MA908 Partial Differential Equations in Finance

EXERCISE SHEET 6: STOCHASTIC OPTIMAL CONTROL

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1. **Deriving the SDE for Mertons example from scratch**
   Consider the following problem of asset allocation: there are two different investment opportunities. One of them is a risk free bond (does anybody still believe such a thing exists?) with price $q(s)$, earning a constant interest rate r. The other is a stock modelled by geometric Brownian motion, so that the price solves

   $$\mathrm{d}p_s = \mu p_s \,\mathrm{d}s + \sigma p_s \,\mathrm{d}\mathcal{W}_s.$$

   At time t, you have wealth x. At time s, let A_s be the quantity of stock you own, and B_s the quantity of bonds. Thus, the value of the portfolio at time s is

   $$y_s = A_s p_s + B_s q_s,$$

   and

   $$\mathrm{d}y_s = A_s \mathrm{d}p_s + B_s \mathrm{d}q_s + (A_s' p_s + B_s' q_s)\,\mathrm{d}s.$$

   The primes denote derivatives.

   (a) Let us look at the situation where we can neither withdraw from nor invest further money into the portfolio after the initial time x. It is then argued that this leads to $A_s' p_s + B_s' q_s = 0$. Try to find out why this is so, and if you can give a rigorous argument. For the latter, you will need an assumption about the speed of transactions relative to the speed of price change that is rarely made explicitly, but is needed. What assumption is that?

   (b) In the situation of a), we thus have

   $$\mathrm{d}y_s = A_s \,\mathrm{d}p_s + B_s \,\mathrm{d}q_s.$$

   Let $\alpha(s)$ be the proportion of your total portfolio that is in stocks. Rewrite the above equation so that it only contains y_s, $\alpha(s)$, $\mathcal{W}_s$ and explicit constants and functions. What special feature of the model do you need to use?

2. Let us continue with the equation that you should have obtained in 1), namely

   $$\mathrm{d}y_s = (1-\alpha(s))y_s r \,\mathrm{d}s + \alpha(s) y_s (\mu \mathrm{d}s + \sigma \,\mathrm{d}\mathcal{W}_s),$$

   with $y_t = x$. At a final time T, the total wealth is weighted with a utility function h. The aim is to maximize the utility, i.e. compute

   $$u(x,t) = \max_{\alpha(s)} \mathbb{E}_{y(t)=x}(h(y_T)).$$

   (a) Find the HJB equation for this problem.

   (b) Find the solution to the HJB equation, and the optimal investment strategy, if $h(y) = y^\gamma$ with $0 < \gamma < 1$. (Hint: try a solution that has the same form as the final condition.)