Ricci flow and Ricci Limit Spaces

Peter M. Topping

6 May 2020

Preface

In these lecture notes I hope to describe some of the developments in Ricci flow theory from the past decade or so in a manner accessible to starting graduate students. I will be most concerned with the understanding of Ricci flows that are permitted to have unbounded curvature in the sense that the curvature can blow up as we wander off to spatial infinity and/or as we decrease time to some singular time. This is an area that has seen substantial recent progress, and it has long been understood that a sufficiently developed theory will be useful in settling important open problems in geometry, particularly to the subject of understanding the topology of manifolds that satisfy certain curvature restrictions. In these notes I will outline an application of the theory as developed so far, but instead to the theory of so-called Ricci limit spaces, as developed particularly by Cheeger-Colding around 20 years ago.

Much of the work I describe is joint work with Miles Simon. Some of the general ideas will be explained in the context of two-dimensional flows – the case in which these were developed – which is partly joint work with Gregor Giesen.

These notes have been written for a summer school at Cetraro in 2018 and I would like to thank Matt Gursky and Andrea Malchiodi for the invitation to explain this theory. A subset of the lectures were given at the Max Planck Institute for Mathematics in Bonn in 2017, and I would like to thank Karl-Theodor Sturm for that invitation.

Warwick, 21 December 2018   *Peter Topping*

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1 Introductory material

1.1 Ricci flow basics

Traditionally, the Ricci flow has been posed as follows. Given a Riemannian manifold (M, g_0), we ask whether we can find a smooth one-parameter family of Riemannian metrics $g(t)$ for $t \in [0, T)$ such that $g(0) = g_0$ and satisfying the PDE

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_g, \tag{1}$$

where Ric_g is the Ricci curvature of g. If you are new to this subject then you should view the right-hand side of (1) morally as a type of Laplacian of g, and the equation as an essentially parabolic PDE for the metric g. One of the simplest examples that is not completely trivial is the shrinking sphere: if (M, g_0) is the round unit sphere of dimension n, then one can easily check that

$$g(t) = \big(1 - 2(n-1)t\big)g_0 \tag{2}$$

is a solution for $t \in [0, \frac{1}{2(n-1)})$. Another standard example that is important to digest is the dumbbell example: If we connect two round three-spheres by a thin neck $\varepsilon S^2 \times [0, 1]$ to give a nonround three-sphere, then Ricci flow will deform it by shrinking the thin neck, which will degenerate before much happens to the two original three-spheres. See [34, Section 1.3.2] for a more involved discussion and pictures.

The Ricci flow is a little simpler if the evolving manifold is Kähler, and the simplest of all is in the lowest possible dimension, i.e. $\dim M = 2$, which corresponds to one complex-dimensional *Kähler Ricci flow*. Then we can write $\mathrm{Ric}_g = Kg$, where K is the Gauss curvature, and the flow can be seen to deform the metric conformally.

The Ricci flow was introduced by Hamilton in 1982 [19] in order to prove the following theorem.[1]

Theorem 1 (Hamilton [19].) *If (M, g_0) is a closed, simply connected Riemannian 3-manifold with* $\mathrm{Ric} > 0$*, then it is diffeomorphic to* S^3*.*

The rough strategy is as follows. First, take (M, g_0) and run the Ricci flow. Second, show that the condition $\mathrm{Ric} > 0$ is preserved for each of the subsequent metrics $g(t)$. (It is worth remarking that a local form of this preservation of lower Ricci control has recently been proved [32] and will be pivotal later on; see Lemma 5.) Finally, show that after appropriate scaling, $(M, g(t))$ converges to a manifold of constant sectional curvature as time increases. This must then be S^3.

For more details of this argument, from a modern point of view, and a more involved introduction to Ricci flow, see [34].

This application is a model for many of the most remarkable subsequent applications of Ricci flow. That is, if we assume that a manifold satisfies a certain curvature condition, then often we can use Ricci flow to deform it smoothly to something whose topology we can identify (or to a *collection* of objects whose topology we can identify). The famous work of Perelman goes one step further, in some sense, by carrying out that programme in three dimensions without any curvature hypothesis whatsoever, thus proving the Poincaré conjecture and Thurston's geometrisation conjecture as originally envisaged by Hamilton and Yau.

In contrast, in these lectures we will use Ricci flow for a completely different task. We will have to pose Ricci flow in a different way, specifying initial data that is a lot rougher than a Riemannian metric, and as a result we will get a refined description of so-called Ricci limit spaces as we explain later. However, the developments in the Ricci flow theory that permit these new applications seem also likely to lead to new applications of the traditional type, i.e. assume a curvature condition and deduce a topological conclusion.

1.2   Traditional theory - closed or bounded curvature

The main difference between the theory we will see in these lectures and the traditional theory of Ricci flow is that most traditional applications concern *closed* manifolds, whereas we are interested in general noncompact manifolds. In the closed case, Hamilton proved existence of a unique solution to Ricci flow, as posed above. In the non-closed case, things become mysterious and involved, unless one reduces essentially to a situation like the closed case by imposing a constraint of *bounded curvature*.

Theorem 2 (Shi [30], Hamilton [19].) *Suppose (M, g_0) is a complete* **bounded curvature** *Riemannian manifold of arbitrary dimension – let's say* $|\mathrm{Rm}|_{g_0} \leq K$ *for some* $K > 0$*. Then there exists a Ricci flow $g(t)$ on M for $t \in [0, \frac{1}{16K}]$ with $g(0) = g_0$ such that* $|\mathrm{Rm}|_{g(t)} \leq 2K$ *for each t.*

Here we are writing Rm for the full curvature tensor. For fixed dimension, controlling $|\mathrm{Rm}|$ is a little like controlling the magnitude of the sectional curvature.

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[1]When we write $\mathrm{Ric} > 0$, we mean that $\mathrm{Ric}(X, X) > 0$ for every nonzero tangent vector X.

Because the Ricci flow of Theorem 2 has bounded curvature, it will automatically inherit the completeness of g_0, i.e. at each time t, the metric $g(t)$ will be complete. This is because the lengths of paths can only grow or shrink exponentially:

Lemma 1 *Given a Ricci flow $g(t)$, $t \in [0, T]$, with $-M_1 \leq \text{Ric}_{g(t)} \leq M_2$, we have*

$$e^{-2M_2 t} g(0) \leq g(t) \leq e^{2M_1 t} g(0),$$

for each t.

Here, when we write $\text{Ric}_g \leq K$, say, we mean that $\text{Ric}_g \leq Kg$ as bilinear forms.

Thus, distances can only expand by a factor at most $e^{M_1 t}$, and shrink by a factor no more extreme than $e^{-M_2 t}$ after a time t. This lemma is a simple computation. For example, for a fixed tangent vector X, we see that

$$\frac{\partial}{\partial t} \ln g(X, X) \leq 2M_1.$$

We also remark that, as proved by Chen-Zhu [8] and simplified by Kotschwar [24], the flow of Theorem 2 is unique in the sense that two smooth bounded-curvature Ricci flows starting at the same smooth, complete, bounded curvature initial metric must agree. We emphasise that by *bounded-curvature* Ricci flow we mean that there is a t-independent curvature bound, not that the curvature of each $g(t)$ is bounded, otherwise the uniqueness fails (see, e.g., Example 1 below).

Remark 1 *For how long does Shi's flow live? The theorem as we gave it tells us that we can at least flow for a time $1/(16K)$, which is, by chance, independent of the dimension of M. But the theorem can be iterated to give a flow that lives until the curvature blows up, and because the theorem provides a curvature bound at the final time, the iterated flow must live for at least a time $1/(16K) + 1/(32K) + 1/(64K) + \cdots = 1/(8K)$. None of these explicit times are claimed to be optimal. Traditionally one called a flow whose curvature blew up at a finite end time a* maximal *flow. However, it was shown in [3] and [17] that one can have flows in this context whose curvature blows up only at spatial infinity at the end time, and which can be extended beyond to exist for all time (see also Section 2.3).*

2   Noncompact and unbounded curvature case

2.1   Difficulties of the noncompact case; examples to reboot intuition

Shi's theorem 2 gives a fine existence statement in the special case that we wish to flow a manifold with bounded curvature. Unfortunately, this is an unreasonable assumption in virtually all applications when we are working on a noncompact manifold. Thus we would like to consider the problem of starting a Ricci flow with a more general manifold, particularly one of unbounded curvature, or more generally considering flows that themselves have unbounded curvature. Unfortunately, such flows can be very weird. They can jump back and forth between being complete and being incomplete.

They can exhibit unusual nonuniqueness properties. They tend to render the ever-useful maximum principle void. They will try to jump to a different underlying manifold if given the chance. Ultimately, they will simply fail to exist in certain cases.

Let us elaborate on some of these vague statements with some specific examples.

Example 1 *If (M, g_0) is the two-dimensional hyperbolic plane, then there is a simple expanding solution $g(t) = (1 + 2t)g_0$ for $t \in [0, \infty)$ that is the direct analogue of the shrinking sphere that we saw in (2). This is the solution that would be given by Shi's theorem 2. However, more surprisingly, there exist infinitely many other smooth Ricci flows also starting with the hyperbolic plane, and with bounded curvature for $t \in (\varepsilon, \infty)$, for any $\varepsilon > 0$. As we will see, these solutions are necessarily not complete, and thus must have unbounded curvature as $t \downarrow 0$, although being smooth they trivially have bounded curvature as $t \downarrow 0$ over any compact subset of the domain. Intuitively they instantly peel away at the boundary, though not necessarily in a rotationally symmetric way. We'll construct them in a moment.*

Example 2 *There exists a rotationally symmetric, complete, bounded curvature conformal metric g_0 on $\mathbb{R}^2$, and a smooth rotationally symmetric Ricci flow $g(t)$ for $t \in [0, 1)$ with $g(0) = g_0$, such that the distance from the origin to infinity (i.e. the length of a radial ray) behaves like $-\log t$ (i.e. it is bounded from above and below by $-\log t$ times appropriate positive constants). In particular, the boundary at infinity is uniformly sucked in making it instantaneously incomplete. The curvature at infinity behaves like t^{-2} (again controlled, up to a constant factor, from above and below).*

Before continuing with our examples, let's construct the two we have already seen. They both have the same building block, which is the concept of a Ricci flow that contracts a hyperbolic cusp. We showed in [36] (though the best way of seeing this now is to use the sharp L^1-L^∞ smoothing estimate proved in [38]) that if we are given a surface with a hyperbolic cusp, then one way of flowing it is to add a point at infinity and let it contract.

