Attractors and Finite-Dimensional Behaviour in the Navier-Stokes Equations

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Chapter 1

Introductory material

Throughout these lectures we will consider the Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = g(x,t) \qquad \nabla \cdot u = 0 \tag{1.1}$$

on a periodic domain $\Omega = [0,L]^d$, so that $u(x+Le_j,t) = u(x,t)$, and in addition make the simplifying assumption that u and g have zero average over Ω,

$$\int_\Omega g(x,t)\,\mathrm{d}x = \int_\Omega u(x)\,\mathrm{d}x = 0.$$

Although we will generally confine our analysis to the case of periodic boundary conditions, many results are also true for the case of Dirichlet boundary conditions,

$$u|_{\partial\Omega} = 0,$$

and will be given in what follows.

In the case of periodic boundary conditions we can expand u as a Fourier series,

$$u(x) = \sum_{k\in\mathbb{Z}^2} u_k \mathrm{e}^{2\pi \mathrm{i}k\cdot x/L} \qquad \text{with} \qquad u_k = \overline{u_{-k}}. \tag{1.2}$$

1.1 The Navier-Stokes equations in functional form

We can rewrite the Navier-Stokes equations in a more convenient way. The ideas go back to Leray (1933, 1934a & b) and are now standard – see, for example, Constantin & Foias (1988), Temam (1979 or 1985), Ladyzhenskaya (1969), Robinson (2001).

The main idea is to remove the pressure by projecting onto the space of all divergence-free vector fields. If we denote by Π the orthogonal projection in

L^2 onto the space of all such functions, then $\Pi\nabla p = 0$, and so we obtain an equation for u alone,

$$\frac{\partial u}{\partial t} + \Pi[(u \cdot \nabla)u] + \nu A u = f(x,t), \tag{1.3}$$

where $A = -\Pi\Delta$ and $f = \Pi g$.

We denote by $\mathbb{H}^k(\Omega)$ the space of d-component vector functions each component of which lies in $H^k(\Omega)$ [the first k derivatives are in $L^2(\Omega)$]. The natural phase space for the problem is

$$H = \left\{ u \in \mathbb{L}^2(Q) : \ \nabla \cdot u = 0, \qquad \text{with} \qquad \int_Q u(x)\,\mathrm{d}x = 0 \right\};$$

we let $V = H \cap \mathbb{H}^1$ and denote by V^* the dual of V. By $D(A^r)$ we denote the domain of A^r, i.e. all those u for which $|A^r u|$ is finite. Note that in the case of periodic boundary conditions this has a simple characterisation as those Fourier series (1.2) for which

$$\sum_{k \in \mathbb{Z}^2} |k|^{2r} |u_k|^2 < \infty$$

(cf. Temam 1985). (In the case of periodic boundary conditions we have $Au = -\Delta u$ for smooth $u \in H$.)

By λ_j we denote the eigenvalues of A, ordered so that $\lambda_{j+1} \geq \lambda_j$. Although in the periodic case we know the eigenfunctions of A explicitly (they are the Fourier components in (1.2)), it can be convenient to label them more abstractly as w_j, so that $Aw_j = \lambda_j w_j$. Using this notation, note that for all $u \in H$ we have the Poincaré inequality

$$\lambda_1 |u|^2 \leq |Du|^2, \tag{1.4}$$

where $\lambda_1 = (2\pi/L)^d$.

We now define a trilinear form from $V \times V \times V$ into $\mathbb{R}$,

$$b(u,v,w) = \int_\Omega [(u \cdot \nabla)v] \cdot w \,\mathrm{d}x,$$

so that $b(u,u,w)$ is the inner product of the nonlinear term in (1.3) with w. Using the Riesz representation theorem we can define a bilinear form $B(u,v) : V \times V \to V^*$ such that

$$(B(u,v),w) = b(u,v,w) \qquad \text{for all} \qquad w \in V.$$

The following properties of b will be useful throughout all that follows. First we have two orthogonality properties: in 2d and 3d we have

$$b(u,v,v) = 0 \qquad \text{for all} \qquad u \in H, \ v,w \in V, \tag{1.5}$$

while in 2d for periodic boundary conditions only we have

$$b(u,u,Au) = 0 \qquad \text{for all} \qquad u \in D(A). \tag{1.6}$$

We will also need some inequalities – we only give the 2d versions here:

$$|b(u,v,w)| \le k|u|^{1/2}|Du|^{1/2}|Dv||w|^{1/2}|Dw|^{1/2} \qquad \text{for all} \qquad u,v,w \in V, \tag{1.7}$$

$$|b(u,v,w)| \le k|u|^{1/2}|Du|^{1/2}|Dv|^{1/2}|Av|^{1/2}|w| \qquad \text{for all} \qquad u \in V,\ v \in D(A),\ w \in H, \tag{1.8}$$

and

$$|b(u,v,w)| \le \|u\|_\infty \|v\| |w| \qquad \text{for all} \qquad u \in L^\infty,\ v \in V,\ w \in H. \tag{1.9}$$

With these definitions we can rewrite the Navier-Stokes equations in the functional form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = f. \tag{1.10}$$

We will refer to this form, in which the pressure is suppressed, as the 'Navier-Stokes evolution equation'.

1.2 Summary of existence and uniqueness results

Standard arguments show that for any $f \in V^*$ the Navier-Stokes evolution equation has a weak solution $u(t)$ with

$$u(t) \in L^\infty(0,T;H) \cap L^2(0,T;V).$$

In 2d this can be improved to a unique solution which is in fact continuous from $[0,T]$ into H.

In 3d the solution is known to be unique if it is strong, i.e. if

$$u(t) \in L^\infty(0,T;V) \cap L^2(0,T;D(A)).$$

1.3 The Navier-Stokes equations as a dynamical system

For many PDEs it is only sensible to consider solutions for $t \ge 0$. For example, backwards solutions of the heat equation ($u_t = \Delta u$) can blow up instantaneously unless the initial data in analytic.

Here we define a dynamical system using the Navier-Stokes equations on the phase space H (defined above), although we could also use $V = H \cap \mathbb{H}^1(Q)$, or indeed $D(A^{k/2})$ for any k. Given an initial condition $u_0 \in H$, we have seen that the equation has a unique solution $u(t;u_0)$ for all positive times. In this case, we can define a C^0 semigroup of solution operators $S(t) : H \to H$ by

$$u(t;u_0) = S(t)u_0.$$

These operator satisfy

$$\begin{array}{l} S(0) = I \\ S(t)S(s) = S(s)S(t) = S(s+t) \\ S(t)x_0 \text{ is continuous in } x_0 \text{ and } t, \end{array}$$

and we can consider the semi-dynamical system

$$\big(H, \{S(t)\}_{t\geq 0}\big).$$

Further regularity results (see the above references) imply that it is also possible to consider the solutions as generating a dynamical system on V, and we consider this briefly in the next section.

1.4 Including the pressure

Taking the divergence of (1.1) we obtain a Poisson equation for the pressure,

$$\Delta p = \nabla \cdot f - \nabla \cdot [(u \cdot \nabla)u].$$

If we impose the additional condition that $\int_Q p = 0$, then this equation has a unique solution which satisfies the estimate

$$|p| \leq C[|Du| + \|f\|_{-1}]$$

(see Simon (1999), for example). We denote by $\mathbb{P}[u]$ the mapping from u to the corresponding pressure p.

It is easy to start to think of the 'Navier-Stokes equations' as the Navier-Stokes evolution equation, and hence of the velocity as the only dependent variable of interest. Although this is a mathematically convenient point of view, it is often the pressure that is of interest in physical problems. We note here that there is no way to talk of a meaningful weak solution of the original problem, including the pressure, unless $f \in \mathbb{H}^{-1}(Q)$ (a smaller space than V^*), as shown by Simon (1999).

A more physical phase space would be the space of all pairs $(u, \mathbb{P}[u])$, i.e. the fluid velocity and the corresponding pressure. In order to give sense to p at every time we require the fluid velocity to be in V (essentially $\mathbb{H}^1$), so here we denote by S_V the semigroup for the Navier-Stokes evolution equation defined on V.

A dynamical system can be defined on the velocity-pressure space V_{VP} in a natural way by setting

$$S_{\mathrm{VP}}(t)(u_0, \mathbb{P}[u_0]) = (S_V(t)u_0, \mathbb{P}[S_V(t)u_0]).$$

Of course, although this extended phase space V_{VP} is in some ways more natural physically, it dresses the problem up as being more complicated than it really is.

Note, in particular, that the phase space V_{VP} is in fact given as a graph over V in the space $V \times L^2(\Omega)$,

$$V_{\mathrm{VP}} = \{(u, \mathbb{P}(u)) : \; u \in H\}.$$

The evolution takes place on a manifold in $V \times L^2(\Omega)$, with the dynamics given more naturally by the projection from V_{VP} onto V.

Chapter 2

Absorbing sets

Central to proving results on existence and unique for the Navier-Stokes equations (and for other PDEs) are various bounds on the norms of solutions. In order to prove the existence of solutions for all time, we have to prove that some norm of the solution is bounded for all time. Because of the strong dissipation in many parabolic problems, it is often a short step from these bounds to time-asymptotic bounds that are independent of the initial conditions.

We first prove a simple result for the Navier-Stokes equations (valid for 2d or 3d weak solutions resulting from the Galerkin procedure) showing that asymptotically the kinetic energy (the $\mathbb{L}^2$ norm of u) is bounded independently of the initial conditions.

