MA3D5 Galois Theory

Homework Assignment 3

The deadline is **2pm Thursday, week 7.** Please hand in your answers to Questions **3, 4, 5, 6** to the MA3D5 Galois Theory box outside the Undergraduate Office.

1. (a) Let M/K be a field extension. Let
$$L = \{\alpha \in M \ : \ \alpha \text{ is algebraic over } K\}.$$
Show that L is a field.
(b) Let $\overline{\mathbb{Q}}$ be the set of elements in $\mathbb{C}$ algebraic over $\mathbb{Q}$. Part (a) tells us that $\overline{\mathbb{Q}}$ is a field. Now let $\beta \in \mathbb{C}$ be algebraic over $\overline{\mathbb{Q}}$. Show that $\beta \in \overline{\mathbb{Q}}$.

2. Let $L = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. In Example 69 in the lecture notes we computed $\mathrm{Aut}(L/\mathbb{Q})$. Write down all its subgroups H, and compute the corresponding fixed fields L^H (**Hint:** see Example 72).

3. Which of the following extensions are normal? Which are separable?
(a) $\mathbb{Q}(\sqrt{-7})/\mathbb{Q}$.
(b) $\mathbb{Q}(\sqrt[4]{-7})/\mathbb{Q}$.
(c) $\mathbb{Q}(\sqrt[4]{-7})/\mathbb{Q}(\sqrt{-7})$.
(d) $K(t^{1/3})/K$ where $K = \mathbb{F}_3(t)$.

4. Let $L = \mathbb{Q}(\sqrt[4]{2})$. Compute $\mathrm{Aut}(L/\mathbb{Q})$ and $L^{\mathrm{Aut}(L/\mathbb{Q})}$.

5. If L/F and F/K are Galois, does L/K have to be a Galois extension? Prove or give a counterexample. (**Big Hint:** Consider L as in question 4.)

6. Let L be a subfield of $\mathbb{C}$ that is a finite Galois extension of $\mathbb{Q}$.
(a) Let $\alpha \in L$ and let $\overline{\alpha}$ be its complex conjugate. Show that $\overline{\alpha} \in L$.
(b) Let
$$\sigma : L \to L, \qquad \sigma(\alpha) = \overline{\alpha}.$$
Show that $\sigma \in \mathrm{Aut}(L/\mathbb{Q})$, and has order 1 or 2.
(c) Show moreover that σ has order 2 if and only if $L \not\subset \mathbb{R}$.
(d) Let $F = L^{\langle \sigma \rangle}$. Show that $[L : F] = 1$ or 2 according to whether $L \subset \mathbb{R}$, $L \not\subset \mathbb{R}$.

7. Let p be an odd prime. Let $\zeta = \exp(2\pi i/p)$.
(a) Show that the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ is Galois.
(b) Define
$$\mu : (\mathbb{Z}/p\mathbb{Z})^* \to \mathrm{Aut}(\mathbb{Q}(\zeta)/\mathbb{Q}), \qquad \mu(\overline{a})(\zeta) = \zeta^a.$$
Show that μ is well-defined and is in fact an isomorphism.
(c) Let σ be as in part (c) of Question 6. Show that
$$\mathbb{Q}(\zeta)^{\langle \sigma \rangle} = \mathbb{Q}(\zeta + 1/\zeta).$$