To see a precise instance of this, consider the punctured two-dimensional disc $D \setminus \{0\}$, equipped with the unique (conformal) complete hyperbolic metric g_h. That is,

$$g_h = \frac{1}{r^2 \log r^2}(dx^2 + dy^2),$$

where $r^2 = x^2 + y^2$. This has a hyperbolic cusp at the origin. Similarly to what we have seen before, there is an explicit Ricci flow starting with g_h given by $g(t) = (1 + 2t)g_h$, and this is the flow that Shi's theorem 2 would give. However, in [39] we constructed a smooth Ricci flow $g(t)$ on the whole disc D, for $t \in (0, \infty)$ such that $g(t) \to g_h$ smoothly locally on $D \setminus \{0\}$ as $t \downarrow 0$. This flow we can restrict to $D \setminus \{0\}$, giving a smooth Ricci flow on $D \setminus \{0\}$ for $t \in [0, \infty)$ such that $g(0) = g_h$, but so that $g(t)$ is incomplete at the origin for $t > 0$.

To construct Example 1, we merely have to lift this flow on $D \setminus \{0\}$ to its universal cover D.

Meanwhile, to construct Example 2, we take a slight variant of the $D \setminus \{0\}$ example above by taking a complete rotationally symmetric metric g_0 on a punctured 2-sphere,

with a hyperbolic cusp. Again, we can find a rotationally symmetric Ricci flow on the whole 2-sphere for $t \in (0, \varepsilon)$, that converges to g_0 on the *punctured* sphere as $t \downarrow 0$, and we can restrict this flow to the punctured sphere, which is conformally $\mathbb{R}^2$. The details and the claimed asymptotics can be found in [36, 38, 39].

In both Examples 1 and 2 we see incompleteness in the flow itself. In the theory we'll see later it turns out to be important to be able to flow *locally*, in which case even the initial data is incomplete. The following example shows how bad that can be.

Example 3 (Taken from [17].) *For all $\varepsilon > 0$, there exists a Ricci flow $g(t)$ on the unit two-dimensional disc D in the plane, defined for $t \in [0, \varepsilon)$, such that $g(0)$ is the standard flat metric corresponding to the unit disc, but*

$$\inf_D K_{g(t)} \to \infty \qquad as \ t \uparrow \varepsilon,$$

and $\mathrm{Vol}_{g(t)} D \to 0$ *as* $t \uparrow \varepsilon$.

If you've ever seen Perelman's famous Pseudolocality theorem before [28, §10], then you might mull over why the previous example does not provide a counterexample.

To construct Example 3, consider a 2-sphere that geometrically looks like a cylinder of length 2 and small circumference, with small hemispherical caps at the ends. See Figure 1. We make the circumference small so that the total area is $8\pi\varepsilon \ll 1$. Thus the circumference of the cylinder is a little below $4\pi\varepsilon$. A theorem of Hamilton [20] tells us that the subsequent Ricci flow $\tilde{g}(t)$ will exist for a time ε, and at that time the volume converges to zero and the curvature blows up everywhere. Just before time ε, the surface looks like a tiny round sphere. Certainly it is easy to see that the flow cannot exist any longer than this time since we can compute that

$$\frac{d}{dt} \mathrm{Vol}_{\tilde{g}(t)} S^2 = \int_{S^2} \frac{1}{2} \mathrm{tr} \frac{\partial \tilde{g}}{\partial t} dV_{\tilde{g}} = -2 \int_{S^2} K_{\tilde{g}} \, dV_{\tilde{g}} = -8\pi,$$

by Gauss-Bonnet. To obtain the Ricci flow $g(t)$ on D of the example, we pull back $\tilde{g}(t)$ under the exponential map $\exp_p$, defined with respect to the initial metric (not the time t metric), restricted to the unit disc D in $T_P M$, where p is any point midway along the cylinder.
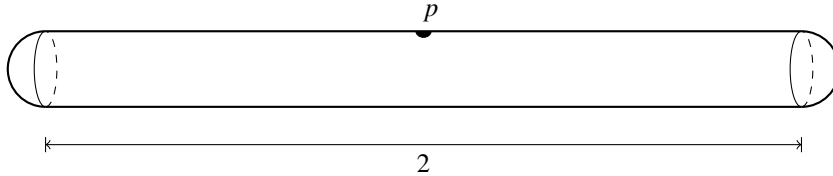

Figure 1: Thin cylinder with caps

Example 4 *Conjecturally, we have an example of a smooth complete three-manifold for which there does not exist a smooth Ricci flow solution even for a short time.*

We construct this by connecting countably many three-spheres together by necks that progressively become longer and thinner. Each neck should want to pinch in a time that converges to zero as the necks get thinner and longer. See Figure 2 for a construction on $S^2 \times \mathbb{R}$. Closing up the left-hand side would give an example on $\mathbb{R}^3$. This construction can be made even under the assumption that the Ricci curvature is bounded below. Making this example rigorous is currently beyond existing technology. One would need (for example) a pseudolocality theorem that is valid in the presence of unbounded curvature, but this is unavailable currently.
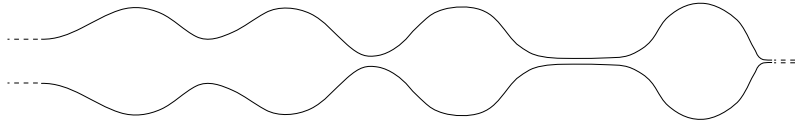

Figure 2: $S^2 \times \mathbb{R}$ with thinner and thinner necks

2.2 What is known in the noncompact case

There are several situations in which we can say a lot about Ricci flows even though the curvature can be unbounded.

The first theory to be developed was the theory for two-dimensional underlying manifolds, which is also the one complex-dimensional Kähler case. This is the easiest case, and gives the strongest results in terms of existence, uniqueness and asymptotic behaviour. The approach developed for this case has turned out to be important to understand the higher dimensional cases. We will see the main results in Section 2.3.

One way that higher dimensional Ricci flow can be made to have a clean existence theory for which there are no 'unnecessary' singularities, just as there are no unnecessary singularities in the general two-dimensional case, is to impose a sufficiently strong positivity of curvature condition. In particular, Cabezas-Rivas and Wilking [3] developed such a theory in the case that the initial manifold is complete with nonnegative complex sectional curvature.

Recently, several higher-dimensional cases that are substantially more singular have been treated. We will be looking particularly at the case in which the initial manifold is complete and three-dimensional, with a lower Ricci curvature bound. Note that Example 4 fits into this category, so we need to find a novel way of flowing. The solution is to flow locally, echoing the (quite different) work of Deane Yang [42] and the concepts of Ricci flow on manifolds with boundary (see [15] and the earlier works referenced therein). Once a solution has been constructed, and we have strong-enough control on the evolution of the Ricci curvature and the distance function, we will obtain applications to the theory of Ricci limit spaces. See our work with Miles Simon [33], [22] and Section 4 for the results, and Section 6 for some applications.

In [4], Bamler, Cabezas-Rivas and Wilking describe a heat kernel method for controlling the curvature in certain cases, which allows the theory of [3] mentioned above to be generalised from nonnegative curvature to curvature bounded below, provided

one assumes a global noncollapsing hypothesis. In [25], this work was combined with that of [33] to generalise both works. See also [27].

Finally, another interesting case in which to consider unbounded curvature is that of Kähler Ricci flow, though we will not attempt to survey that field here other than the one complex-dimensional case.

2.3 The two-dimensional theory

In 2D we will be able to flow any smooth initial metric. We do not require the curvature to be bounded, so Shi's theorem 2 does not help. We do not even require that the initial metric is complete. Even if we can settle the existence problem, the level of generality here poses a potentially serious problem of nonuniqueness of solutions, and to understand this let us consider the PDE more carefully.

We already saw that in 2D, the Ricci flow is the equation

$$\frac{\partial g}{\partial t} = -2Kg,$$

where K is the Gauss curvature of g. If we choose local isothermal coordinates x, y, i.e. so the metric g can be written $e^{2u}(dx^2 + dy^2)$ for a locally-defined scalar function u, then we can write the Gauss curvature as $K = -e^{-2u}\Delta u$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is only locally defined. Thus the Ricci flow can be written locally as

$$\frac{\partial u}{\partial t} = e^{-2u}\Delta u, \tag{3}$$

which is the so-called logarithmic fast diffusion equation (up to a change of variables). Given such a parabolic equation, the accepted wisdom is that one should specify initial and boundary data in order to obtain existence and uniqueness. However here we abandon the idea of specifying boundary data in the traditional sense and instead replace it with the the global condition that the flow should be complete for all positive times. Miraculously this completeness condition is just weak enough to permit existence, and just strong enough to force uniqueness.

Theorem 3 *Given any smooth (connected) Riemannian surface* (M, g_0)*, possibly incomplete and possibly with unbounded curvature, define* $T \in (0, \infty]$ *depending on the conformal type of* (M, g_0) *by*

$$T := \begin{cases} \frac{1}{4\pi} \operatorname{Vol}_{g_0} M & \text{if } (M, g_0) \cong \mathbb{C} \text{ or } \mathbb{R}P^2, \\ \frac{1}{8\pi} \operatorname{Vol}_{g_0} M & \text{if } (M, g_0) \cong \mathcal{S}^2, \\ \infty & \text{otherwise.} \end{cases}$$

Then there exists a Ricci flow $g(t)$ *on* M *for* $t \in [0, T)$ *such that*

1. $g(0) = g_0$, *and*

2. $g(t)$ *is complete for all* $t \in (0, T)$.

If $T < \infty$ then $\mathrm{Vol}_{g(t)} M \to 0$ as $t \uparrow T$. Moreover, this flow is unique in the sense that if $\tilde{g}(t)$ is any other complete Ricci flow on M with $\tilde{g}(0) = g_0$, existing for $t \in [0, \tilde{T})$, then $\tilde{T} \leq T$ and $\tilde{g}(t) = g(t)$ for all $t \in [0, \tilde{T})$.

The existence of this theorem was proved with Giesen in [16], following earlier work in [35]. The compact case, where the metric is automatically complete and the curvature bounded, was already proved by Hamilton and Chow [20, 10]. Other prior results include the extensive theory of the logarithmic fast diffusion equation, typically posed on $\mathbb{R}^2$, for example [12, 14, 40]. The uniqueness assertion was proved in [37] following a large number of prior partial results (see [37] for details).

   We illustrate the theorem with two simple examples.

