $\eta = \gamma$ gives us

$$\mathcal{A}(W(\gamma), u) \equiv \int_{W(\gamma)} \det(\nabla u) = \int_\gamma^\infty \left(\int_{V(\xi)} \frac{\det(\nabla u)}{|\nabla G|} \right) d\xi,$$

which we may differentiate to find that

$$-\frac{d\mathcal{A}(W(\gamma),u)}{d\gamma}=\int_{V(\gamma)}\frac{\det(\nabla u)}{|\nabla G|}.$$

Combining these two threads we obtain

$$\varphi(a)=-\int_0^\infty\frac{d\mathcal{A}(W(\gamma),u)}{d\gamma}\,\gamma\,d\gamma.$$

We would like to integrate by parts, though we must first prove that the boundary term at infinity is negligible - in other words that $\mathcal{A}(W(\gamma),u)\gamma\to 0$ as $\gamma\to\infty$. To see this, we first observe that for $x\in\mathbb{B}_r(a)$,

$$\det(\nabla u)(x)=\det(\nabla u)(a)+o(1),$$

as $r\to 0$, and then, by virtue of (6), that

$$W(\gamma)\subset\mathbb{B}_{\eta(\gamma)}(a),\qquad\text{where}\qquad\eta(\gamma)=e^{-2\pi(\gamma-\sup h)}=Ce^{-2\pi\gamma}.$$

Combining, we find that

$$\mathcal{A}(W(\gamma),u)=A(W(\gamma))\det(\nabla u)(a)+o(\eta(\gamma)^2),$$

and so

$$|\mathcal{A}(W(\gamma),u)|\gamma\leq\eta(\gamma)|\det(\nabla u)(a)|+o(1)=o(1),$$

as $\gamma\to\infty$.

We can now integrate by parts as planned, to get

$$\varphi(a)=\int_0^\infty\mathcal{A}(W(\gamma),u)d\gamma. \tag{9}$$

Applying Lemma 2 (ignoring the irrelevant case $\gamma\notin\mathcal{R}_+$) we find that

$$|\varphi(a)|\leq\frac{1}{4\pi}\int_0^\infty\mathcal{L}(V(\gamma),u)^2d\gamma. \tag{10}$$

We proceed by using the definition of $\mathcal{L}$ to estimate

$$\mathcal{L}(V(\gamma),u)\leq\int_{V(\gamma)}|\nabla u.\tau|\leq\left(\int_{V(\gamma)}|\nabla G|\right)^{\frac{1}{2}}\left(\int_{V(\gamma)}\frac{|\nabla u.\tau|^2}{|\nabla G|}\right)^{\frac{1}{2}}.$$

However, denoting outwards normal differentiation by $\frac{\partial}{\partial\nu}$, we observe that

$$\int_{V(\gamma)}|\nabla G|=-\int_{\partial W(\gamma)}\frac{\partial G}{\partial\nu}=\int_{W(\gamma)}(-\Delta G)=1,$$

which together with Lemma 3 leads to

$$\mathcal{L}(V(\gamma),u)^2\leq\int_{V(\gamma)}\frac{\frac{1}{2}|\nabla u|^2+|\omega(u)|}{|\nabla G|}.$$

Returning to (10) a final application of the coarea formula Corollary 1 with $f = \frac{1}{2}|\nabla u|^2 + |\omega(u)|$ and $\eta = 0$ delivers the concluding estimate

$$|\varphi(a)| \leq \frac{1}{4\pi}\int_0^\infty \left(\int_{V(\gamma)} \frac{\frac{1}{2}|\nabla u|^2 + |\omega(u)|}{|\nabla G|}\right) d\gamma = \frac{1}{4\pi}\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + |\omega(u)|\right) = \frac{1}{4\pi}\left(E(u) + \Theta(u)\right).$$

■

For historical reasons, we give an equivalent of Theorem 1 with u expressed in coordinates (a, b), and with coordinates (x, y) on the domain.