We say that B is an absorbing set if, for any bounded set $X \in H$,

$$S(t)X \subseteq B \qquad \text{for all} \qquad t \geq t(X),$$

i.e. orbits starting in X eventually enter and never leave B. Note that the 'absorbtion time' is required to be uniform for all initial conditions in X.

2.1 Absorbing set bounded in $\mathbb{L}^2$

The existence of an absorbing set in $\mathbb{L}^2$ is straightforward.

Proposition 2.1. *For the 2d Navier-Stokes evolution equation there exists an absorbing set that is bounded in* $\mathbb{L}^2(\Omega)$.

[As remarked above this is also valid for the weak solutions in 3d that result as limits of the Galerkin procedure.]

Proof. We take the inner product of

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = f$$

with u to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|Du|^2 + (B(u,u),u) = (f,u).$$

Since $(B(u,u),u) = 0$ this gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|Du|^2 \leq |f||u|. \tag{2.1}$$

We now use the Poincaré inequality on the $|Du|$ term,

$$|Du| \geq \lambda_1^{1/2}|u|$$

and Young's inequality on the right-hand side to write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu\lambda_1|u|^2 \leq \frac{\nu\lambda_1}{2}|u|^2 + \frac{1}{2\nu\lambda_1}|f|^2.$$

Tidying this up gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 \leq -\nu\lambda_1|u|^2 + \frac{1}{\nu\lambda_1}|f|^2,$$

and then Gronwall's inequality gives

$$|u(t)|^2 \leq |u_0|^2\mathrm{e}^{-\nu\lambda_1 t} + \frac{|f|^2}{\nu^2\lambda_1^2}(1-\mathrm{e}^{-\nu\lambda_1 t}).$$

So for t large enough (depending only on ϵ and $|u_0|$)

$$|u(t)|^2 \leq \rho_0^2 := \frac{2|f|^2}{\nu^2\lambda_1^2}. \tag{2.2}$$

The number 2 could be replaced by $1+\epsilon$ for any $\epsilon > 0$. □

[For an alternative estimate we can instead use

$$|(f,u)| \leq \|f\|_{-1}|Du|,$$

where $\|f\|_{-1}$ denotes the norm of f in $\mathbb{H}^{-1}$ (we have $\|f\|_{-1} = |A^{-1/2}f|$). Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|Du|^2 \leq \|f\|_{-1}|Du| \leq \frac{\nu}{2}|Du|^2 + \frac{1}{2\nu}\|f\|_{-1}^2,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|Du|^2 \leq \frac{\|f\|_{-1}^2}{\nu}. \tag{2.3}$$

From here we obtain, for t large enough,

$$|u(t)|^2 \leq [\rho_0^*]^2 := \frac{2\|f\|_{-1}^2}{\nu^2\lambda_1}.$$

Now, note that if f has expansion $f = \sum_{j=1}^{\infty} f_j w_j$, where w_k are the eigenfunctions of A, then it follows that

$$|f|^2 = \sum_{j=1}^{\infty} |f_j|^2$$

and

$$\|f\|_{-1}^2 = |A^{-1/2} f|^2 = \sum_{j=1}^{\infty} \lambda_j^{-1} |f_j|^2.$$

Since

$$\frac{\|f\|_{-1}^2}{|f|^2} = \sum_{j=1}^{\infty} \lambda_j^{-1} \frac{|f_j|^2}{\sum_i |f_i|^2} \tag{2.4}$$

it follows that $\|f\|_{-1} \leq \lambda_1^{-1/2} |f|$, and so

$$\frac{\|f\|_{-1}^2}{\nu^2 \lambda_1} \leq \frac{|f|^2}{\nu^2 \lambda_1^2},$$

and we have improved on the bound in (2.2).]

2.2 Absorbing set bounded in $\mathbb{H}^1$

The existence of an absorbing set in $\mathbb{H}^1$ for the 2d equations was first shown (in a different terminology) by Foias & Prodi (1967). This is the crucial ingredient for proving the existence of a global attractor (see next chapter), and is (essentially) the missing estimate that would allow us to prove regularity for the 3d equations.

Proposition 2.2. *The 2d Navier-Stokes equations have an absorbing set that is bounded in* $\mathbb{H}^1(\Omega)$.

Before proving the proposition we make an additional estimate from

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |u|^2 + \nu |Du|^2 \leq |f||u|.$$

(this was (2.1)). Integrating both sides between t and $t+1$ we obtain

$$|u(t+1)|^2 + \int_t^{t+1} |Du(s)|^2 \, \mathrm{d}s \leq |u(t)|^2 + |f| \int_t^{t+1} |u(s)| \, \mathrm{d}s.$$

Since $|u(t)| \leq \rho_0$ for t large enough, for all such t we also have

$$\int_t^{t+1} |Du(s)|^2 \, \mathrm{d}s \leq I_1 := \rho_0^2 + |f| \rho_0. \tag{2.5}$$

Although in the proof we use the orthogonality condition $b(u, u, Au) = 0$, which is only valid for periodic boundary conditions in 2d, the same result (with a slightly more involved argument and weaker estimates) holds for Dirichlet boundary conditions.

Proof. To prove the existence of this absorbing set we use a 'trick', which can be formalised as the 'uniform Gronwall lemma' (see Temam (1988), for example, although the statement of this as a formal lemma tends to hide the underlying idea). We take the inner product of (1.10) with Au to give

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Du|^2 + \nu|Au|^2 + b(u,u,Au) = (f, Au).$$

We now use an orthogonality condition (1.6), $b(u,u,Au) = 0$ and the Cauchy-Schwarz inequality to rewrite this as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Du|^2 + \nu|Au|^2 \le \frac{|f|^2}{2\nu} + \frac{\nu}{2}|Au|^2.$$

Dropping the $|Au|^2$ terms we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Du|^2 \le \frac{|f|^2}{2\nu}.$$

We integrate this equation between s and $t+1$, with $t \le s < t+1$, which gives

$$|Du(t+1)|^2 \le \frac{|f|^2}{2\nu} + |Du(s)|^2$$

(since $0 < t+1-s \le 1$). We now integrate both sides with respect to s between $t-1$ and t, and obtain

$$|Du(t+1)|^2 \le \frac{|f|^2}{2\nu} + \int_t^{t+1} |Du(s)|^2\,\mathrm{d}s.$$

We can now use (2.5): if t is large enough then we have

$$|Du(t+1)|^2 \le \rho_1^2 := \frac{|f|^2}{2\nu} + I_1.$$

□

Chapter 3

The global attractor: existence

We have seen that not only is there an absorbing set in $\mathbb{L}^2$, but there is a bounded absorbing set in $\mathbb{H}^1$. Since $\mathbb{H}^1$ is compactly embedded in $\mathbb{L}^2$, this gives us a compact absorbing set in $\mathbb{L}^2$.

In order to prove the existence of a global attractor we make a slightly weaker assumption. We say that a set B is attracting if

$$\text{dist}(S(t)X, B) \to 0 \qquad \text{as} \qquad t \to \infty,$$

where $\text{dist}(X, Y)$ is the Hausdorff semi-distance between two sets,

$$\text{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} |x - y|.$$

Note that this distance does *not* define a metric - indeed, if $\text{dist}(X, Y) = 0$ then one only has $X \subset Y$. To obtain a metric on subsets of H, we need to use the symmetric Hausdorff metric,

$$\text{dist}_{\mathcal{H}}(X, Y) = \max(\text{dist}(X, Y), \text{dist}(Y, X)). \tag{3.1}$$

(One can show that the space of all compact subsets of $\mathbb{R}^m$ is a complete space when endowed with this metric.)

We say that a semiflow is *dissipative* if it has a compact attracting set. It is easy to see that any semigroup with a compact absorbing set is dissipative.

Here we will prove a general result that a dissipative semigroup has a compact, globally attracting, invariant set, the 'global attractor'. The first result along these lines seems to be due to Bilotti & LaSalle (1971). General treatments of the theory of attractors are given in Babin & Vishik (1992), Hale (1988), Ladyzhenskaya (1991), Robinson (2001), and Temam (1988).

First we define the ω-limit set of a set X, which consists of all the limit points of the orbit of X:

$$\omega(X) = \{y : \exists\ t_n \to \infty,\ x_n \in X \text{ with } S(t_n)x_n \to y\}. \tag{3.2}$$

This can be also be characterised as

$$\omega(X) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)X}. \qquad (3.3)$$

$\omega(X)$ in some sense captures all the recurrent dynamics of the orbit through X.

Definition 3.1. *The global attractor $\mathcal{A}$ is the maximal compact invariant set*

$$S(t)\mathcal{A} = \mathcal{A} \qquad \text{for all } t \geq 0 \qquad (3.4)$$

which attracts all bounded sets;

$$\text{dist}(S(t)X, \mathcal{A}) \to 0 \qquad \text{as } t \to \infty, \qquad (3.5)$$

for any bounded set $X \subset H$.

That $\mathcal{A}$ is maximal means that if Y is a bounded invariant set then $\mathcal{A} \supset Y$. (3.5) says that $\mathcal{A}$ attracts all orbits, at a rate uniform on any bounded set. Without the compactness condition we could just take $\mathcal{A} = H$. Note that while $\mathcal{A}$ is the maximal compact invariant set, it is also the minimal set that attracts all bounded sets. Confusion is possible, since various authors refer to $\mathcal{A}$ as the 'minimal attractor' and others as the 'maximal attractor'.