Example 5 *Suppose (M, g_0) is the flat unit disc. Finding a Ricci flow starting with (M, g_0) is equivalent to solving (3) with the initial condition $u|_{t=0} = 0$. There is the obvious solution that has $u \equiv 0$ for all time, and we also saw the solution of Example 3. In general, standard PDE theory tells us that if we specify boundary data on the disc as a function of time (and space) then there exists a unique solution that approaches that data at the boundary. None of these solutions give complete metrics, so Theorem 3 is giving another solution that blows up at the boundary. For this instantaneously complete flow, at small positive time $t > 0$ the flat metric gets adjusted asymptotically near the boundary to the Poincaré metric scaled to have constant curvature $-\frac{1}{2t}$, but remains almost flat on the interior. See Figure 3.*
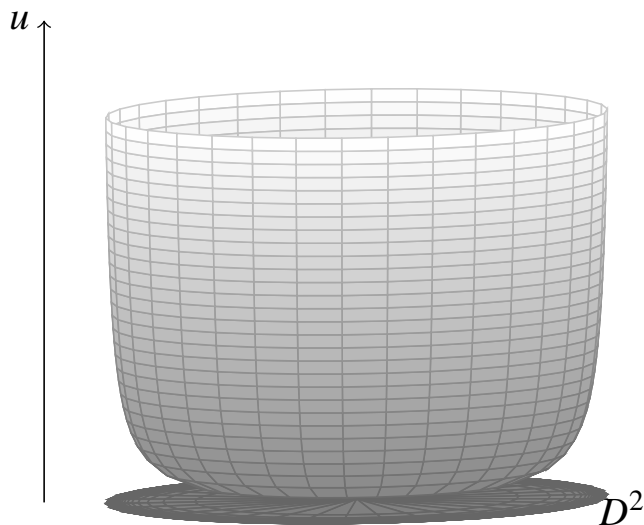

Figure 3: Metric for positive time stretches at infinity in Example 5

Example 6 *Alternatively, suppose that (M, g_0) is the flat Euclidean plane with one point removed to make it incomplete. This time the instantaneously complete solution stretches the metric immediately near the puncture to make it asymptotically like a hyperbolic cusp scaled to have curvature $-\frac{1}{2t}$. See Figure 4.*
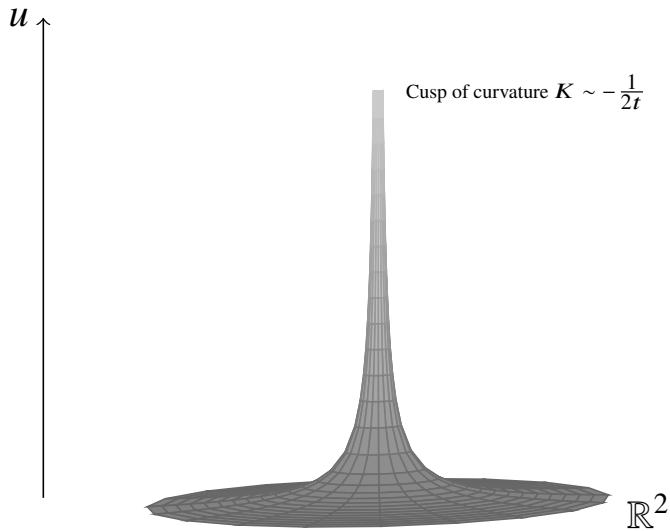

Figure 4: Puncture turns into a hyperbolic cusp in Example 6

It is important to appreciate that the flows themselves in Theorem 3 can have unbounded curvature at spatial infinity at each time [17] as well as initially. More precisely, the curvature can be unbounded above; it follows from [9] that our solutions always satisfy the Gauss curvature estimate $K_{g(t)} \geq -\frac{1}{2t}$. As alluded to in Remark 1, they might alternatively start as complete metrics of bounded curvature (so Shi's theorem 2 applies), develop blow-up of curvature in finite time (at spatial infinity) but then continue smoothly for all time [18]. These types of behaviour can even happen under strong positivity of curvature conditions, if one permits higher dimensional flows [3].

2.4 Proof ideas for the 2D theory

It will be helpful to outline some of the ideas in the proof of the existence of Theorem 3; we'll emphasise the ones that ended up being useful in the higher dimensional theory. A rough principle is that we would like to bring Shi's result into play despite the unboundedness of the curvature. We achieve this through the following steps, where

we are free to assume that M is not closed (or we would be in the classical situation of Hamilton).

1. First we take an exhaustion of M by sets $\Omega_1 \subset \Omega_2 \subset \cdots \subset M$, each compactly contained in M and with smooth boundary.

2. Next, for each $i \in \mathbb{N}$ we restrict g_0 to Ω_i and blow up the metric conformally near the boundary to make the metric complete and of very large constant negative curvature near the boundary. Call the resulting metric g_i.

3. Since each (Ω_i, g_i) is complete, with bounded curvature, we can apply Shi's theorem 2 to obtain Ricci flows $g_i(t)$.

4. Next, we prove that each $g_i(t)$ exists for a uniform time (in fact for *all* time in this case) and enjoys uniform curvature estimates (independent of i).

5. The uniform estimates from the previous step give us compactness: a subsequence converges to a limit flow $g(t)$, this time on the whole of M.

6. Finally we argue that the limit flow $g(t)$ is complete for all positive times, and has g_0 as initial data.

We'll reuse the same strategy for higher dimensions, but the results are a little different.

3 Ricci curvature and Ricci limit spaces

Most of the remaining material we will cover in these lectures will be geared to understanding Ricci flow in higher dimensions in the unbounded curvature case. The new ideas we wish to explain are best illustrated in the case of three-dimensional manifolds that have a lower Ricci curvature bound. These ideas generalise to higher dimensions if we assume correspondingly stronger curvature bounds. This theory has a number of potential applications to the understanding of the topology and geometry of manifolds with lower curvature bounds, but the application on which we focus in these lectures will be to so-called Ricci limit spaces, which we shall shortly define as some sort of limits of Riemannian manifolds with lower Ricci bounds. Before that, we survey some of the most basic consequences of a manifold having a lower Ricci bound.

3.1 Volume comparison

Ricci curvature controls the growth of the volume of balls as the radius increases. The simplest result of this form is:

Theorem 4 (Bishop-Gromov, special case.) *Given a complete Riemannian n-manifold (M, g) with* $\mathrm{Ric} \geq 0$, *and a point $x_0 \in M$, the* volume ratio *function*

$$r \mapsto \frac{\mathrm{VolB}_g(x_0, r)}{r^n} \tag{4}$$

is a weakly decreasing (i.e. nonincreasing) function of $r > 0$.

Since the limit of the function (4) as $r \downarrow 0$ is necessarily equal to ω_n, the volume of the unit ball in Euclidean n-space, this implies that $\text{VolB}_g(x_0, r) \leq \omega_n r^n$. But of course, the theorem also gives us *lower* volume bounds: For $0 < r \leq R$, we have

$$\text{VolB}_g(x_0, r) \geq (r/R)^n \, \text{VolB}_g(x_0, R).$$

In fact, we do not even need to have the same centres here. Later it will be useful to observe that for any $x \in M$ we can set $R = r + 1 + d(x, x_0)$ and estimate, as in Figure 5, that

$$\frac{\text{VolB}_g(x, r)}{r^n} \geq R^{-n} \, \text{VolB}_g(x, R) \geq R^{-n} \, \text{VolB}_g(x_0, 1), \tag{5}$$

and thus knowing that one unit ball $\text{VolB}_g(x_0, 1)$ has a specific lower volume bound then gives a lower bound for the volume of any other ball, though that lower bound gets weaker as x drifts far from x_0.
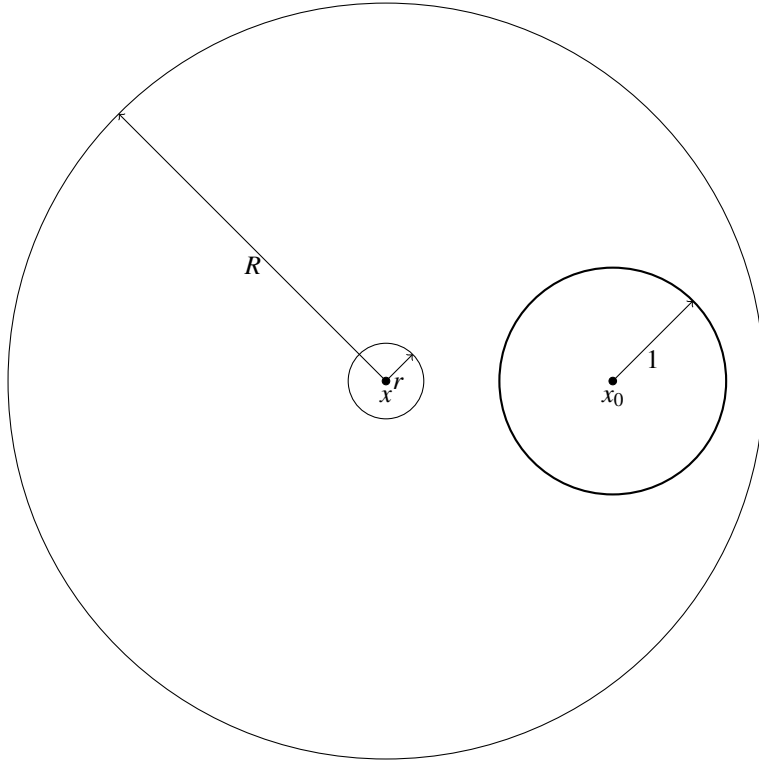

Figure 5: Comparing the volumes of nonconcentric balls.

Theorem 4 also allows us to define a finite *asymptotic volume ratio*

$$\text{AVR} := \lim_{r \to \infty} \frac{\text{VolB}_g(x_0, r)}{r^n}.$$

In practice it is often particularly relevant whether AVR equals zero or is positive.

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Remark 2 *The Bishop-Gromov theorem as stated extends easily to manifolds with more general Ricci lower bounds* $\text{Ric} \geq -\alpha_0$*, but now the volume of a ball of radius r can grow as fast as it does in the complete model space of constant sectional curvature where the constant is chosen so that the Ricci curvature equals* $-\alpha_0$*.*

3.2 What is a Ricci limit space?

Suppose that (M_i^n, g_i, x_i) is a sequence of pointed complete Riemannian manifolds, i.e. a sequence of complete Riemannian n-manifolds (M_i^n, g_i) together with points $x_i \in M_i$, and suppose that the sequence satisfies the properties that

$$\begin{cases} \text{Ric}_{g_i} \geq -\alpha_0 \\ \text{VolB}_{g_i}(x_i, 1) \geq v_0 > 0. \end{cases} \tag{6}$$

Such a sequence of manifolds with lower Ricci curvature control has some sort of compactness property. That is, a subsequence will converge in a suitable sense. What it converges to will be more general than a Riemannian manifold, but will be a very special type of metric space. These limits will be what we mean by Ricci limit space. Before giving any details, let's imagine an example.

