Algebraic Number Theory
Example Sheet 1

Hand in the answers to questions 4, 5, 7. Deadline 2pm Thursday, Week 3.

1. Which of the following are algebraic numbers and which are algebraic integers [1]:
$$23/7, \quad \sqrt{-5}, \quad \sqrt{2}+\sqrt{7}, \quad \exp(2\pi i/5), \quad \cos(2\pi/7), \quad \frac{\sqrt[3]{3}}{2}.$$
Write down their minimal polynomials, conjugates [2], norms and traces.

2. You probably know that e and π are transcendental. Show that e and π are algebraic over the field $\mathbb{Q}(e+\pi, e\cdot\pi)$.

3. Let L/K be a finite extension.
   (i) Show that L/K is algebraic.
   (ii) $\deg(\mu_\alpha) \mid [L:K]$ for all $\alpha \in L$.

4. Let K be a number field of degree 2.
   (i) Show that $K = \mathbb{Q}(\sqrt{d})$ where $d \neq 0$, 1 is a squarefree integer.
   (ii) Write down the matrix for $a+b\sqrt{d}$ ($a, b \in \mathbb{Q}$) with respect to the basis 1, $\sqrt{d}$. Deduce the trace, norm and characteristic polynomial of $a+b\sqrt{d}$.

5. Suppose d_1, d_2 are squarefree integers $\neq 0$, 1.
   (i) Show $\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})$ if and only if $d_1 = d_2$.
   (ii) Suppose $d_1 \neq d_2$ and let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.
   - Write down a basis for $K/\mathbb{Q}$.
   - Show that $K = \mathbb{Q}(\sqrt{d_1}+\sqrt{d_2})$.

6. Let K be the splitting field of X^3-5 over $\mathbb{Q}$. Write down a basis for $K/\mathbb{Q}$.

7. Let $f(X) = X^3 - 2X - 2$.
   (i) Show that f is irreducible.
   (ii) Let θ be a root of f. Find the minimal polynomial for $1+\theta+\theta^2$.

8. Let L/K be an extension of number fields. Let $\alpha \in K$. Show that
$$\mathrm{Trace}_{L/\mathbb{Q}}(\alpha) = [K:\mathbb{Q}] \cdot \mathrm{Trace}_{K/\mathbb{Q}}(\alpha),$$
and
$$\mathrm{Norm}_{L/\mathbb{Q}}(\alpha) = \mathrm{Norm}_{K/\mathbb{Q}}(\alpha)^{[K:\mathbb{Q}]}.$$

9. In the lectures we used the Cayley–Hamilton Theorem to show than any algebraic number is the root of its characteristic polynomial. If you were traumatised by the use of Cayley–Hamilton Theorem then this exercise is

[1] An algebraic number α is called an **algebraic integer** if its minimal polynomial μ_α belongs to $\mathbb{Z}[X]$.

[2] Two algebraic numbers α, β are **conjugates** if they share the same minimal polynomial.

for you. Let $f = a_0 + a_1 X + \cdots + a_{n-1}X^{n-1} + X^n$ be a monic polynomial. We define the **companion matrix** of f to be

$$C_f = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

(i) Use induction [3] on the degree n to show that the characteristic polynomial of C_f is f.
(ii) Let α have minimal polynomial f and let $K = \mathbb{Q}(\alpha)$. Show that the matrix for α with respect to the basis $1, \alpha, \dots, \alpha^{n-1}$ is C_f and deduce that $\chi_{K,\alpha} = f$.
(iii) Finally let L be a number field and $\alpha \in L$. Show that $\chi_{L,\alpha}(\alpha) = 0$.

10. Let a, $b > 0$. Compute the area of the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1.$$
This might seem irrelevant to algebraic number theory, but we'll need such a formula later when we come to compute class numbers.

11. Let α be an algebraic number with minimal polynomial μ of degree n. We know that
$$\mathbb{Q}(\alpha) = \{a_0 + a_1 + \cdots + a_{n-1}\alpha^{n-1} \ : \ a_i \in \mathbb{Q}\}.$$
Thus every element of $\mathbb{Q}(\alpha)$ can be written (in fact uniquely) as $f(\alpha)$ where $f \in \mathbb{Q}[X]$ has degree $\leq n-1$. It is obvious how to add and subtract elements of $\mathbb{Q}(\alpha)$ when expressed in this form.
(i) Let f, $g \in \mathbb{Q}[X]$ have degree $\leq n-1$. Then $f(\alpha)g(\alpha) = h(\alpha)$ for some $h \in \mathbb{Q}[X]$ with degree $\leq n-1$. Explain how the Euclidean algorithm allows you to compute h.
(ii) Let $f \in \mathbb{Q}[X]$ have degree $\leq n-1$ and suppose $f(\alpha) \neq 0$. Then $f(\alpha)^{-1} = h(\alpha)$ for some $h \in \mathbb{Q}[X]$ with degree $\leq n-1$. Explain how the Euclidean algorithm allows you to compute h.
(iii) Let $\mu = X^3 - 2X - 2$. You showed this to be irreducible in Exercise 7. Let θ be a root of μ. Compute $1/(\theta^2 + 1)$.

12. A field K is **algebraically closed** if every β that is algebraic over K belongs to K.
(i) Explain why $\mathbb{C}$ is algebraically closed.
(ii) Show that $\overline{\mathbb{Q}}$ is algebraically closed.

[3]**Hint:** For the inductive step write $g = a_1 + a_2 X + \cdots + a_{n-1}X^{n-2} + X^{n-1}$ and show that $\mathrm{Det}(XI_n - C_f) = X \cdot \mathrm{Det}(XI_{n-1} - C_g) + a_0$.