We give the result in the following elegant version (inspired by a similar result due to Crauel (2001) for random dynamical systems; Hale (1988) and Babin & Vishik (1992) have similar results).

Theorem 3.2. *A semigroup $S(t)$ has a global attractor $\mathcal{A}$ if and only if it has a compact attracting set K, and then $\mathcal{A} = \omega(K)$.*

Note that the condition of a compact attracting set is much weaker than the existence of a compact absorbing set. The proof requires the following simple lemma:

Lemma 3.3. *If K is a compact set and x_n is a sequence such that*

$$\text{dist}(x_n, K) \to 0$$

then $\{x_n\}$ has a convergent subsequence whose limit lies in K.

As a first step to proving this new theorem first we prove the following properties of ω limit sets.

Proposition 3.4. *If there exists a compact attracting set K then the ω-limit set $\omega(X)$ of any bounded set X is a non-empty, invariant, closed subset of K. Furthermore $\omega(X)$ attracts X.*

Proof. To see that $\omega(X)$ is non-empty choose some point $x \in X$. Then since K is attracting

$$\text{dist}(S(n)x, K) \to 0.$$

It follows that for some sequence $n_j \to 0$

$$S(n_j)x \to x^* \in K.$$

As the intersection of a decreasing sequence of closed sets $\omega(X)$ is clearly closed. To show that $\omega(X) \subset K$ suppose that $t_n \to \infty$, $x_n \in X$ and

$$S(t_n)x_n \to y.$$

Then since K is attracting

$$\text{dist}(S(t_n)x_n, K) \to 0,$$

implying that a subsequence of $S(t_n)x_n$ converges to a point in K. Since the sequence itself converges it follows that $y \in K$. So $\omega(X)$ is compact.

Now suppose that $\omega(X)$ does not attract X. Then there exists a $\delta > 0$ and a sequence of t_n such that

$$\text{dist}(S(t_n)X, \omega(X)) > \delta,$$

and hence $x_n \in X$ such that

$$\text{dist}(S(t_n)x_n, \omega(X)) > \delta. \tag{3.6}$$

However, the argument above shows that a subsequence of $\{S(t_n)x_n\}$ converges to some point z. By (3.6) we should have

$$\text{dist}(z, \omega(X)) \geq \delta,$$

while by definition $z \in \omega(X)$. So $\omega(X)$ attracts X. □

Now observe that

$$A \subseteq B \qquad \Longrightarrow \qquad \omega(A) \subseteq \omega(B), \tag{3.7}$$

and that since $\omega(X)$ is invariant

$$\omega[\omega(X)] = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)\omega(X)} = \omega(X). \tag{3.8}$$

Proof. (Proof of theorem 3.2). It follows from the previous proposition that $\omega(K)$ is non-empty, compact, invariant, and attracts K. So all we have to prove is that $\omega(K)$ attracts X. Since $\omega(X)$ attracts X it suffices to show that $\omega(X) \subset \omega(K)$. But this follows immediately from (3.7) and (3.8). The only if part is clear, taking $K = \mathcal{A}$. □

Note that since the 2d Navier-Stokes equations have a compact absorbing set they have a compact attracting set, and hence a global attractor.

Theorem 3.5. *The 2d Navier-Stokes equations have a compact global attractor that is bounded in* $\mathbb{H}^1(Q)$.

Although the existence of an absorbing set in V for the 2D equations was first shown (in different terminology) by Foias & Prodi (1967), the proof of the existence of a global attractor for the 2D Navier-Stokes equations was first published by Ladyzhenskaya (1972), and later, along with many other important results, by Foias & Temam (1979).

We will see later that with higher regularity of f we can obtain much better bounds on the functions in the attractor.

3.1 The 3d Navier-Stokes equations

At present we cannot show that the 3D Navier-Stokes equations generated unique weak solutions, nor could we show that the strong solutions, which are unique, exist for all time. Trying to investigate the existence of attractors without the guarantee of a sensible semigroup seems futile.

However, the result given here shows that if we are prepared to assume that the equations generate a semigroup on V, i.e. if we assume the existence of strong solutions, then we can show that the equations must have a global attractor. In fact the result here just shows the existence of an absorbing set bounded in V, and to show that there is a global attractor we would need an absorbing set that is compact in V. A relatively straightforward argument can be used to prove the existence of an absorbing set that is bounded in $D(A)$ once we have the absorbing set in V, and hence of a global attractor.

What we are doing here is making a physically reasonable assumption in a mathematically precise way, and then deducing an entirely mathematical consequence. It allows us to consider the asymptotic regimes of the "true" Navier-Stokes equations, and so fully-developed turbulence, within a mathematical framework.

Another way to view this theorem, which does not require us to make any "unjustified" assumptions, is as a description of the way in which the 3D Navier-Stokes equations must break down if they are not well-posed. The theorem shows that existence and uniqueness fail only if there is some solution $u(t)$ such that $|Du(t)|$ becomes infinite in some finite time.

Theorem 3.6. *Suppose that the 3D Navier-Stokes equations are well-posed on V, so that for any $f \in H$ and $u_0 \in V$,*

$$du/dt + Au + B(u,u) = f$$

has a strong solution $u(t)$, i.e. a solution u with

$$u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$$

for all $T > 0$. Then there exists an absorbing set in V.

The theorem is due to Constantin et al. (1985); the proof also appears in Constantin & Foias (1988) and in Temam (1988).

Chapter 4

The global attractor: properties

4.1 A dynamical system on the attractor

Importantly, we can show that if the semigroup $S(t)$ is injective on $\mathcal{A}$ (we make this precise in the statement of the next theorem) then the dynamics, restricted to $\mathcal{A}$, actually define a dynamical system, i.e. $S(t)|_{\mathcal{A}}$ makes sense for all $t \in \mathbb{R}$. This is one good reason for investigating the dynamics on the attractor. The importance of this result is emphasised in Hale (1988).

Theorem 4.1. *If the semigroup is injective on $\mathcal{A}$, so that*

$$S(t)u_0 = S(t)v_0 \in \mathcal{A} \text{ for some } t > 0 \qquad \Rightarrow \qquad u_0 = v_0, \tag{4.1}$$

then every trajectory on $\mathcal{A}$ is defined for all $t \in \mathbb{R}$, and (3.4) holds for all $t \in \mathbb{R}$. In particular,

$$\big(\mathcal{A}, \{S(t)\}_{t\in\mathbb{R}}\big)$$

is a dynamical system.

Proof. For each $u \in \mathcal{A}$ we know that $u \in S(t)\mathcal{A}$, and so there exists a unique $v \in \mathcal{A}$ with $S(t)v = u$. We define $S(-t)u = v$ to give $S(t)$ for all $t \in \mathbb{R}$, and hence obtain (3.4) for $t < 0$ also. Since $\mathcal{A}$ is compact, it follows that $S(-t)$ as defined here is continuous on $\mathcal{A}$. Thus $S(t)$ is a continuous map from $\mathcal{A}$ into itself for all $t \in \mathbb{R}$. Finally, it follows that $S(t)|_{\mathcal{A}}$ satisfies the semigroup properties in (1.11) for all $t, s \in \mathbb{R}$. □

4.2 Structure of the attractor

We now want to examine the attractor itself in more detail, first investigating its structure. We show that it consists of all complete bounded orbits, and contains the unstable manifolds of all fixed points and periodic orbits. This gives us a better idea of the kind of dynamics we can expect to understand if we restrict our attention to the attractor.

Theorem 4.2. *All complete bounded orbits lie in $\mathcal{A}$. If $S(t)$ is injective on $\mathcal{A}$ (as in (4.1)) then $\mathcal{A}$ is the union of all the complete bounded orbits.*

A "complete" orbit $u(t)$ is a solution of the PDE (or ODE) which is defined for all $t \in \mathbb{R}$. In general we do not expect the solutions of a PDE to lie on a complete orbit, since we cannot define $S(t)$ for $t < 0$.

Proof. Let $\mathcal{O}$ be a complete bounded orbit, and assume that $\mathcal{O}$ is not contained in $\mathcal{A}$; then for some $\epsilon > 0$ there is a point $x \in \mathcal{O}$ with $x \notin N(\mathcal{A}, \epsilon)$. However, since $\mathcal{A}$ attracts bounded sets, for t large enough

$$\operatorname{dist}(S(t)z, \mathcal{A}) < \epsilon \qquad \text{for all } z \in \mathcal{O}. \tag{4.2}$$

Since $\mathcal{O}$ is a complete orbit, $x = S(t)\tilde{x}$ for some $\tilde{x} \in \mathcal{O}$; (4.2) now gives a contradiction.

If $x \in \mathcal{A}$ and $S(t)$ is injective then the orbit through x is defined for all $t \in \mathbb{R}$ (theorem 4.1) and is contained in $\mathcal{A}$ since $\mathcal{A}$ is invariant. Thus $\mathcal{A}$ consists only of complete bounded orbits. □

Note from the proof that we only use the injectivity to show that every point in $\mathcal{A}$ lies on a complete bounded orbit. Even without the injectivity we know that every complete bounded orbit lies in $\mathcal{A}$.