Consider the two-dimensional cone $C := (\mathbb{R}^2 \setminus \{0\}, dr^2 + \alpha r^2 d\theta^2)$, where $\alpha \in (0, 1)$ and we are working in polar coordinates. Intuitively there is a delta function of positive curvature at the vertex, i.e. at the origin, scaled by a factor $2\pi(1 - \alpha)$. If we view C as a metric space, and add in the origin, we have an example of a Ricci limit space with the origin as a singular point (these terms will be defined in a moment). This space will be the limit of a sequence of surfaces (M_i, g_i) that are constructed by smoothing out the cone point of C by a smaller and smaller amount as i gets large. Each of these surfaces can be assumed to have nonnegative Gauss curvature, and thus Ricci curvature bounded below. Moreover, if we let x_i be the origin for each i, we can arrange that $\text{VolB}_{g_i}(x_i, 1)$ converges to the area of the unit ball in C centred at the origin, i.e. to $\pi\alpha$, which is positive. Thus the conditions (6) will be satisfied, and we see that the cone C is a Ricci limit space; see Figure 6.
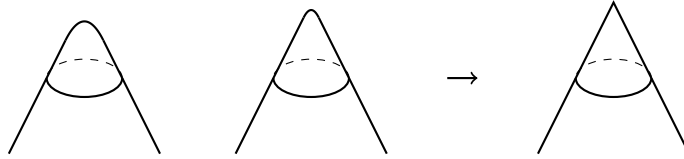
Figure 6: Smoothed out cones approaching a Ricci limit space.

So far this is very vague. The notion of convergence we are considering here is that of *pointed Gromov-Hausdorff* convergence. Before defining that, it makes sense to recall the usual notion of Gromov-Hausdorff convergence. A sequence of metric spaces (X_i, d_i) converges to a limit metric space (X, d) in the Gromov-Hausdorff sense if for all $\varepsilon > 0$ and for sufficiently large i (depending also on ε) there exists a map $f : X_i \to X$ such that

1. for all $x, y \in X_i$ we have

$$|d(f(x), f(y)) - d_i(x, y)| < \varepsilon,$$

2. for every $z \in X$, there exists $x \in X_i$ such that

$$d(f(x), z) < \varepsilon.$$

This is clearly a very weak notion of closeness. For example, a finer and finer lattice of points in the plane would converge to the plane itself.

We'll loosely call any such map f an ε-Gromov-Hausdorff approximation.

The general definition of *pointed* Gromov-Hausdorff convergence is similar: Given a complete metric space (X, d) and a point $x_\infty \in X$, i.e. a pointed metric space (X, d, x_∞), we say that a sequence of pointed metric spaces (X_i, d_i, x_i) converges to (X, d, x_∞) in the pointed Gromov-Hausdorff sense if given any radius $r > 0$ and any $\varepsilon > 0$, then for sufficiently large i (depending also on r and ε) there exists a map $f : B_{d_i}(x_i, r) \to X$ such that

1. $f(x_i) = x_\infty$,

2. for all $x, y \in B_{d_i}(x_i, r)$ we have

$$|d(f(x), f(y)) - d_i(x, y)| < \varepsilon,$$

3. the ε-neighbourhood of the image $f(B_{d_i}(x_i, r))$ contains the ball $B_d(x_\infty, r - \varepsilon) \subset X$.

In practice, we will essentially only consider this definition in the special case that the metric spaces (X_i, d_i, x_i) are in fact Riemannian manifolds (M_i^n, g_i, x_i). This restriction removes some annoying pathology by virtue of restricting us to working with length spaces and boundedly compact spaces (see [2] for definitions).

Indeed, working only with boundedly compact spaces, our limit (X, d, x_∞) is unique in the sense that any other (complete) limit will be isometric to this via an isometry that fixes the marked points [2, Theorem 8.1.7].

Another property that follows with this restriction to Riemannian manifolds, by virtue of restricting us to length spaces, is that for each $r > 0$, the balls $B_{g_i}(x_i, r)$ converge to the ball $B_d(x_\infty, r)$ in the Gromov-Hausdorff sense, which is not true in general. Beware that the converse fails: Gromov-Hausdorff convergence of open balls of arbitrary radius r does not imply the pointed convergence as given here. Nor does convergence of closed balls of arbitrary radius. (Just consider metric spaces each consisting of two points only.)

See [2], p.271*ff* for further details on this general topic.

Let's return to the original sequence of pointed manifolds satisfying conditions (6). By virtue of the lower Ricci curvature bound, and the resulting volume comparison discussed in Section 3.1, a compactness result of Gromov given in Theorem 12 of Appendix 7 tells us that we can pass to a subsequence in i and obtain pointed Gromov-Hausdorff convergence

$$(M_i, g_i, x_i) \to (X, d, x_\infty) \tag{7}$$

to some complete metric space (X, d) (even a length space) and $x_\infty \in X$.

Definition 1 *A complete metric space* (X, d) *arising in a limit* (7) *of a sequence* (M_i, g_i, x_i) *of pointed complete Riemannian n-manifolds satisfying* (6) *is called a (non-collapsed) Ricci limit space.*

In these lectures we will not consider so-called collapsed Ricci limit spaces, i.e. we will always assume the volume lower bound of (6).

Such Ricci limit spaces were studied extensively starting with Cheeger and Colding around the late 90s. They can be considered as rough spaces that can be viewed as having some sort of lower Ricci bound. As such they are analogues of Alexandrov spaces, which are metric spaces with a notion of lower *sectional* curvature bound. A synthetic notion of rough space with lower Ricci curvature bound (i.e. defined directly rather than as a limit of smooth spaces) is provided by the so-called RCD spaces, for example; see [41] for background. The exact link between Ricci limit spaces and RCD spaces is a topic of intense current study. One of the basic questions in that direction will be settled by the Ricci flow methods we will see in the next section; see Remark 4.

3.3 How irregular can a Ricci limit space be?

Let's return to the example of a Ricci limit space that we have already seen, i.e. the two-dimensional cone C. Intuitively it has precisely one singular point; we now justify that by giving a definition of *singular point*. At each point p in a Ricci limit space (X, d), we can define a *tangent cone* to be a (pointed Gromov-Hausdorff) limit of rescalings $(X, \lambda_i d, p)$ for some $\lambda_i \to \infty$. We can always find a tangent cone by picking an arbitrary sequence $\lambda_i \to \infty$ and passing to a subsequence using Gromov compactness as discussed in Theorem 12. Strictly speaking we have only talked about Gromov compactness for sequences of Riemannian manifolds rather than metric spaces as here, but $(X, \lambda_i d, p)$ itself is the pointed Gromov-Hausdorff limit of $(M_j, \lambda_i^2 g_j, y_j)$ as $j \to \infty$, for some sequence of points $y_j \in M_j$, and the tangent cone can be viewed as a limit of rescaled and rebased pointed manifolds $(M_i, \mu_i g_i, y_i)$.

There may be different tangent cones corresponding to different sequences (see e.g. [6, Theorem 8.41]). Tangent cones have a lot of structure, see [5] for details.

Definition 2 *A point p in a Ricci limit space (X, d) is singular if at least one tangent cone is not Euclidean space.*

In fact, an elegant argument shows that if any one tangent cone is Euclidean space, then any other tangent cone (i.e. for any other sequence $\lambda_i \to \infty$) is Euclidean space too, see [5].

In our cone example, the tangent cones are Euclidean space away from the vertex (thus regular) and a nontrivial cone at the vertex (hence singular).

The singular set could be a lot worse in size, for example it could be dense. Let's see at least how one can construct a singular set with an accumulation point. To do this, start with a round sphere. Select a spherical cap that is strictly smaller than a hemisphere, and replace it with the top of a cone so that the surface is C^1, i.e. the tangent spaces match up. Thus we end up with an 'ice cream cone' as in Figure 7.

We can now pick a countable collection of further spherical caps, of size converging to zero, that are mutually disjoint. Each of them can be replaced by a cone as before.
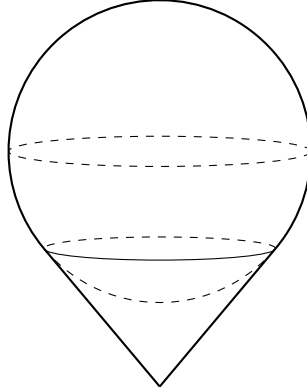
Figure 7: Ice cream cone

As the caps become smaller, the cones become 'flatter'. Each cone vertex will be a singular point. Clearly we have constructed another Ricci limit space, this time with accumulating singular points, which we call (X_0, d_0).

Although the singular set here is infinite, it nevertheless has Hausdorff dimension zero, which is optimal:

Theorem 5 (Cheeger-Colding [6].) *A noncollapsed Ricci limit space has Hausdorff dimension n and its singular set has Hausdorff dimension no larger than $n - 2$.*

By considering the product of (X_0, d_0) with $\mathbb{R}^{n-2}$, which is clearly another Ricci limit space, we see that the control on the size of the singular set of Theorem 5 is optimal in all dimensions.

Our example (X_0, d_0) may have an infinite singular set, but it is *topologically* regular in the sense that it is *homeomorphic* to a smooth manifold, in this case S^2. The simple conical points present in this example have neighbourhoods homeomorphic to flat discs. However even this milder form of regularity fails if we increase the dimension from 2 to 4 because of the following example.

Example 7 *There exists a four-dimensional complete Riemannian manifold (M, g_0), known as the Eguchi-Hanson manifold, that is Ricci-flat, and which looks asymptotically like $\mathbb{R}^4/\sim$, where $\sim$ is the equivalence relation that identifies each point $x \in \mathbb{R}^4$ with $-x$. By picking an arbitrary point $x_0 \in M$ and taking pointed manifolds $(M_i, g_i, x_i) := (M, \frac{1}{i} g_0, x_0)$ for each i, which satisfy the conditions (6), and converge in the pointed Gromov-Hausdorff sense to $\mathbb{R}^4/\sim$, we find that $\mathbb{R}^4/\sim$ is a Ricci limit space. But this is a cone over $\mathbb{R}P^3$, and being a cone over something other than a sphere, it cannot be homeomorphic to a manifold.*

Remark 3 *The term* cone *can refer to many different things. Here when we say cone over $\mathbb{R}P^3$ we can start with the Riemannian cone defined on $(0, \infty) \times \mathbb{R}P^3$, with the warped product metric $dr^2 + r^2 g_1$, where $r \in (0, \infty)$ and g_1 is the standard metric on*

$\mathbb{R}P^3$, *then view the space as a metric space and complete it by adding one point at the tip of the cone.*

By taking a product of the Eguchi-Hanson manifold and $\mathbb{R}^{n-4}$, we see that in general the best regularity one can hope for is for our Ricci limit space to be homeomorphic to a manifold off a set of dimension $n - 4$. It is a long-standing conjecture, normally attributed to Anderson, Cheeger, Colding and Tian (ACCT) that in some sense this is the case. We'll give a precise formulation of the conjecture in general dimension in a moment, once we have introduced some more theory. But we can immediately state it precisely in the three-dimensional case: The whole Ricci limit space should be homeomorphic to a topological manifold. With Miles Simon we prove more:

Theorem 6 (with Miles Simon [33].) *In the case n = 3, any Ricci limit space is locally bi-Hölder homeomorphic to a smooth manifold.*

The proof of this will require a string of new ideas, some of which are due to Hochard [22]. In fact, a slight improvement is possible.