Corollary 2 *Suppose Ω is a bounded domain in $\mathbb{R}^2$ with regular boundary, and $a, b \in H^1(\Omega, \mathbb{R})$. Then if φ is the unique solution in $W_0^{1,1}(\Omega, \mathbb{R})$ to*

$$\begin{cases} -\Delta\varphi = a_x b_y - a_y b_x & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

then we have the estimate

$$\|\varphi\|_{L^\infty(\Omega)} \leq \frac{1}{2\pi}\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}.$$

Proof. For $\lambda > 0$ let us define $\hat{u} = (\lambda a, \frac{1}{\lambda}b)$. Observing that $\det(\nabla u) = \det(\nabla \hat{u})$ we may use Theorem 1 to estimate

$$\|\varphi\|_{L^\infty} \leq \frac{1}{4\pi}\|\nabla\hat{u}\|_{L^2}^2 \leq \frac{1}{4\pi}\left(\|\nabla(\lambda a)\|_{L^2}^2 + \|\nabla(\frac{b}{\lambda})\|_{L^2}^2\right) = \frac{1}{4\pi}\left(\lambda^2\|\nabla a\|_{L^2}^2 + \frac{1}{\lambda^2}\|\nabla b\|_{L^2}^2\right).$$

The corollary follows by setting

$$\lambda^2 = \frac{\|\nabla b\|_{L^2}}{\|\nabla a\|_{L^2}}$$

in the case that both $\|\nabla a\|_{L^2} \neq 0$ and $\|\nabla b\|_{L^2} \neq 0$, or by taking an appropriate limit if not. ■

Finally, let us note that although Theorem 1 gives us an estimate for $\|\nabla\varphi\|_{L^2}$ via the calculation

$$\|\nabla\varphi\|_{L^2(\Omega)} = \left(-\int_\Omega \varphi\,\Delta\varphi\right)^{\frac{1}{2}} \leq (\|\varphi\|_{L^\infty}\|\det(\nabla u)\|_{L^1})^{\frac{1}{2}} \leq \left(\frac{1}{4\pi}\|\nabla u\|_{L^2}^2\frac{1}{2}\|\nabla u\|_{L^2}^2\right)^{\frac{1}{2}} \leq \sqrt{\frac{1}{8\pi}}\|\nabla u\|_{L^2}^2,$$

the constant $\sqrt{\frac{1}{8\pi}}$ is not optimal (though it may be improved using Theorem 2). The optimal constant in this case is given in the following result of Ge [7].

Theorem 3 *With Ω, u and φ as in Theorem 1, we have the inequality*

$$\|\nabla\varphi\|_{L^2(\Omega)} \leq \frac{1}{8}\sqrt{\frac{3}{\pi}}\|\nabla u\|_{L^2(\Omega)}^2.$$

4 Generalisations of the domain

Although we have only considered the case in which Ω is a domain in $\mathbb{R}^2$, we remark that the proof carries through in exactly the same way if Ω is a compact Riemannian surface with boundary. Note now that the equation

$$-\Delta\varphi = \det(\nabla u) \qquad \text{in } \Omega \tag{11}$$

is to be satisfied with respect to local isothermal coordinates, and that this is a well defined notion owing to the conformal invariance of (11). Moreover the quantities E and Θ do not depend on the local isothermal coordinates with which they are calculated.

We may also extend to the case that Ω is a compact Riemannian surface without boundary. Without boundary conditions, a solution of (11) is now only unique up to a constant, and so what we wish to control is the oscillation of φ

$$\operatorname{osc}(\varphi) = \operatorname*{ess\,sup}_{x,y\in\Omega} |\varphi(x) - \varphi(y)|.$$

Although the statement of the following result appears to require a Riemannian metric on Ω, we observe that all the quantities and notions involved are dependent only on its conformal structure, and we therefore allow Ω to be merely a Riemann surface.

Theorem 4 *Suppose Ω is a compact Riemann surface and $u \in H^1(\Omega, \mathbb{R}^2)$. Then if φ is a solution in $W^{1,1}(\Omega, \mathbb{R})$ to*

$$-\Delta\varphi = \det(\nabla u) \qquad \text{in } \Omega,$$

we have the estimate

$$\operatorname{osc}(\varphi) \le \frac{1}{4\pi}\left(E(u) + \Theta(u)\right). \tag{12}$$

Proof. As in the proof of Theorem 2 we need only consider the case that u is smooth. Let $x \in \Omega$ be a point at which φ attains its minimum value, which we may assume to be zero. Fixing some $\varepsilon > 0$, let us then choose a small ball $B \subset \Omega$ around x such that $\varphi|_B \le \varepsilon$.

We will compare φ with the unique solution v of

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega\backslash B, \\ v = \varphi & \text{on } \partial B, \end{cases}$$

which satisfies $v \le \varepsilon$ throughout $\Omega\backslash B$. Indeed applying the above-mentioned extension of Theorem 2 on the domain $\Omega\backslash B$ to $\varphi - v$, we see that

$$\|\varphi - v\|_{L^\infty(\Omega\backslash B)} \le \frac{1}{4\pi}\left(E(u) + \Theta(u)\right),$$

and hence that

$$0 \le \varphi \le \varepsilon + \frac{1}{4\pi}\left(E(u) + \Theta(u)\right).$$

Since ε was an arbitrary positive number, the proof is complete. ∎

5 Immersed surfaces of constant mean curvature

With the extensions of our results discussed in the previous chapter, we can obtain restrictions of immersed surfaces of constant mean curvature in $\mathbb{R}^3$. This application was inspired by Wente's use of his original inequality.