To investigate the structure of the attractor further, we need to recall the definition of stable and unstable manifolds. If z is a fixed point, then

Definition 4.3. *The* unstable manifold of z *is the set*

$$W^u(z) = \{u_0 \in H : \ S(t)u_0 \text{ is defined for all } t,\ S(-t)u_0 \to z \quad \text{as} \quad t \to \infty.\}$$

One can define the unstable manifold of a general invariant set X just as in definition 4.3, replacing "$S(-t)u_0 \to z$" by

$$\operatorname{dist}(S(-t)u_0, X) \to 0.$$

Now, the unstable manifolds of all invariant sets (in particular of all the fixed points and periodic orbits) are contained in the attractor:

Theorem 4.4. *If X is a compact invariant set, then*

$$W^u(X) \subset \mathcal{A}.$$

Proof. Let $u \in W^u(X)$. Then by definition (definition 4.3) u lies on the complete orbit $Y = \cup_{t \in \mathbb{R}} u(t)$. As $t \to -\infty$ we know that $\operatorname{dist}(u(t), X) \to 0$, and as $t \to \infty$ we know that $\operatorname{dist}(u(t), \mathcal{A}) \to 0$, so the orbit $u(t)$ is bounded. Thus u lies on a complete bounded orbit, and by theorem 4.2, $u \in \mathcal{A}$. □

Thus the commonly drawn picture of the attractor for the Lorenz equations is not in fact the whole of the global attractor. There are unstable fixed points at the centre of the "eyes", and their unstable manifolds will fill out these eyes.

4.3 Continuity properties of the attractor

We now show that the attractor cannot "explode", a property known as upper semicontinuity. This is a relatively easy result to prove, but essential in guaranteeing some stability for our notion of an attractor. We then discuss when it is possible to show that the attractor does not implode.

4.3.1 Upper semicontinuity

We treat a semigroup $S_0(t)$ with a global attractor $\mathcal{A}_0$, and a family of perturbed semigroups $S_\eta(t)$ with attractors $\mathcal{A}_\eta$. Such a situation could arise (for example) by making small changes to the right-hand side of an ODE or changing the forcing term in the Navier-Stokes equations. We first prove upper semi-continuity. Since $\text{dist}(\mathcal{A}_\eta, \mathcal{A}_0) < \epsilon$ means that $\mathcal{A}_\eta$ lies within a small ϵ-neighbourhood of $\mathcal{A}_0$, it is clear that the result of the following theorem means that $\mathcal{A}_0$ cannot "explode".

Theorem 4.5. *Assume that for $\eta \in [0, \eta_0)$ the semigroups S_η each have a global attractor A_η, and that there exists a bounded set X such that*

$$\bigcup_{0 \leq \eta < \eta_0} A_\eta \subset X.$$

If in addition the semigroups S_η converge to S_0 in that $S_\eta(t)x \to S_0(t)x$ uniformly on bounded time intervals $[0,T]$ and uniformly on bounded subsets Y of H,

$$\sup_{u_0 \in Y} \sup_{t \in [0,T]} |S_\eta(t)u_0 - S_0(t)u_0| \to 0 \qquad \text{as} \qquad \eta \to 0,$$

then

$$\text{dist}(\mathcal{A}_\eta, \mathcal{A}_0) \to 0 \qquad \text{as} \qquad \eta \to 0.$$

For similar results see Babin & Vishik (1992), Hale et al. (1988) or Temam (1988).

Proof. Given $\epsilon > 0$ choose T such that $S_0(T)X \subset N(\mathcal{A}_0, \epsilon/2)$, which is possible since $\mathcal{A}$ is attracting. Now choose $\eta_0 > 0$ such that

$$\sup_{t \in [0,T]} \sup_{x_0 \in X} |S_\eta(t)x_0 - S_0(t)x_0| < \epsilon/2 \qquad \text{for all} \qquad \eta \leq \eta_0.$$

It follows that

$$\mathcal{A}_\eta = S_\eta(T)\mathcal{A}_\eta \subset S_\eta(T)X \subset N(\mathcal{A}_0, \epsilon),$$

and we are done. □

Much use of this result has been made by Stuart and coworkers (see Stuart & Humphries, 1997) to address the long-time approximation of dynamical systems by numerical schemes.

4.3.2 Lower semicontinuity

The result that would give full continuity would be that

$$\text{dist}(\mathcal{A}_0, \mathcal{A}_\eta) \to 0 \qquad \text{as} \qquad \eta \to 0.$$

This is not known except in a few special cases. The most general result holds for a generalisation of gradient (Lyapunov) systems, and is due to Humphries (for antecedents see Hale et al., 1989). We give the proof here since the result does not seem to be widely known in this form.

Theorem 4.6. *(Stuart & Humphries, 1997). Let the assumptions of theorem 4.5 hold and in addition let $\mathcal{A}_0$ be given by the closure of the unstable manifolds of a finite number of fixed points, so that*

$$\mathcal{A}_0 = \bigcup_{z \in \mathcal{E}} \overline{W^u(z)}.$$

Provided that the unstable manifolds vary continuously with η near $\eta = 0$ in some neighbourhood of each fixed point, then the attractor is lower semicontinuous,

$$\text{dist}(\mathcal{A}_0, \mathcal{A}_\eta) \to 0 \qquad \text{as} \qquad \eta \to 0. \tag{4.3}$$

It follows that the attractor is continuous in the Hausdorff metric,

$$\text{dist}_{\mathcal{H}}(\mathcal{A}_\eta, \mathcal{A}_0) \to 0 \qquad \text{as} \qquad \eta \to 0.$$

Proof. We need to prove (4.3), i.e. that within ϵ of any point in $\mathcal{A}_0$ there is a point in $\mathcal{A}_\eta$ for all $\eta \leq \eta^*$, with $\eta^* > 0$. Since $\mathcal{A}_0$ is compact this reduces to showing that there are points in $\mathcal{A}_\eta$ within ϵ of some finite set of points $\{x_k\}$ in $\mathcal{A}_0$. Since $\mathcal{A}_0$ is the closure of the unstable manifolds, there are points $\{y_k\}$, lying in the unstable manifolds in $\mathcal{A}_0$, with

$$|x_k - y_k| \leq \epsilon/2.$$

We write $y_k = S_0(t_k)z_k$, with each z_k within a small neighbourhood of the fixed points. Since the set of $\{y_k\}$ is finite, $t_k \leq T$, for some $T > 0$.

Now choose $\delta > 0$ such that

$$|z_k - u| \leq \delta \qquad \Rightarrow \qquad |S_0(t)z_k - S_0(t)u| \leq \epsilon/4 \qquad \text{for all} \qquad t \in [0, T],$$

using the continuity of S_0, and then pick η small enough ($\eta \leq \eta_1$) such that

$$|S_0(t)u - S_\eta(t)u| \leq \epsilon/4 \qquad \text{for all} \qquad u \in N(\mathcal{A}_0, \delta), \; t \in [0, T].$$

Since the local unstable manifolds perturb smoothly, provided that η is small enough ($\eta \leq \eta^* \leq \eta_1$) there are points z_k^η in the unstable manifolds within $\mathcal{A}_\eta$ that satisfy

$$|z_k^\eta - z_k| < \delta.$$

It follows that

$$\begin{aligned}
|S_\eta(t_k)z_k^\eta - y_k| &= |S_\eta(t_k)z_k^\eta - S_0(t_k)z_k| \\
&\leq |S_\eta(t_k)z_k^\eta - S_0(t_k)z_k^\eta| + |S_0(t_k)z_k^\eta - S_0(t_k)z_k| \\
&\leq \epsilon/4 + \epsilon/4 = \epsilon/2.
\end{aligned}$$

Therefore

$$|S_\eta(t_k)z_k^\eta - x_k| \leq \epsilon,$$

and since $S_\eta(t_k)z_k^\eta \in \mathcal{A}_\eta$, we have obtained lower semicontinuity. Coupled with the result of theorem 4.5 we now have continuity of $\mathcal{A}_\eta$ at $\eta = 0$ in the Hausdorff metric. □

Chapter 5

Gevrey regularity

One property of solutions lying on the attractor is that they are more regular than arbitrary solutions. In particular, if the forcing function f is real analytic then the functions lying in the attractor are real analytic, in a uniform way.

A function $f(x)$ is real analytic [it can be represented locally by its Taylor series expansion] if and only if its derivatives satisfy

$$|D^\beta f| \leq M|\beta|!\tau^{-|\beta|}$$

for some M and τ. This motivates the definition of the analytic Gevrey class $D(\mathrm{e}^{\tau A^{1/2}})$: this consists of functions such that

$$|\mathrm{e}^{\tau A^{1/2}} u| < +\infty,$$

where $\mathrm{e}^{\tau^{1/2} A}$ is defined using the power series for exponentials,

$$\mathrm{e}^{\tau A^{1/2}} = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} A^{n/2}.$$

Writing u as a Fourier expansion

$$u = \sum_{j \in \mathbb{Z}^2} u_j \mathrm{e}^{\mathrm{i} \cdot jx}$$

we have

$$|\mathrm{e}^{\tau A^{1/2}} u|^2 = \sum_{j \in \mathbb{Z}^2} \mathrm{e}^{2\tau|j|} |u_j|^2.$$

In particular, therefore, if $u \in D(\mathrm{e}^{\tau A^{1/2}})$ the Fourier coefficients of u must decay exponentially fast.

Foias & Temam (1989) [see also Doering & Gibbon (1995)] showed that if $f \in D(\mathrm{e}^{\sigma A^{1/2}})$ for some $\sigma > 0$ then $u(t)$ is bounded in $D(A^{1/2}\mathrm{e}^{\tau A^{1/2}})$,

$$|A^{1/2}\mathrm{e}^{\tau A^{1/2}} u| \leq K \qquad \text{for all} \qquad t \geq T,$$

T and K depend only on $|Du(0)|$.