Theorem 7 (with Andrew McLeod [26], using mainly the theory from [32, 33].) *In the case n = 3, any Ricci limit space is globally homeomorphic to a smooth manifold via a homeomorphism that is locally bi-Hölder.*

Prior to these results, it was only possible to give topological regularity statements about the *regular set* of a Ricci limit space (i.e. the complement of the singular set) or suitable generalisations thereof. This reduces considerations to spaces that are almost-Euclidean in some sense, which is a completely different task. However, it does have the advantage of having a chance of working in general dimension.

Theorem 8 (Cheeger-Colding [6].) *There exists an open neighbourhood of the regular set of a general Ricci limit space that is locally bi-Hölder homeomorphic to a smooth manifold.*

In Theorem 8 we can make the Hölder exponent as close as we like to 1 by shrinking the set (still containing the regular set). It is unknown whether we can make the homeomorphism Lipschitz.

We now return to formulate the ACCT conjecture in general dimension. For each $k \in \{0, \dots, n-1\}$, consider the subset $\mathcal{S}_k$ of the singular set $\mathcal{S}$ consisting of points at which no tangent cone can be written as a product of $\mathbb{R}^{k+1}$ with something else. (At most they can be a product of $\mathbb{R}^k$ with something else.) Note that

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{n-1} = \mathcal{S}.$$

Cheeger-Colding's theory [6] tells us that the Hausdorff dimension of $\mathcal{S}_k$ is at most k. As usual, we are considering noncollapsed Ricci limit spaces [6]. In fact, in this case, Cheeger-Colding also proved that $\mathcal{S}_{n-2} = \mathcal{S}$, which is how one proves Theorem 5.

The full ACCT conjecture is:

Conjecture 1 ([6, Conjecture 0.7].) *Given any Ricci limit space X, the interior of $X \setminus \mathcal{S}_{n-4}$ is homeomorphic to a topological manifold.*

Remark 4 *One consequence of Theorem 6 is that we can settle the most obvious questions concerning whether synthetically-defined metric spaces with lower Ricci bounds always arise as noncollapsed Ricci limit spaces. One can see the cone over $\mathbb{R}P^2$ (plus the vertex) as such a metric space with non-negative Ricci curvature (e.g. a $\mathrm{RCD}^*(0, 3)$ space, as shown in [23]) but it is not a manifold, being a cone over something other than a sphere, and thus cannot be a Ricci limit space by our theorem. More significantly, our theory in [33] gives a local distinction in that a unit ball centred at the vertex of the cone cannot arise as the pointed Gromov-Hausdorff limit of a sequence of possibly incomplete Riemannian manifolds (M_i^3, g_i, x_i) each with the property that $B_{g_i}(x_i, 1) \subset\subset M_i$ (i.e. each manifold can fit in a unit ball) and $\mathrm{Ric}_{g_i} \geq 0$, say. Note that it is much easier (and much weaker) to show that the cone over $\mathbb{R}P^2$ cannot arise as the pointed Gromov-Hausdorff limit of a sequence of* closed *Riemannian 3-manifolds each with $\mathrm{Ric}_{g_i} \geq 0$. A global restriction of this form imposes very strong rigidity on the geometry and topology of the approximating closed manifolds. It is not even clear that a* smooth *complete Riemannian 3-manifold with non-negative Ricci curvature can be approximated by closed 3-manifolds with the same curvature restriction.*

3.4 How to apply Ricci flow to Ricci limit spaces

The central question when trying to apply Ricci flow to Ricci limit spaces is:

> Can we start the Ricci flow with a Ricci limit space as initial data?

Clearly we will have to revisit the traditional viewpoint of how to pose Ricci flow that we described at the start of the lectures. Now, when we are given initial data that is a metric space (X, d) rather than a Riemannian manifold, we are asking Ricci flow to create not only a one-parameter family of metrics $g(t)$, but also the underlying manifold on which they live! The family $g(t)$ will now only live for positive times (i.e. not at $t = 0$) and we will want $(M, d_{g(t)})$ to converge in a sufficiently strong sense to the initial data (X, d) as $t \downarrow 0$.

Slightly more precisely, we will want that the distance metric $d_{g(t)}$ converges nicely to a distance metric d_0 on M as $t \downarrow 0$, and that (M, d_0) is isometric to the Ricci limit space (X, d). Here *nice* convergence will be more than local uniform convergence on the domain $M \times M$ of the distance metrics.

Luckily we are not trying to flow a general metric space, but instead a Ricci limit space, which has a lot more structure. We can therefore see a potential strategy that essentially follows the strategy that worked in the 2D case. (We will need more ideas from the 2D case in due course.) For a Ricci limit space (X, d) arising as the pointed limit of (M_i, g_i, x_i) we might try to:

1. Flow each (M_i, g_i) to give Ricci flows $(M_i, g_i(t))$, and argue that they exist on a time interval $[0, T]$ that is independent of i. (T is allowed to depend on α_0 and v_0.)

2. Derive uniform estimates on the full curvature tensor of $g_i(t)$ that are independent of i, but can depend on $t \in (0, T]$ as well as α_0 and v_0.

3. Use parabolic regularity to improve these bounds to bounds on all derivatives of the full curvature tensor.

4. Appeal to the smooth 'Cheeger-Gromov-Hamilton' compactness that one obtains from these curvature bounds. More precisely, after passing to a subsequence, there exist a smooth manifold M, a smooth Ricci flow $g(t)$ for $t \in (0, T]$ and a point $x_\infty \in M$ such that

$$(M_i, g_i(t), x_i) \to (M, g(t), x_\infty)$$

smoothly as $i \to \infty$. Smooth convergence means that there exist an exhaustion $x_\infty \in \Omega_1 \subset \Omega_2 \subset \cdots \subset M$ and diffeomorphisms $\varphi_i : \Omega_i \to M_i$ such that $\varphi_i(x_\infty) = x_i$ and

$$\varphi_i^* g_i(t) \to g(t)$$

smoothly locally on $M \times (0, T]$. See [34, §7] for a more detailed description of Cheeger-Gromov convergence.

5. Show that the distance metric $d_{g(t)}$ converges to some limit metric d_0 at least uniformly as $t \downarrow 0$. (In practice, stronger convergence will be required.)

6. Show that having passed to a subsequence, the sequence (M_i, g_i, x_i) converges in the pointed Gromov-Hausdorff sense to (M, d_0, x_∞). But this sequence also converges to the Ricci limit space we are trying to flow, so that must be isometric to (M, d_0, x_∞).

Note that the smooth manifold M that we require has been *created* by the Ricci flow.

If this programme could be completed, then we would have succeeded in starting the Ricci flow with a Ricci limit space. Additionally, if we could prove that the distance metric $d_{g(t)}$ is sufficiently close to d_0, depending on t, in a strong enough sense, we can hope that the identity map from (M, d_0) to $(M, d_{g(t)})$, i.e. to the smooth Riemannian manifold $(M, g(t))$, will be bi-Hölder. This would give a proof of Theorem 6.

Unfortunately, this strategy is a little naive as written. Even the first step fails at the first hurdle: Example 4 showed that we cannot expect to even start the Ricci flow with a general smooth complete manifold even if the Ricci curvature is bounded below. Miles Simon carried the programme through in the case that the approximating manifolds are closed [31] so this problem doesn't arise. For the general case this problem forces us to work locally. Once we do that, the strategy will work.

4 Local Ricci flow in 3D

4.1 Local Ricci flow theorem with estimates

The following theorem, proved with Miles Simon, gives the local Ricci flow that we need. We can apply it to each of the (M_i, g_i, x_i) considered in the previous section in order to make the strategy there work. As mentioned earlier, local Ricci flow under different hypotheses or with weaker conclusions on curvature and distance has been considered by several authors including D. Yang [42], P. Gianniotis [15] and R. Hochard [22].

Theorem 9 (Special case of results in [33].) *Suppose (M^3, g_0) is a complete Riemannian manifold, and $x_0 \in M$ so that*

$$\begin{cases} \operatorname{Ric}_{g_0} \geq -\alpha_0 \\ \operatorname{VolB}_{g_0}(x_0, 1) \geq v_0 > 0. \end{cases} \tag{8}$$

Then for all $r \geq 2$ there exists a Ricci flow $g(t)$ (incomplete) defined on $B_{g_0}(x_0, r)$ for $t \in [0, T]$, with $g(0)$ equal to (the restriction of) g_0 such that

$$B_{g(t)}(x_0, r - 1) \subset\subset B_{g_0}(x_0, r), \tag{9}$$

and with Ricci and volume bounds persisting in the sense that

$$\begin{cases} \operatorname{Ric}_{g(t)} \geq -\alpha \\ \operatorname{VolB}_{g(t)}(x_0, 1) \geq v > 0, \end{cases} \tag{10}$$

but additionally so that

$$|\operatorname{Rm}|_{g(t)} \leq \frac{c_0}{t}, \tag{11}$$

where $T, \alpha, v, c_0 > 0$ depend only on α_0, v_0 and r. Moreover, we have distance function convergence $d_{g(t)} \to d_{g_0}$ as $t \downarrow 0$ in the sense that (for example) for all $x, y \in B_{g_0}(x_0, r/8)$ we have

$$d_{g_0}(x, y) - \beta\sqrt{c_0 t} \leq d_{g(t)}(x, y) \leq e^{\alpha t} d_{g_0}(x, y) \tag{12}$$

and

$$d_{g_0}(x, y) \leq \gamma [d_{g(t)}(x, y)]^{\frac{1}{1+4c_0}}, \tag{13}$$

where β is universal, α is as above, and γ depends only on c_0, thus on α_0, v_0 and r.