Theorem 5 *Let Ω be a compact orientable surface and suppose $X : \Omega \to \mathbb{R}^3$ describes an immersed surface of constant mean curvature H. Hence turning Ω into a Riemann surface with the conformal structure which makes X conformal, the equation*

$$-\Delta X = 2H \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} \tag{13}$$

is satisfied where $x + iy$ is a local complex coordinate on Ω. Then we have the inequality

$$Diameter(X(\Omega)) \leq \frac{Area(X(\Omega))H}{2\pi}. \tag{14}$$

This inequality is optimal in the sense that equality holds when X is the inclusion $S^2 \hookrightarrow \mathbb{R}^3$, in which case both sides are equal to 2.

Proof. Let us write $X = (X^1, X^2, X^3)$ and $u = (X^1, X^2)$. As X is conformal, we see that

$$0 = \frac{1}{2}\left(\left|\frac{\partial X}{\partial x}\right|^2 - \left|\frac{\partial X}{\partial y}\right|^2 - 2i\left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}\right\rangle\right) = \omega(u) + \frac{1}{2}\left(\left|\frac{\partial X^3}{\partial x}\right|^2 - \left|\frac{\partial X^3}{\partial y}\right|^2 - 2i\frac{\partial X^3}{\partial x}\frac{\partial X^3}{\partial y}\right),$$

and hence that

$$|\omega(u)| = \frac{1}{2}|\nabla X^3|^2.$$

In particular,

$$\Theta(u) = E(X^3). \tag{15}$$

Now let us take the third component of the equation (13)

$$-\Delta X^3 = 2H \det(\nabla u).$$

Applying Theorem 4 and using (15) we see that

$$\operatorname{osc}(X^3) \leq 2H\frac{1}{4\pi}\left(E(u) + \Theta(u)\right) = \frac{H}{2\pi}\left(E(u) + E(X^3)\right) = \frac{H}{2\pi}E(X) = \frac{Area(X(\Omega))H}{2\pi}$$

where the final equality uses again the conformality of X. Without loss of generality, the direction of the third component in $\mathbb{R}^3$ is the direction of maximum oscillation, and so the proof is complete. ∎

6 Generalisations of the target

It is clear from the proof given in section 3 that the reason for the compensation phenomena discussed in this paper is the existence of an isoperimetric inequality on the target $\mathbb{R}^2$ of the map u. We would therefore expect to be able to generalise the target providing we preserve this property.

Let us consider the case that the target of u is a compact Riemannian surface without boundary $\mathcal{N}$, and let us denote its volume form by Λ. The problem we would now like to solve is

$$\begin{cases} (-\Delta\varphi)dx \wedge dy = u^*(\Lambda) & \text{in } \Omega, \\ \varphi = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{16}$$

Let us assume that $\Omega = D$. How we progress depends on the genus of $\mathcal{N}$.

If the genus of $\mathcal{N}$ is at least 1, then an isoperimetric inequality holds, and our results extend. Indeed, we may lift the map u to the universal cover of $\mathcal{N}$, which is either $\mathbb{R}^2$ or D with an appropriately periodic metric, and apply our results as before. Of course we can expect a worse constant in the inequalities than we had for the flat metric on the target $\mathbb{R}^2$.

However, if $\mathcal{N}$ is S^2, no suitable isoperimetric inequality holds and the results fail as stated. A counterexample is the map $u : D \to S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (using stereographic projection) given by $u(x) = \lambda x$ for $\lambda > 0$. We find that $E(u) = Area(u(D)) < Area(S^2) = 4\pi$ (note that u is conformal) for any $\lambda > 0$ but that as $\lambda \to \infty$ the value of $\varphi(0)$ tends to infinity. This may be seen by calculation, but is most easily seen using the representation (9) in the proof of Theorem 2.

In contrast, if $\mathcal{N}$ is the round 2-sphere and $E(u) < 4\pi$ then the isoperimetric inequality on the sphere $L^2 \geq A(4\pi - A)$ gives us the inequality

$$L^2 \geq A(4\pi - E(u)),$$

and our results extend with the constant $\frac{1}{4\pi}$ in Theorems 1 and 2 replaced by $\frac{1}{4\pi - E(u)}$.

7 Acknowledgements

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