We give the proof here, following Foias & Temam's paper closely. We assume the following result (which is lemma 2.1 in Foias & Temam (1989)):

Lemma 5.1. *If u, v, and $w \in D(A\mathrm{e}^{\tau A^{1/2}})$ for some $\tau > 0$ then $B(u,v) \in D(\mathrm{e}^{\tau A^{1/2}})$ and*

$$|(\mathrm{e}^{\tau A^{1/2}} B(u,v), \mathrm{e}^{\tau A^{1/2}} Aw)| \\ \leq c|\mathrm{e}^{\tau A^{1/2}} A^{1/2} u|^{1/2} |\mathrm{e}^{\tau A^{1/2}} Au|^{1/2} |\mathrm{e}^{\tau A^{1/2}} A^{1/2} v||\mathrm{e}^{\tau A^{1/2}} Aw|,$$

for some $c > 0$.

In order to make the notation more compact, we can write

$$(u,v)_\tau = (\mathrm{e}^{\tau A^{1/2}} u, \mathrm{e}^{\tau A^{1/2}} v)$$

and

$$((u,v))_\tau = (A^{1/2}\mathrm{e}^{\tau A^{1/2}} u, A^{1/2}\mathrm{e}^{\tau A^{1/2}} v)).$$

The result of lemma 5.1 is now

$$|(B(u,v), Aw)_\tau| \leq c|A^{1/2}u|_\tau^{1/2} |Au|_\tau^{1/2} |A^{1/2}v|_\tau |Aw|_\tau, \tag{5.1}$$

cf. (1.8).

We now show:

Theorem 5.2. *If $f \in D(\mathrm{e}^{\sigma A^{1/2}})$ then for $t \leq T(|f|_\sigma + |A^{1/2}u(0)|)$ we have*

$$|A^{1/2}\mathrm{e}^{\phi(t)A^{1/2}} u(t)| \leq K(|f|_\sigma, |A^{1/2}u(0)|) \qquad \text{for all} \qquad 0 \leq t \leq T,$$

where $\phi(t) = \min(\sigma, t)$.

Proof. Taking the scalar product with Au (or u) in $D(\mathrm{e}^{\tau A^{1/2}})$ leads to an equation for $y = |A^{1/2}u|_\tau$ like $\dot{y} \leq Ky^3$. Not only do the solutions of this equation blow up in a finite time, but also we need to control $|A^{1/2}u(0)|_\tau$ in order to control $|A^{1/2}u(t)|_\tau$: we would need to start with analyticity in order to prove it.

The trick to get round this is to define $\phi(t) = \min(t, \sigma)$, and take the scalar product of

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = f$$

with $\mathrm{e}^{2\phi(t)A^{1/2}} Au$ to obtain

$$\begin{aligned} &\left(\frac{\mathrm{d}u}{\mathrm{d}t}, \mathrm{e}^{2\phi(t)A^{1/2}} Au\right) + \nu|\mathrm{e}^{\phi(t)A^{1/2}} Au|^2 \\ &\quad = (\mathrm{e}^{\phi(t)A^{1/2}} f, \mathrm{e}^{\phi(t)A^{1/2}} Au) - (\mathrm{e}^{\phi(t)A^{1/2}} B(u,u), \mathrm{e}^{\phi(t)A^{1/2}} Au) \\ &\quad = (f, Au)_\phi - (B(u,u), Au)_\phi \\ &\quad \leq |f|_\phi |Au|_\phi + c|A^{1/2}u|_\phi^{3/2} |Au|_\phi^{3/2} \\ &\quad \leq \frac{\nu}{4}|Au|_\phi^2 + c|A^{1/2}u|_\phi^{3/2} |Au|_\phi^{3/2} \\ &\quad \leq \frac{\nu}{4}|Au|_\phi^2 + \frac{2}{\nu}|f|_\phi^2 + \frac{c}{\nu^3}|A^{1/2}u|_\phi^6. \end{aligned}$$

The left hand side of the equation we can bound as

$$\begin{aligned}
&\left(\mathrm{e}^{\phi(t)A^{1/2}}\frac{\mathrm{d}u}{\mathrm{d}t}, \mathrm{e}^{\phi(t)A^{1/2}}Au\right) \\
&\quad = \left(A^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\phi(t)A^{1/2}}u(t)) - \frac{\mathrm{d}\phi}{\mathrm{d}t}A\mathrm{e}^{\phi(t)A^{1/2}}u(t), \mathrm{e}^{\phi(t)A^{1/2}}A^{1/2}u(t)\right) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A^{1/2}\mathrm{e}^{\phi(t)A^{1/2}}u(t)|^2 - \frac{\mathrm{d}\phi}{\mathrm{d}t}(A\mathrm{e}^{\phi(t)A^{1/2}}u, \mathrm{e}^{\phi(t)A^{1/2}}A^{1/2}u) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\phi(t)} - \frac{\mathrm{d}\phi}{\mathrm{d}t}(Au, A^{1/2}u)_{\phi(t)} \\
&\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\phi(t)} - |Au|_{\phi(t)}\|u\|_{\phi(t)} \\
&\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\phi(t)} - \frac{\nu}{4}|Au|^2_{\phi(t)} - \frac{1}{\nu}\|u\|^2_{\phi(t)}.
\end{aligned}$$

We therefore have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|A^{1/2}u|_\phi + \nu|Au|^2_\phi &\leq \frac{4}{\nu}|f|^2_\phi + \frac{2}{\nu}|A^{1/2}u|^2_\phi + \frac{c}{\nu^3}|A^{1/2}u|^6_\phi \\
&\leq \frac{4}{\nu}|f|^2_\phi + c + \frac{c}{\nu^3}|A^{1/2}u|^6_\phi.
\end{aligned}$$

Now we can set

$$y(t) = 1 + |A^{1/2}u(t)|^2_{\phi(t)},$$

and we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leq Ky^3 \tag{5.2}$$

with

$$K = \frac{4}{\nu}|f|^2_\sigma + c + \frac{c}{\nu}.$$

The solution of (5.2) is

$$y(t) \leq \frac{1}{y(0)^{-2} - 2Kt},$$

and so $y(t) \leq 2y(0)$ for $t \leq (4Ky(0)^2)^{-1}$. Since $\phi(0) = 0$, we have

$$y(0) = 1 + |A^{1/2}u(0)|^2,$$

and so for $t \leq T(|A^{1/2}u(0)|, |f|_\sigma)$, we have

$$|A^{1/2}u(t)| \leq K(|A^{1/2}u(0)|, |f|_\sigma).$$

□

It now follows that solutions on the attractor are uniformly bounded in $D(A^{1/2}\mathrm{e}^{\tau A^{1/2}})$:

Corollary 5.3. *The global attractor for the 2d Navier-Stokes evolution equation with periodic boundary conditions is uniformly bounded in* $D(A^{1/2}e^{\tau A^{1/2}})$, *i.e. there exists a constant* K *such that if* $u \in \mathcal{A}$ *then*

$$|A^{1/2}e^{\tau A^{1/2}}u| \leq K. \tag{5.3}$$

In particular the attractor consists of real analytic functions.

Proof. The attractor is bounded in $\mathbb{H}^1(Q)$, with $|A^{1/2}u| \leq M$ for all $u \in \mathcal{A}$. Set $T = T(M, |f|_\sigma)$. Since the attractor is invariant, if $u \in \mathcal{A}$ then there exists a $u_0 \in \mathcal{A}$ such that $u = S(T)u_0$. It follows from the above theorem that

$$|A^{1/2}e^{\phi(T)}S(T)u_0| \leq K(M, |f|_\sigma).$$

Now set $\tau = \phi(T)$ and $K = K(M, |f|_\sigma)$, which gives (5.3). □

Chapter 6

Finite-dimensional attractors

6.1 Fractal (box-counting) dimension

The "fractal" dimension, which we will write as $d_f(X)$, is based on counting the number of closed balls of a fixed radius ϵ needed to cover X.

We denote the minimum number of balls in such a cover by $N(X, \epsilon)$. If X were a line, we would expect $N(X, \epsilon) \sim \epsilon^{-1}$, if X a surface we would have $N(X, \epsilon) \sim \epsilon^{-2}$, and for a (3-) volume we would have $N(X, \epsilon) \sim \epsilon^{-3}$. So one possible method for obtaining a general measure of dimensions would be to say that X has dimension d if $N(X, \epsilon) \sim \epsilon^{-d}$. Accordingly, we make the following definition. (The fractal dimension is also known as the (upper) box-counting dimension[1] and the entropy dimension.)

Definition 6.1. *The* fractal dimension *of X, $d_f(X)$ is given by*

$$d_f(X) = \limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{\log(1/\epsilon)}, \tag{6.1}$$

where we allow the limit in (6.1) to take the value $+\infty$.

Note that it follows from the definition that if $d > d_f(X)$, then for sufficiently small ϵ,

$$N(X, \epsilon) \leq \epsilon^{-d}. \tag{6.2}$$

The fractal dimension has the following properties:

[1]At least in the finite-dimensional spaces $\mathbb{R}^m$ definition 6.1 is equivalent to the "upper box-counting dimension". The idea here is take a grid of cubes of side ϵ which cover $\mathbb{R}^m$, and let $N(X, \epsilon)$ be the number of cubes which intersect X. The dimension is then defined exactly as in 6.1. Although these definitions coincide in $\mathbb{R}^m$, the "unit cube" $[0, 1]^\infty$ in an infinite-dimensional Hilbert space has elements with arbitrarily large norm, so one cannot "count boxes" in this context.