Example 3 illustrates the importance of (9) in the context of incomplete Ricci flows. Our flows have stable balls that do not instantly collapse. This property, together with the refined information from (12) and (13) relies crucially on the curvature estimates of (10) and (11). In fact, it will be useful to split off what control one obtains, in slightly greater generality, in the following remark.

Remark 5 (Distance estimates.) *Given a Ricci flow $g(t)$ of arbitrary dimension n, defined only for* positive *times $t \in (0, T]$, and satisfying the curvature bounds*

$$\begin{cases} \operatorname{Ric}_{g(t)} \geq -\alpha \\ \operatorname{Ric}_{g(t)} \leq \dfrac{(n-1)c_0}{t} \end{cases} \tag{14}$$

we automatically obtain curvature estimates like in Theorem 9 that are strong enough to extend the distance metric to $t = 0$. Indeed, for any $0 < t_1 \leq t_2 \leq T$, we have

$$d_{g(t_1)}(x, y) - \beta\sqrt{c_0}(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g(t_2)}(x, y) \leq e^{\alpha(t_2 - t_1)} d_{g(t_1)}(x, y), \tag{15}$$

where $\beta = \beta(n)$. In particular, the distance metrics $d_{g(t)}$ converge locally uniformly to some distance metric d_0 as $t \downarrow 0$, and

$$d_0(x, y) - \beta\sqrt{c_0 t} \leq d_{g(t)}(x, y) \leq e^{\alpha t} d_0(x, y), \tag{16}$$

for all $t \in (0, T]$. Furthermore, there exists $\eta > 0$ depending on n, c_0 and the size of the region in space-time we are considering such that

$$d_{g(t)}(x, y) \geq \eta \left[d_0(x, y)\right]^{1 + 2(n-1)c_0}, \tag{17}$$

for all $t \in (0, T]$. Here we are being vague as to where the points x, y live, and the dependencies of η. In the local version of this result that we need, these issues require some care. See [33, Lemma 3.1] for full details.

Remark 6 *The distance estimates of Theorem 9 imply that over appropriate local regions Ω, we have $id : (\Omega, d_{g(t)}) \to (\Omega, d_{g_0})$ is Hölder and $id : (\Omega, d_{g_0}) \to (\Omega, d_{g(t)})$ is Lipschitz. Similar statements follow from the estimates of Remark 5, with d_0 in place of d_{g_0}. In particular, the identity is a bi-Hölder homeomorphism from the potentially-rough initial metric to the positive-time smooth metric.*

Remark 7 (Parabolic rescaling of Ricci flows.) *It will be essential to digest the natural parabolic rescaling of a Ricci flow to obtain a new Ricci flow. Given a Ricci flow $g(t)$ and a constant $\lambda > 0$, we can define a scaled Ricci flow*

$$g_\lambda(t) := \lambda g(t/\lambda),$$

on the appropriately scaled time interval. When we do this rescaling, the curvature is scaled, for example the sectional curvatures are all multiplied by a factor λ^{-1}. One significant aspect of the uniform curvature estimate (11) obtained in Theorem 9 is that it is invariant *under rescaling. That is, we get the same estimate for the rescaled flow with the same c_0.*

Remark 8 *A well-known consequence of the curvature decay (11), via parabolic regularity, known as Shi's estimates in this context [21, §13], is that for all $k \in \mathbb{N}$ we have*

$$|\nabla^k \mathrm{Rm}|_{g(t)} \leq \frac{C}{t^{1 + k/2}}, \tag{18}$$

where C is allowed to depend on k as well as α_0, v_0 and r.

4.2 Getting the flow going

The target of Theorem 9 is to start the Ricci flow locally for a controlled amount of time, with a selection of quantitative estimates. But first we want to focus on the problem of merely starting the flow locally, without estimates and for an uncontrolled time interval. We follow the general strategy we used in the 2D theory, as in Section 2.3. To flow on a ball $B_{g_0}(x_0, r)$ as in the theorem, we would like to try to conformally blow up the metric in order to obtain a complete metric with constant negative curvature asymptotically, and hence one of bounded curvature. In 2D, this construction is straightforward [35].

In higher dimensions we use a lemma of Hochard [22], developed precisely for this purpose, following Simon [31, Theorem 8.4].

The following lemma basically says that given a local region of a Riemmanian manifold with curvature bounded by $|\mathrm{Rm}| \leq \rho^{-2}$ on that local region, we can conformally blow up the metric near the boundary so that the resulting metric is *complete* and has curvature bounded by $|\mathrm{Rm}| \leq \gamma \rho^{-2}$, where the factor γ only depends on the dimension. It is asymptotically like a scaled hyperbolic metric.

Lemma 2 (Variant of Hochard, [22, Lemma 6.2].) *Let (N^n, g) be a smooth Riemmanian manifold (that need not be complete) and let $U \subset N$ be an open set. Suppose that for some $\rho \in (0, 1]$, we have $\sup_U |\mathrm{Rm}|_g \leq \rho^{-2}$, $B_g(x, \rho) \subset\subset N$ and $\mathrm{inj}_g(x) \geq \rho$ for all $x \in U$. Then there exist an open set $\tilde{U} \subset U$, a smooth metric $\tilde{g}$ defined on $\tilde{U}$ and a constant $\gamma = \gamma(n) \geq 1$, such that each connected component of $(\tilde{U}, \tilde{g})$ is a complete Riemmanian manifold such that*

(1) $\sup_{\tilde{U}} |\mathrm{Rm}|_{\tilde{g}} \leq \gamma \rho^{-2}$,

(2) $U_\rho \subset \tilde{U} \subset U$, *and*

(3) $\tilde{g} = g$ *on* $\tilde{U}_\rho \supset U_{2\rho}$,

where $U_s = \{x \in U \mid B_g(x, s) \subset\subset U\}$.

In order to achieve our modest task of at least starting the flow locally on a ball $B_{g_0}(x_0, r)$, we can appeal to Lemma 2 with (N, g) of the lemma equal to (M, g_0), with U of the lemma equal to $B_{g_0}(x_0, r + 2) \subset M$, and with $\rho \in (0, 1]$ sufficiently small so that $\sup_U |\mathrm{Rm}|_g \leq \rho^{-2}$ and $\mathrm{inj}_g(x) \geq \rho$ for all $x \in U$. The output of the lemma is a complete Riemannian manifold $(\tilde{U}, \tilde{g})$ of bounded curvature, where $\tilde{U} \subset M$, such that $\tilde{g} = g_0$ on $B_{g_0}(x_0, r)$. To obtain the local Ricci flow, we simply apply Shi's theorem 2 to $(\tilde{U}, \tilde{g})$ to obtain a Ricci flow $g(t)$, $t \in [0, T]$, and then restrict the flow to $B_{g_0}(x_0, r)$.

We have succeeded in starting the flow, but there is a major problem that the existence time T depends on $\sup_{\tilde{U}} |\mathrm{Rm}|_{\tilde{g}}$, which in turn depends on $\sup_{B_{g_0}(x_0, r+2)} |\mathrm{Rm}|_{g_0}$. This is not permitted. To be useful, the existence time can only ultimately depend on a lower Ricci bound, a lower bound on the volume of $B_{g_0}(x_0, 1)$ and the radius r.

To resolve this problem, we need to extend the flow we have found. This will be acheived with the extension lemma of the following section, after which we'll pick up the proof of Theorem 9 in Section 4.4.

4.3 The Extension Lemma

The extension lemma will tell us that if we have a reasonably controlled Ricci flow on some local region $B_{g_0}(x_0, r_1)$, with reasonably controlled initial metric, then we can extend the Ricci flow to a longer time interval, albeit on a smaller ball, whilst retaining essentially the same estimates on the extension. This will be iterated to prove Theorem 9.

Lemma 3 (Extension lemma [33].) *Given $v_0 > 0$, there exist $c_0 \geq 1$ and $\tau > 0$ such that the following is true. Let $r_1 \geq 2$, and let (M, g_0) be a smooth three-dimensional Riemannian manifold with $B_{g_0}(x_0, r_1) \subset\subset M$, and*

(i) $\mathrm{Ric}_{g_0} \geq -\alpha_0$ for some $\alpha_0 \geq 1$ on $B_{g_0}(x_0, r_1)$, and

(ii) $\mathrm{VolB}_{g_0}(x, s) \geq v_0 s^3$ for all $s \leq 1$ and all $x \in B_{g_0}(x_0, r_1 - s)$.

Assume additionally that we are given a smooth Ricci flow $(B_{g_0}(x_0, r_1), g(t))$, $t \in [0, \ell_1]$, *where* $\ell_1 \leq \frac{\tau}{200\alpha_0 c_0}$, *with* $g(0)$ *equal to the restriction of* g_0, *for which*

(a) $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *and*

(b) $\mathrm{Ric}_{g(t)} \geq -\frac{\tau}{\ell_1}$

on $B_{g_0}(x_0, r_1)$ *for all* $t \in (0, \ell_1]$. *Then, setting* $\ell_2 = \ell_1(1 + \frac{1}{4c_0})$ *and* $r_2 = r_1 - 6\sqrt{\frac{\ell_2}{\tau}} \geq 1$, *the Ricci flow* $g(t)$ *can be extended smoothly to a Ricci flow over the longer time interval* $t \in [0, \ell_2]$, *albeit on the smaller ball* $B_{g_0}(x_0, r_2)$, *with*

(a') $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *and*

(b') $\mathrm{Ric}_{g(t)} \geq -\frac{\tau}{\ell_2}$

throughout $B_{g_0}(x_0, r_2)$ *for all* $t \in (0, \ell_2]$.

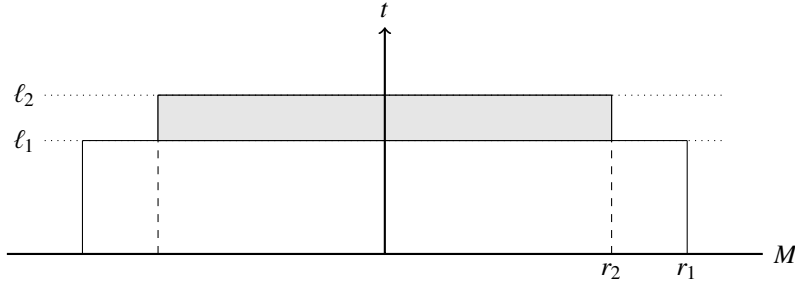See Figure 8 for the domains where the Ricci flows are defined.

Figure 8: Domain of original Ricci flow In the Extension Lemma 3 and (shaded) additional domain to which we extend.