Proposition 6.2. *(Properties of fractal dimension).*
(i) Stability under finite unions:

$$d_f\left(\bigcup_{k=1}^{N} X_k\right) \leq \max_k \; d_f(X_k), \tag{6.3}$$

(ii) if $f : H \to H$ is Hölder continuous with exponent θ, i.e.

$$|f(x) - f(y)| \leq L|x - y|^{\theta}, \tag{6.4}$$

then $d_f\big(f(X)\big) \leq d_f(X)/\theta$,
(iii) $d_f(X \times Y) \leq d_f(X) + d_f(Y)$, and
(iv) if $\overline{X}$ is the closure of X in H, then $d_f(\overline{X}) = d_f(X)$,

For more details see Eden et al. (1994), Falconer (1985 & 1990), C. Robinson (1995), Robinson (2001).

6.2 Dimension estimates

There is an analytical method by which we can estimate the fractal dimension of the global attractor. The idea is to study the evolution of infinitesimal n-dimensional volumes as they evolve under the flow, and try to find the smallest dimension n at which we can guarantee that all such n-volumes contract asymptotically. We will not give the analysis in detail, but merely in outline.

We will consider an abstract problem, written as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u(t)) \qquad u(0) = u_0,$$

with u_0 contained in a Hilbert space H, whose norm we denote by $|\cdot|$. We assume that the equation has unique solutions given by $u(t; u_0) = S(t)u_0$, and a compact global attractor $\mathcal{A}$.

We want to start off with an orthogonal set of infinitesimal displacements near an initial point $u_0 \in \mathcal{A}$, and then watch how the volume they form evolves under the flow.

To study the evolution of this volume we have to study the evolution of a set of infinitesimal displacements $\delta x^{(i)}(t)$ about the trajectory $u(t)$. We suppose that the evolution of these displacements is given by the linearised equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = F'(u(t))U(t) \qquad U(0) = \xi,$$

which we write as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = L(t; u_0)U(t) \qquad U(0) = \xi. \tag{6.5}$$

The validity of such a linearisation is one of the main points to check when applying this theory rigorously. To this end we make the following definition:

Definition 6.3. *We say that* $S(t)$ *is* uniformly differentiable on $\mathcal{A}$ *if for every* $u \in \mathcal{A}$ *there exists a linear operator* $\Lambda(t,u)$*, such that, for all* $t \geq 0$*,*

$$\sup_{u,v\in\mathcal{A};\,0<|u-v|\leq\epsilon} \frac{|S(t)v - S(t)u - \Lambda(t,u)(v-u)|}{|v-u|} \to 0 \text{ as } \epsilon \to 0 \tag{6.6}$$

$$\text{and} \qquad \sup_{u\in\mathcal{A}} \|\Lambda(t,u)\|_{\mathrm{op}} < \infty \qquad \textit{for each } t \geq 0.$$

Although this is straightforward to check for ordinary differential equations, its proof in the PDE will often involve technical difficulties.

Heuristically speaking the growth rate of each infinitesimal displacement $\delta x^{(j)}$ will be related to the eigenvalues of L. In particular, the length of an infinitesimal displacement in the λ-eigendirection is $\mathrm{e}^{\lambda t}\delta(0)$ at time t: the growth rate is λ, attached to the eigen-direction of L with eigenvalue λ. The size of a small 2-volume with sides in two different eigendirections would be

$$\mathrm{e}^{(\lambda_1+\lambda_2)t},$$

and so the growth "rate" is $\lambda_1 + \lambda_2$. The growth rate of an n-volume made of infinitesimal directions in n eigendirections would be

$$\lambda_1 + \ldots + \lambda_n.$$

If we can makes sure that this growth rate must be negative then we know that n-volumes contract. It is true, although by no means immediate, that if all n-volumes contract then the dimension of the attractor must be smaller than n. In the finite-dimensional setting this result is due to Douady & Oesterlé (1980), while in the infinite-dimensional case it was proved by Constantin & Foias (1985) and Constantin et al. (1985).

To extract the "growth rate", we consider

$$\sum_{j=1}^{n} (\phi_j, L(t;u_0)\phi_j), \tag{6.7}$$

over all possible orthonormal collections of n elements $\{\phi_j\}_{j=1}^n$ of H. The idea is, essentially, that the maximum over all choices of ϕ_j gives the largest possible growth rate, i.e. the sum of the n largest eigenvalues of L. A more compact notation for (6.7) is

$$\mathrm{Tr}(L(t;u_0)P),$$

where Tr denotes the trace in H and P is the orthogonal projection onto the space spanned by the $\{\phi_j\}_{j=1}^n$.

The following theorem is given in a form suitable for calculations. We denote by $\langle f\rangle$ the time average of $f(t)$, namely

$$\langle f\rangle = \limsup_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,\mathrm{d}s.$$

Theorem 6.4. *Suppose that $S(t)$ is uniformly differentiable on $\mathcal{A}$, and there exists a t_0 such that $\Lambda(t, u_0)$ is compact for all $t \geq t_0$. If for some n we have*

$$\sup_{u_0 \in \mathcal{A}} \left\langle \sup_{\{\phi_j\}_{j=1}^n} \sum_{j=1}^n (\phi_j, L(u(t))\phi_j) \right\rangle < 0$$

for any choice of n orthogonal elements $\{\phi_j\}$ of H, then $d_f(\mathcal{A}) < n$. [Here $u(t)$ is the solution through u_0.]

[The result of Constantin & Foias gives $d_f(\mathcal{A}) < 2n$ and $d_H(\mathcal{A}) < n$ (where d_H denotes the Hausdorff dimension); this was improved by Hunt (1999, Appendix B in Robinson 2001 , see also Chepyzhov & Ilyin, 2001) to show that $d_f(\mathcal{A}) < n$. Constantin, Foias, & Temam in fact developed the 'trace formula' for dimension estimation: the important quantity is in fact

$$\sup_{u_0 \in \mathcal{A}} \sup_{P^{(n)}(0)} \langle \mathrm{Tr}(L(t; u_0)P^{(n)}(t) \rangle,$$

where $P^{(n)}(t)$ is the projection onto the space spanned by the solutions

$$\{\delta x^{(1)}(t), \ldots, \delta x^{(n)}(t)\}.$$

Requiring a bound on the trace over all *fixed projections* $P^{(n)}$ for each t leads to the formulation of theorem 6.4. An alternative method is given in Mallet-Paret (1976).]

Before we apply this result to give dimension bounds for reaction-diffusion equations and for the Navier-Stokes equations, we will need an auxiliary lemma which gives a lower bound on

$$\sum_{j=1}^n (\phi_j, -\Delta\phi_j),$$

valid for any choice of n orthogonal elements of H.

Indeed, we use exactly this idea in the next lemma, which gives a bound on this quantity over all choices of ϕ_j when $L(t; u_0) = -\Delta$.

Lemma 6.5. *For any choice of n orthogonal elements $\{\phi_j\}_{j=1}^n$ of $\mathbb{L}^2(Q)$,*

$$\sum_{j=1}^n |D\phi_j|^2 = \sum_{j=1}^n (\phi_j - \Delta\phi_j) \geq CL^{-m} n^{(m+2)/m}. \tag{6.8}$$

The same result is also valid for any bounded C^2 domain $\Omega \subset \mathbb{R}^m$.

Proof. Write $A = -\Delta$, and denote its orthonormal eigenfunctions as w_j, with corresponding eigenvalues λ_j ordered so that $\lambda_{j+1} \geq \lambda_j$. We show first that

$$\sum_{j=1}^n (\phi_j, A\phi_j) \geq \sum_{j=1}^n \lambda_j. \tag{6.9}$$

We rewrite the left-hand side of (6.9), expanding the ϕ_j in terms of the eigenbasis $\{w_j\}$, and obtain

$$\begin{aligned}\mathrm{Tr}(AP_n) &= \sum_{j=1}^{n}\sum_{k=1}^{\infty}\lambda_k|(\phi_j,w_k)|^2 \\ &= \sum_{k=1}^{\infty}\lambda_k\Bigg(\sum_{j=1}^{n}|(\phi_j,w_k)|^2\Bigg).\end{aligned}$$

Now, since $|\phi_j| = 1$ we have

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty}|(w_k,\phi_j)|^2 = n,$$

and since $\{\phi_j\}$ are orthonormal but do not span H we have

$$\sum_{j=1}^{n}|(w_k,\phi_j)|^2 \le 1.$$

(6.9) now follows.

The explicit bound in (6.8) follows easily, using the property of the eigenvalues of the Laplacian

$$cj^{2/m} \le \lambda_j \le Cj^{2/m},$$

since then

$$\sum_{j=1}^{n}\lambda_j \ge c\sum_{j=1}^{n}j^{2/m} \ge cn^{(2/m)+1} = cn^{(m+2)/m}.$$

□

6.3 Dimension estimate for the 2d Navier-Stokes equation

We first state the differentiability result.

Theorem 6.6. *The solutions of the Navier-Stokes equations in 2d satisfy (6.6) with $\Lambda(t;u_0)\xi$ the solution of the equation*

$$\frac{\mathrm{d}U}{\mathrm{d}t} + \nu AU + B(u,U) + B(U,u) = 0 \qquad U(0) = \xi. \tag{6.10}$$

Furthermore, $\Lambda(t;u_0)$ is compact for all $t > 0$.

For a proof see Constantin & Foias (1985), Constantin et al. (1985), Temam (1988), or Robinson (2001).

With the differentiability ensured we can apply the trace formula to find a bound on the dimension.

Theorem 6.7. *The attractor for the 2d periodic Navier-Stokes equations is finite dimensional, with*

$$d_f(\mathcal{A}) \leq \frac{c}{\lambda_1^{1/2}\nu}\langle |Du|_{L^2}^2\rangle^{1/2}. \tag{6.11}$$

The result, due to Constantin et al. (1985), is also valid as stated for Dirichlet boundary conditions.

Proof. The correct form of the linearised equation is given in (6.10), and so

$$L(u)w = \nu Aw - B(w,u) - B(u,w).$$

Thus the time-averaged trace $\langle P_n L(u(t))\rangle$ is bounded by

$$\begin{aligned}\langle P_n L(u)\rangle &= \left\langle \sum_{j=1}^n (L(u)\phi_j,\phi_j)\right\rangle \\ &= -\left\langle \sum_{j=1}^n (-\nu\Delta\phi_j,\phi_j)\right\rangle - \left\langle \sum_{j=1}^n b(\phi_j,u,\phi_j)\right\rangle.\end{aligned}$$

In order to bound the contribution from the nonlinear term we could proceed using the standard bound on $b(u,v,w)$ in (1.7), which leads to an estimate similar to (6.11) but with a factor of ν^2 rather than ν.

However, a better estimate can be obtained as follows. Note that we have

$$\sum_{j=1}^n b(\phi_j,u,\phi_j) = \int_\Omega \sum_{j=1}^n \sum_{i,k=1}^2 \phi_{ji}(x)\frac{\partial u_k}{\partial x_i}(x)\phi_{jk}(x)\,\mathrm{d}x,$$

and we have for each $x \in Q$

$$\left|\sum_{i,k=1}^2 \phi_{ji}(x)\frac{\partial u_k}{\partial x_i}(x)\phi_{jk}(x)\right| \leq \left(\sum_{i=1}^2\sum_{j=1}^n \phi_{ji}(x)^2\right)\left(\sum_{i,k=1}^2 \left|\frac{\partial u_k}{\partial x_i}(x)\right|^2\right)^{1/2}.$$

It follows use the Cauchy-Schwarz inequality that

$$|\sum_{j=1}^n b(\phi_j,u,\phi_j)| \leq |Du||\rho|_{L^2},$$

where

$$\rho(x) = \sum_{i=1}^2\sum_{j=1}^n \phi_{ji}(x)^2.$$

An inequality due to Lieb & Thirring (1976), adapted appropriately to this case (details are given in Temam, 1988) allows us to bound $|\rho|_{L^2}$ by

$$|\rho|_{L^2}^2 \leq c\sum_{j=1}^n |D\phi_j|^2.$$

It follows that

$$P_nL(u) \leq -\nu \sum_{j=1}^{n} |D\phi_j|^2 + c|Du| \left(\sum_{j=1}^{n} |D\phi_j|^2 \right)^{1/2},$$

and so, using the Cauchy-Schwarz inequality we obtain

$$P_nL(u) \leq -\frac{\nu}{2} \sum_{j=1}^{n} |D\phi_j|^2 + \frac{c}{\nu}|Du|^2.$$

Now taking the time average and using lemma 6.5 we obtain

$$\langle P_nL(u)\rangle \leq -\frac{\nu}{2}n^2 + \frac{c}{\nu}\langle|Du|\rangle^2.$$

We therefore have $\langle P_nL(u)\rangle < 0$ provided that $n > \lambda_1^{-1/2}\nu^{-1}\langle|Du|^2\rangle^{1/2}$ as claimed. □

In order to make the dimension estimate more explicit, we define the dimensionless Grashof number G, which measures the relative strength of the forcing and viscosity by

$$G = \frac{|f|_{L^2}}{\nu^2\lambda_1},$$

and estimate $\langle|Du|^2\rangle$ in terms of G.

Returning to the equation (2.1) that we obtained on the way to finding an absorbing set in $\mathbb{L}^2$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|Du|^2 \leq |f||u| \tag{6.12}$$

we can use the Poincaré inequality and Cauchy-Schwarz inequalities on the right-hand side to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|Du|^2 \leq \lambda_1^{-1/2}|f||Du| \leq \frac{|f|^2}{2\nu\lambda_1} + \frac{\nu}{2}|Du|^2,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|Du|^2 \leq \frac{|f|^2}{\nu\lambda_1}.$$

Integrating between 0 and t we obtain

$$\nu\int_0^t |Du(s)|^2\,\mathrm{d}s \leq \left(|u(0)|^2 - |u(t)|^2\right) + \frac{t|f|^2}{\nu\lambda_1},$$

which gives

$$\limsup_{t\to\infty} \langle|Du|^2\rangle \leq \frac{|f|^2}{\nu^2\lambda_1}.$$

Therefore

$$d_f(\mathcal{A}) \leq \frac{c}{\lambda_1^{1/2}\nu}\frac{|f|}{\nu\lambda_1^{1/2}} = c\frac{|f|}{\nu^2\lambda_1} = cG.$$

We have shown:

Theorem 6.8. *The global attractor for the 2d Navier-Stokes equations is finite-dimensional, and*

$$d_f(\mathcal{A}) \leq cG.$$

This is the best bound in the case of Dirichlet boundary conditions. By working with the equation for $\omega = \nabla \wedge u$ and using the identity $(B(u,u), Au) = 0$, Constantin et al. (1988) were able to improve this for periodic boundary conditions to

$$d_f(\mathcal{A}) \leq cG^{2/3}(1 + \log G)^{1/3}. \tag{6.13}$$

This bound is known to be sharp (see Babin & Vishik (1988), Liu (1993), Ziane (1997)).

6.4 Reflecting the lengthscales in the forcing

Note that G reflects only the amount of energy being put into the flow, and says nothing of the scales at which the energy is supplied. The following simple calculation, inspired by the paper of Olson & Titi (2003), shows that it is in fact possible to improve on the estimate in the Dirichlet boundary condition case, and on the estimate (6.13) in the periodic case when the forcing is at very small scales, by making a small modification to the above argument (see Robinson 2003). If we estimate $\langle |Du|^2 \rangle$ from

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \nu|Du|^2 \leq \frac{\|f\|_{-1}^2}{\nu}.$$

(this was (2.3)) rather than from (6.12) as above, then we obtain (the argument is similar)

$$\langle |Du|^2 \rangle \leq \frac{\|f\|_{-1}^2}{\nu^2} = \lambda_1 \nu^2 G_*^2,$$

where now we have defined an alternative Grashof number G_* based on $\|f\|_{-1}$,

$$G_* := \frac{\|f\|_{-1}}{\nu^2 \lambda_1^{1/2}}.$$

It follows that

$$d_f(\mathcal{A}) \leq cG_*.$$

We have already shown in (2.4) that $\|f\|_* \leq \lambda_1^{-1/2}|f|_{L^2}$, and it follows that $G_* \leq G$.

6.4.1 The 'effective lengthscale' of the forcing

In the case of periodic boundary conditions, the eigenvalue corresponding to the (un-normalised) eigenfunction $\mathrm{e}^{2\pi \mathrm{i}\underline{k}\cdot x/L}$ is

$$\lambda_{\underline{k}} = \frac{4\pi^2|\underline{k}|^2}{L^2};$$

it follows that $\lambda_k^{-1/2}$, which has the dimension of a length, is essentially the lengthscale on which the eigenfunction varies. Carrying this idea over to the case of Dirichlet boundary conditions, it is a convenient and suggestive shorthand to describe a forcing of the form

$$f = \sum_{k_1 \le j \le k_2} f_j w_j \tag{6.14}$$

as being confined to lengthscales between $\lambda_{k_1}^{-1/2}$ and $\lambda_{k_2}^{-1/2}$; it is clear from (2.4) that for a forcing of this form

$$\lambda_{k_2}^{-1/2} |f|_{L^2} \le \|f\|_* \le \lambda_{k_1}^{-1/2} |f|_{L^2}. \tag{6.15}$$

More suggestively one can say that if the forcing is confined to lengthscales between $l_{\min}$ and $l_{\max}$ then

$$l_{\min}|f| \le \|f\|_* \le l_{\max}|f|.$$

Using $l_{\rm ref} = \lambda_1^{-1/2}$ as a reference length, this gives

$$\frac{l_{\min}}{l_{\rm ref}}\, G \le G_* \le \frac{l_{\max}}{l_{\rm ref}}\, G.$$

Returning to (2.4), the ratio $\|f\|_*^2/|f|^2$ appears as an average of squared lengthscales, weighted according to the amount of energy injected at each scale. Accordingly, it is natural to define an effective lengthscale of the forcing by

$$l_{\rm eff} = \|f\|_*/|f|,$$

in which case the relationship

$$G_* = \frac{l_{\rm eff}}{l_{\rm ref}}\, G \tag{6.16}$$

is essentially a tautology. (Ratios of successive Sobolev norms have already been used to define lengthscales; for example, a whole hierarchy of lengths is defined in Doering & Gibbon (1995) using (essentially) the ratios of successive norms of the solution u.)

Note that this implies that the dimension of the attractor is less than one if the forcing is applied at sufficiently small scales. Since the attractor is a compact connected set, this in fact implies that the attractor is a point (Falconer, 1985), so that the dynamics is trivial (no matter how 'complicated' the forcing is at these scales).