4.4   Using the Extension lemma to construct a local Ricci flow

Assuming the Extension lemma, we can now pick up from where Section 4.2 ended, and give a sketch of the proof of the existence and curvature bounds of Theorem 9. See the end of Section 4 of [33] for the fine details. For the estimates on the distance function, the crucial thing is to have the lower Ricci and upper Rm bounds of the theorem. The estimates then follow from [33, Lemma 3.1], cf. Remark 5.

Before we start, it will be useful to reformulate the conditions (8) of Theorem 9 a little. The claim is that taking r from Theorem 9, the conditions (8) imply that for $r_1 = r + 1$,

$$\mathrm{VolB}_{g_0}(x, s) \geq \tilde{v}_0 s^3 \quad \text{for all } s \leq 1 \text{ and all } x \in B_{g_0}(x_0, r_1 - s), \tag{19}$$

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where $\tilde{v}_0 > 0$ depends on v_0, α_0 and r. To prove this is a simple comparison geometry argument; if we had (8) with $\alpha_0 = 0$, then this was done in Section 3.1 – see (5) in particular. The general case is a minor variant.

This observation means that we may as well replace the second condition of (8) by the condition (19). We will do that below, dropping the tilde on $\tilde{v}_0$. Of course, this means that we satisfy the conditions (i) and (ii) of Lemma 3.

Now we have this new v_0, we can use it to obtain constants c_0 and τ from Extension lemma 3.

The next step is to get the flow going as in Section 4.2, except that we construct the flow on the ball $B_{g_0}(x_0, r + 1)$ of slightly larger radius than before. This flow will exist on some nontrivial but otherwise uncontrolled time interval $[0, \ell_1]$. By reducing $\ell_1 > 0$ if necessary we can ensure that conditions (a) and (b) of the Extension lemma are satisfied.

At this point, if $\ell_1 > \frac{\tau}{200\alpha_0 c_0}$, then we have existence of a Ricci flow on a time interval with the right dependencies. But in general, we will have existence for an uncontrollably short time interval. In this latter case, we apply the Extension lemma. This gives us existence over a longer time interval, albeit on a smaller ball, and with the same estimates. Thus we can iterate this process to extend the flow more and more.

We stop iterating only if either ℓ_1 is large enough as above, or if the radius of the ball on which we get estimates becomes smaller than the radius r on which we would like our ultimate solution. Here is a potential worry: If the original existence time were extremely small, then we would need a huge number of iterations to get a reasonable existence time. Each time we extend on a smaller ball, so there is a concern that the whole space is quickly exhausted as the balls get too small. Luckily, the smaller the original existence time, the smaller the reduction in the size of the ball need be. It turns out that we can ask that the radius of the balls where we obtain existence never drops below r, but still be guaranteed a good time of existence.

Section 4 of [33] gives the details.

4.5 Proof of the Extension lemma 3

We sketch some of the main ingredients and ideas of the proof of the Extension lemma. The objective is to make the full proof in [33, §4] accessible.

The first result expresses that a Ricci flow that has an initial volume lower bound, and a Ricci lower bound for each time, must regularise itself over short time periods. The lemma has many variants; in particular, a global version can be found in the earlier work of Miles Simon [31].

Lemma 4 (Local lemma, cf. [32, Lemma 2.1].) *Let $(N^3, g(t))$, for $t \in [0, T]$, be a Ricci flow such that for some fixed $x \in N$ we have $B_{g(t)}(x, 1) \subset\subset N$ for all $t \in [0, T]$, and so that*

(i) $\mathrm{Vol}B_{g(0)}(x, 1) \geq v_0 > 0$, *and*

(ii) $\mathrm{Ric}_{g(t)} \geq -1$ *on* $B_{g(t)}(x, 1)$ *for all* $t \in [0, T]$.

Then there exist $C_0 \geq 1$ and $\hat{T} > 0$, depending only on v_0, such that $|\mathrm{Rm}|_{g(t)}(x) \leq C_0/t$, and $\mathrm{inj}_{g(t)}(x) \geq \sqrt{t/C_0}$ for all $0 < t \leq \min(\hat{T}, T)$.

The key point to digest here is that we obtain the curvature decay $|\mathrm{Rm}|_{g(t)} \leq C_0/t$, which is an estimate that is invariant under parabolic rescaling of the flow as in Remark 7. The main idea of the proof is as follows. If the curvature estimate failed in a shorter and shorter time and with disproportionately larger and larger C_0, for some sequence of flows, then we can parabolically rescale each flow so that they exist for a longer and longer time, now with almost nonnegative Ricci curvature, but still end up with $|\mathrm{Rm}| = 1$ at the end. It then makes sense to pass to a limit of the flows (in the smooth Cheeger-Gromov sense, see, for example, [34, §7] and the discussion in Section 3.4) and obtain a limit smooth Ricci flow that existed since time $-\infty$, has $\mathrm{Ric} \geq 0$, and has $|\mathrm{Rm}| = 1$ at time 0, say, but which has bounded curvature. Moreover it is possible to argue that the hypothesis $\mathrm{VolB}_{g(0)}(x, 1) \geq v_0 > 0$ eventually tells us that this limit Ricci flow has positive asymptotic volume ratio in the sense that the volume of a ball of radius $r > 0$ is at least εr^3 for some $\varepsilon > 0$, however large we take r. Finally, a result of Chow and Knopf [11, Corollary 9.8] tells us that such a 3D complete ancient Ricci flow with bounded curvature must have nonnegative sectional curvature.

This puts our limit flow in the framework of Perelman's famous κ-solutions [28, §11] that are pivotal in understanding the structure of Ricci flow singularities. The output of Perelman's work is that the asymptotic volume ratio of such a nonflat solution must in fact be *zero*, contradicting what we already saw, and completing the (sketch) proof. See [32, §5] for the argument in full detail.

The following key ingredient gives a local version of the preservation of nonnegative Ricci curvature that we saw at the beginning of these lectures. The Ricci lower bounds it provides will be critical in the applications we have in mind.

Lemma 5 (cf. Double Bootstrap Lemma [32, Lemma 9.1].) *Let* $(N^3, g(t))$, *for* $t \in [0, T]$, *be a Ricci flow such that* $B_{g(0)}(x, 2) \subset\subset N$ *for some* $x \in N$, *and so that throughout* $B_{g(0)}(x, 2)$ *we have*

(i) $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *for some* $c_0 \geq 1$ *and all* $t \in (0, T]$, *and*

(ii) $\mathrm{Ric}_{g(0)} \geq -\delta_0$ *for some* $\delta_0 > 0$.

Then there exists $\hat{S} = \hat{S}(c_0, \delta_0) > 0$ *such that for all* $0 \leq t \leq \min(\hat{S}, T)$ *we have*

$$\mathrm{Ric}_{g(t)}(x) \geq -100\delta_0 c_0.$$

Equipped with these two lemmata, we will attempt to outline some of the central ideas of the Extension lemma 3. Amongst the various details we suppress is the estimation of the exact domain on which we can extend the flow. For the precise details of the proof, see Section 4 of [33].

Step 1: Choose the constants

Let v_0 be the constant from the Extension lemma 3, and plug it in to the Local lemma 4 to obtain constants C_0 and $\hat{T}$. Extracting the constant $\gamma \geq 1$ from Hochard's lemma 2, we are already in a position to specify $c_0 := 4\gamma C_0$.

In turn this allows us to specify $\delta_0 := \frac{1}{100c_0}$, and appeal to the Double Bootstrap lemma 5 to obtain $\hat{S}$. We can then insist to work on time intervals $[0, \tau]$ where $\tau \leq \hat{T}, \hat{S}$ so that all our supporting lemmata apply.

Step 2: Improvement of the curvature decay

The flow we are given in the Extension lemma already enjoys c_0/t curvature decay, where we recall that $c_0 := 4\gamma C_0 > C_0$. By parabolically scaling up our flow, scaling up time by a factor τ/ℓ_1, and distances by a factor $\sqrt{\tau/\ell_1}$, our flow ends up with a lower Ricci bound of -1, and the Local lemma 4 applies over all unit balls to give C_0/t curvature bounds (and injectivity radius bounds) for $t \in (0, \tau]$. Scaling back, we get again C_0/t curvature bounds for the original flow, holding now up until time $\tau \times \frac{\ell_1}{\tau} = \ell_1$, as desired.

Hidden a little here is that the details of this step require the assumed c_0/t curvature decay in order to verify the hypotheses of the Local lemma.

Step 3: Continuation of the flow

The previous step has obtained improved control on the curvature of the originally given Ricci flow on an appropriate interior region of $B_{g_0}(x_0, r_1)$, and in particular at time ℓ_1 we have

$$|\text{Rm}|_{g(\ell_1)} \leq \frac{C_0}{\ell_1}.$$

We can now follow Hochard, Lemma 2, and modify the metric around the outside in order to make it complete and with a curvature bound

$$|\text{Rm}| \leq \gamma \frac{C_0}{\ell_1} = \frac{c_0}{4\ell_1},$$

by definition of c_0. Having a complete, bounded curvature metric, we can apply Shi's theorem 2 to extend the flow for a time $\frac{1}{16} \times \frac{4\ell_1}{c_0} = \frac{\ell_1}{4c_0}$, maintaining a curvature bound

$$|\text{Rm}| \leq 2 \times \frac{c_0}{4\ell_1} \leq \frac{c_0}{\ell_2},$$

where $\ell_2 := \ell_1(1 + \frac{1}{4c_0}) \leq 2\ell_1$. Thus we have proved a little more than

$$|\text{Rm}|_{g(t)} \leq \frac{c_0}{t}$$

for the extended Ricci flow $g(t)$, which is part (a') of the desired conclusion.

Step 4: Ricci lower bounds

Now we have indirectly obtained c_0/t curvature decay for the extended flow, we are in a position to obtain the crucial lower Ricci bound conclusion of part (b'). Similarly to before, we parabolically scale up the extended flow, scaling time by a factor τ/ℓ_2 so that the new flow lives for a time τ. This preserves our c_0/t curvature decay, but improves the initial Ricci lower bound from $-\alpha_0$ to

$$-\alpha_0 \frac{\ell_2}{\tau} \geq -2\alpha_0 \frac{\ell_1}{\tau} \geq -\frac{1}{100c_0} = -\delta_0.$$

Thus we can apply the Double Bootstrap lemma 5 to conclude that the rescaled flow has a Ricci lower bound of $-100c_0\delta_0 = -1$. Rescaling back, we obtain that our original extended flow has a Ricci lower bound of $-\tau/\ell_2$ as required.

This completes the sketch of the proof of the Extension lemma 3.

Remark 9 *The Pyramid Ricci flow construction that we briefly discuss in Section 5 requires its own extension lemma. In that result we are unable to follow this strategy of applying Shi's theorem to extend, since the curvature bounds available are insufficient.*

5 Pyramid Ricci flows

In Section 4 we stated and sketched the proof of the Local Ricci flow theorem 9. This result is enough to prove Theorem 6 and hence settle the three-dimensional Anderson-Cheeger-Colding-Tian conjecture, but instead of elaborating on the details, in this section we give a different Ricci flow existence result that will not only give the slightly stronger result Theorem 7, but will allow us a slightly less technical description of the proof.

The existence result we will give is for so-called Pyramid Ricci flows. To understand the idea, let us revisit the Local Ricci flow theorem 9 in the case that the radius r is a natural number k. For each $k \in \mathbb{N}$ we obtain a Ricci flow on a parabolic cylinder $B_{g_0}(x_0, k) \times [0, T_k]$, and in general the final existence time T_k converges to zero as $k \to \infty$ as demonstrated by Example 4. The Ricci flows corresponding to different values of k will exist on cylinders with nontrivial intersection, but it is unreasonable to expect that they agree on these intersections.

The point of Pyramid Ricci flows is to fix this problem. Instead of obtaining a countable number of Ricci flows on individual cylinders, indexed by k, we obtain one Ricci flow that lives throughout the *union* of all cylinders.

Theorem 10 (with McLeod [26], using [32, 33].) *Suppose (M^3, g_0) is a complete Riemannian manifold, and $x_0 \in M$ so that*

$$\begin{cases} \mathrm{Ric}_{g_0} \geq -\alpha_0 \\ \mathrm{VolB}_{g_0}(x_0, 1) \geq v_0 > 0. \end{cases} \tag{20}$$

Then there exist increasing sequences $C_j \geq 1$ and $\alpha_j > 0$ and a decreasing sequence $T_j > 0$, all defined for $j \in \mathbb{N}$, and depending only on α_0 and v_0, for which the following is true. There exists a smooth Ricci flow $g(t)$, defined on a subset of spacetime

$$\mathcal{D}_k := \bigcup_{k \in \mathbb{N}} B_{g_0}(x_0, k) \times [0, T_k],$$

satisfying that $g(0) = g_0$ throughout M, and that for each $k \in \mathbb{N}$,

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_k \\ |\mathrm{Rm}|_{g(t)} \leq \dfrac{C_k}{t}, \end{cases} \tag{21}$$

throughout $\mathbb{B}_{g_0}(x_0, k) \times (0, T_k]$.

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The idea of defining a Ricci flow on a subset of spacetime was used by Hochard [22], prior to this work, when he defined *partial* Ricci flows. Our flows differ in that we concede having a uniform c/t curvature bound in return for strong uniform control on the shape of the spacetime region on which the flow is defined, which is essential for the compactness arguments that use this theorem.

The theory that allows this pyramid construction revolves around a variant of the Extension lemma 3 known as the Pyramid Extension Lemma; see [26, Lemma 2.1] and [27, Lemma 4.1].

6 Proof of the 3D Anderson-Cheeger-Colding-Tian conjecture

In this section we apply what we have discovered about Ricci flow in order to complete the programme of Section 3.4 and start the Ricci flow with a Ricci limit space in order to prove Theorems 6 and 7.

The theory can be carried out with Theorem 9, but here we will describe the strategy of [33] using instead Theorem 10, with one benefit being a slight reduction in the technical aspects of considering distance metrics on incomplete manifolds.

To reiterate the argument, we use Theorem 10 to flow each (M_i, g_i) generating our Ricci limit space (X, d). We call the resulting pyramid Ricci flows $g_i(t)$. The estimates on the curvature from Theorem 10 allow us to invoke a local version of Cheeger-Gromov-Hamilton compactness to get a type of smooth convergence

$$(M_i, g_i(t), x_i) \to (M, g(t), x_\infty) \tag{22}$$

to some smooth limit Ricci flow $g(t)$, which inherits the curvature bounds of the pyramid flows $g_i(t)$ and lives on a subset of spacetime similar to the domains of the pyramid flows. By keeping track of the evolution of distances, see Remark 5, and [33, Lemma 3.1] for the precise local statement, we obtain the following.

Theorem 11 (Ricci flow from a Ricci limit space.) *Given a Ricci limit space* $(X, d, \hat{x}_\infty)$ *that is approximated by a sequence* (M_i^3, g_i, x_i) *of complete, smooth, pointed Riemannian three-manifolds such that*

$$\begin{cases} \mathrm{Ric}_{g_i} \geq -\alpha_0 \\ \mathrm{VolB}_{g_i}(x_i, 1) \geq v_0 > 0, \end{cases} \tag{23}$$

there exist increasing sequences $C_k \geq 1$ *and* $\alpha_k > 0$ *and a decreasing sequence* $T_k > 0$, *all defined for* $k \in \mathbb{N}$, *and depending only on* α_0 *and* v_0, *for which the following holds.*

There exist a smooth three-manifold M, *a point* $x_\infty \in M$, *and a complete distance metric* $d_0 : M \times M \to [0, \infty)$ *generating the same topology as we already have on* M, *such that* (M, d_0, x_∞) *is isometric to the Ricci limit space* $(X, d, \hat{x}_\infty)$.

Moreover, there exists a smooth Ricci flow $g(t)$ *defined on a subset of spacetime* $M \times (0, \infty)$ *that contains* $\mathbb{B}_{d_0}(x_0, k) \times (0, T_k]$ *for each* $k \in \mathbb{N}$, *with* $d_{g(t)} \to d_0$ *locally*

uniformly on M as t ↓ 0. For any $k \in \mathbb{N}$ we have

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_k \\ |\mathrm{Rm}|_{g(t)} \leq \dfrac{C_k}{t} \end{cases} \tag{24}$$

throughout $B_{d_0}(x_0, k) \times (0, T_k]$. Finally, for all $\Omega \subset\subset M$, there exists $\sigma > 1$ depending on Ω, α_0 and v_0 such that for $t \in (0, \min\{\frac{1}{\sigma}, T\}]$ and for all $x, y \in \Omega$, we have

$$\frac{1}{\sigma} d_{g(t)}(x, y) \leq d_{g_0}(x, y) \leq \sigma [d_{g(t)}(x, y)]^{\frac{1}{\sigma}}. \tag{25}$$

Of course, the existence of the distance metric d_0 and its relation to the original Ricci limit space is established by virtue of having the Ricci flow $g(t)$, not the other way round as the phrasing above may suggest.

The simplified distance estimate (25) tells us immediately that for sufficiently small $t \in (0, T]$, the identity map $id : (\Omega, d_{g(t)}) \to (\Omega, d_{g_0})$ is Hölder and $id : (\Omega, d_{g_0}) \to (\Omega, d_{g(t)})$ is Lipschitz.

This is a very rough outline. See [33, 26] for further details, and [27] for a discussion of subsequent developments and simplifications.

7 An open problem

Despite all the progress in this general area, there are some very basic problems that remain open. An example of an intriguing long-standing open problem at the heart of the matter, which contrasts with Example 4, is the following.

Conjecture 2 *Given a complete Riemannian three-manifold with $\mathrm{Ric} \geq 0$, there exists a smooth complete Ricci flow continuation.*

Appendix - Gromov compactness

When defining Ricci limit spaces we used the following elementary but important compactness result.

Theorem 12 (Gromov compactness, special case) *Given a sequence of complete pointed Riemannian n-manifolds (M_i, g_i, x_i) with a uniform lower Ricci curvature bound, we may pass to a subsequence to obtain pointed Gromov-Hausdorff convergence*

$$(M_i^n, g_i, x_i) \to (X, d, x_\infty),$$

to some pointed limit metric space (X, d, x_∞).

To illustrate the (simple) proof, let's see how one can prove the even simpler statement that for any $r > \varepsilon > 0$, there exists a metric space (X, d, x_∞) such that after passing to a subsequence, all elements of our sequence (M_i^n, g_i, x_i) are ε-close to (X, d, x_∞) in the sense that there exist maps $f_i : B_{g_i}(x_i, r) \to X$ such that

1. $f_i(x_i) = x_\infty$,

2. for all $x, y \in B_{d_i}(x_i, r)$ we have

$$|d(f_i(x), f_i(y)) - d_i(x, y)| < \varepsilon,$$

3. the ε-neighbourhood of the image $f_i(B_{d_i}(x_i, r))$ contains the ball $B_d(x_\infty, r - \varepsilon) \subset X$.

To see this, for each fixed $i \in \mathbb{N}$ pick a maximal set of points $x_i =: p_i^1, p_i^2, \ldots, p_i^m$ in $B_{g_i}(x_i, r)$ such that the balls $B_{g_i}(p_i^j, \varepsilon/9)$ lie in $B_{g_i}(x_i, r)$ and are pairwise disjoint. Maximal means that we cannot adjust the points to squeeze in even one more point with this disjointness property. That way, we can be sure that the balls $B_{g_i}(p_i^j, \varepsilon/3)$ cover all of $B_{g_i}(x_i, r)$.

The claim is that the number of such points is necessarily bounded independently of i. Assuming this is true for the moment, we can pass to a subsequence so that for each i we have the same number m of these points. The distance between each fixed pair of points is bounded above by $2r$, and below by $2\varepsilon/9$, so by passing to enough subsequences (but finitely many) we have convergence of the distances

$$d_i(p_i^j, p_i^k) \to d^{jk} > 0$$

as $i \to \infty$, where $j \neq k$. The metric space X can be chosen to be a finite collection of points $X = \{q^1, \ldots, q^m\}$, with distance

$$d(q^j, q^k) := d^{jk},$$

and we can choose $x_\infty := q^1$ and a suitable map f_i such that $f_i(p_i^j) = q^j$, giving

$$|d(f_i(p_i^j), f_i(p_i^k)) - d_i(p_i^j, p_i^k)| = |d(q^j, q^k) - d_i(p_i^j, p_i^k)| \to 0,$$

as required.

To show that the number of points $\{p_i^j\}$ is bounded independently of i, we use Bishop-Gromov and consider volumes. On the one hand, the lower Ricci bound and volume comparison (cf. Theorem 4 and Remark 2) tell us that the volume of $B_{g_i}(x_i, r)$ is uniformly bounded above independently of i. On the other hand, an argument similar to that giving (5) (but extended to handle lower Ricci bounds other than zero) tells us that the volume of each pairwise disjoint ball $B_{g_i}(p_i^j, \varepsilon/9)$ is uniformly bounded below by some i-independent positive number. Combining these two facts gives an i-independent upper bound on the total number of such balls that can be pairwise disjoint and fit within $B_{g_i}(x_i, r)$.

For further aspects of the proof, see for example [5, Chapter 3].

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