6.5 Physical interpretation of the attractor dimension

One way of interpreting the physical significance of an attractor is as a means of giving a rigorous notion of the number of independent "degrees of freedom" of the asymptotic dynamics of the system. Following Doering & Gibbon (1995) [also Constantin et al. (1988)] we relate this estimate to a lengthscale in the underlying physical problem

Suppose that there is a smallest physically relevant length-scale l in the problem, the idea being that interactions on scales of less than l do not affect the dynamics (for example, in fluid mechanics the viscosity has a large effect on the very small scales, and we hope that this means that fluctuations on these scales have negligible effects). A heuristic indication of the number of degrees of freedom would then given by how many "boxes" of side l fit into Ω,

$$n_{\text{heuristic}} \sim |\Omega| l^{-m}.$$

If we assume that $n_{\text{heuristic}}$ is a good estimate of the true number of degrees of freedom, and in turn (!) that this is well estimated by the attractor dimension, we can isolate a length scale, given in terms of $d_f(\mathcal{A})$ by

$$l \sim \left(\frac{|\Omega|}{d_f(\mathcal{A})} \right)^{1/m}. \tag{6.17}$$

Notice that tighter bounds on $d_f(\mathcal{A})$ raise the estimate of the smallest length scale. Recall that the best current estimates for the dimension of the 2d Navier-Stokes attractor are

$$d_f(\mathcal{A}) \leq cG^{2/3}(1 + \log G)^{1/3} \tag{6.18}$$

in the case of periodic boundary conditions, and

$$d_f(\mathcal{A}) \leq cG$$

in the case of Dirichlet boundary conditions, both due to Constantin *et al.* (1988a) [a simpler proof of (6.18) can be found in Doering & Gibbon (1995)].

What is remarkable about the bound in (6.18) is that, using our "very heuristic" estimate in (6.17), it corresponds to a length scale l which satisfies

$$\frac{l}{L} \sim G^{-1/3}, \tag{6.19}$$

to within logarithmic corrections (L denotes the size of one side of our 2D periodic domain). The length scale in (6.19) is precisely the "Kraichnan length", derived by other (also heuristic) methods as the natural minimum scale in two-dimensional turbulent flows (see Kraichnan, 1967). This links the rigorous analytical bound on the attractor dimension with an "intuitive" estimate from fluid dynamics.

Chapter 7

Embedding theorems and finite-dimensional dynamics

What does it mean for a set to be finite-dimensional? And, in particular, what does it mean for the attractor of a PDE (or ODE) to be finite-dimensional?

7.1 Embedding and parametrisation

We will show that, if k is large enough (roughly twice the dimension of the attractor), then the attractor can be embedded into $\mathbb{R}^k$. This result can be considered in two ways. Firstly, the idea is that we can take the attractor "out" of the infinite-dimensional space and map it, using some linear map L, homeomorphically onto a subset of $\mathbb{R}^k$. This makes sense of the idea that $\mathcal{A}$ is a finite-dimensional set. Secondly, and perhaps more importantly, it follows that L^{-1} provides a way of parametrising the attractor using a finite set of coordinates. We will make use of both interpretations in what follows.

The following theorem is due to Hunt & Kaloshin (1999). The first such embedding result for arbitrary finite-dimensional sets was proved by Mañé (1981). Ben-Artzi et al. (1993) showed that the inverse of the projection is Hölder continuous in the finite-dimensional case, along with strict bounds on the Hölder exponent. Foias & Olson (1996) subsequently proved that the inverse of the projections is Hölder continuous in the infinite-dimensional case, but without bounds on the Hölder exponent, and Hunt & Kaloshin introduced the notion of the 'thickness' of a set and gave the strict bounds on the exponent given in the following theorem.

In the statement of the theorem, the "thickness" of X, $\tau(X)$, is given by

$$\tau(X) = \limsup_{\epsilon\to 0} \frac{\log d(X,\epsilon)}{\log(1/\epsilon)},$$

where $d(X,\epsilon)$ is the minimum dimension of all finite-dimensional subspaces, V, of B such that every point of X lies within ϵ of V. [Foias & Olson implicitly use the bound $\tau(X) \leq d_f(X)$ in their proof.]

Note that the following alternative definition (whose equivalence is proved in Kukavica & Robinson, 2003) is often more useful in applications: let X be a subset of a Banach space $\mathcal{B}$, and denote by $\varepsilon_{\mathcal{B}}(X, n)$ the minimum distance between X and any n-dimensional linear subspace of $\mathcal{B}$. Then

$$\tau(X, \mathcal{B}) = \lim_{n \to \infty} \frac{-\log n}{\log \varepsilon_{\mathcal{B}}(X, n)}. \tag{7.1}$$

Theorem 7.1. *[Hunt & Kaloshin] If X is a compact subset of a Banach space B then, provided that D is an integer with $D > 2d_f(X)$ and*

$$\theta < \frac{D - 2d_f(X)}{D(1 + \tau(X))},$$

a dense set of linear maps from B into $\mathbb{R}^D$ have the following properties:

(i) they are injective on X, and

(ii) their inverse is Hölder continuous from LX into X with exponent θ,

$$|L^{-1}x - L^{-1}y| \leq C|x - y|^{\theta} \qquad \text{for all} \qquad x, y \in LX. \tag{7.2}$$

(We note here that their result is in fact slightly stronger: in a Hilbert space the factor of τ in the denominator can be changed to $\tau/2$, while they in fact obtain a "prevalent" set of linear maps with this Hölder property, see Hunt *et al.* (1992).)

It is relatively easy to show that if X is a compact subset of $L^2(\Omega)$ that is bounded in $H^k(\Omega)$ then $\tau(X) \leq d/k$ when $\Omega \subset \mathbb{R}^d$, see Friz & Robinson (1999).

7.2 A proof in the finite-dimensional case

We will prove an analogous (though weaker) result for subsets of a (large) finite-dimensional space. The proof is based on that in Eden et al. (1994).

Theorem 7.2. *If X is a compact subset of $\mathbb{R}^N$ with $d_f(X) \leq d$ ($d \in \mathbb{N}$) then there exists an orthogonal projection P, of rank $2d+2$, which is injective on X, and whose inverse is continuous on PX.*

The theorem is a corollary of the following result.

Proposition 7.3. *If Y is a compact subset of $\mathbb{R}^N$ with $d_H(Y) \leq k \leq N - 3$ then there exists an orthogonal projection P, of rank $k+2$, with*

$$\operatorname{Ker} P \cap Y = \{0\}.$$

Given this proposition, the proof of theorem 7.2 is straightforward:

Proof. (Theorem 7.2) Let

$$Y = X - X = \{x_1 - x_2 : \ x_1 \in X, \ x_2 \in X\}.$$

Then since $d_f(X) \leq d$, we have $d_f(Y) \leq 2d$ (by results from proposition 6.2). By proposition 7.3 there is an orthogonal projection P, of rank $2d + 2$, with

$$\ker P \cap Y = \{0\}.$$

Thus, if $x_1, x_2 \in X$, $P(x_1 - x_2) = 0$ implies that $x_1 - x_2 = 0$, i.e.

$$Px_1 = Px_2 \qquad \text{implies that} \qquad x_1 = x_2.$$

so P is injective on X.

To show that the inverse is continuous, suppose not. Then there exists a sequence $\{x_n\} \in PX$ with $x_n \to y \in PX$ but $|P^{-1}x_n - P^{-1}y| \geq 0 > 0$. However, $P^{-1}x_n \in X$, and since X is compact there exists a subsequence x_{n_j} such that $P^{-1}x_{n_j} \to z$. Since P is continuous, it follows that $x_{n_j} \to Pz$. Since P is injective, it follows from $Pz = y$ that $z = P^{-1}y$, which is a contradiction. So P^{-1} is continuous on PX. □

To prove proposition 7.3 we need the following simple lemma.

Lemma 7.4. *Let Y be a compact subset of $\mathbb{R}^N$ with $d_f(Y) \leq N-3$. Then there exists a unit vector b such that $b \notin \mathbb{R}Y$ (i.e. $b \neq \alpha y$ for any $\alpha > 0$, $y \in Y$).*

Proof. (Lemma **??**) $d_f(\mathbb{R}Y) \leq N - 2$, but $d_f(\{b : |b| = 1\}) = N - 1$. □

Proof. (Proposition 7.3) By lemma 7.4, there exists a unit vector $a \notin \mathbb{R}Y$. Then if

$$Q_1 = I - aa^*, \qquad \text{where} \qquad (aa^*)x = a(a, x),$$

suppose that $y \in \ker Q_1 \cap Y$; this means that

$$(a, y)a = y.$$

But $a \notin \mathbb{R}Y$, so $y = 0$. Thus

$$\ker Q_1 \cap Y = \{0\}.$$

We now proceed by induction. Suppose that we have an orthogonal projector Q_m of rank $N - m$, with

$$\ker Q_m \cap Y = \{0\}$$

and $k \leq N - m - 3$. We apply lemma 7.4 to $Q_m\mathbb{R}^N \simeq \mathbb{R}^{N-m}$, so that there is a unit vector $a \notin \mathbb{R}[Q_mY]$, and consider

$$Q_{m+1} = Q_m - aa^*.$$

Again, if $y \in \ker Q_{m+1} \cap Y$, we have

$$Q_my = (a, y)a,$$

and as before this implies that $y = 0$, so that

$$\ker Q_{m+1} \cap Y = \{0\}.$$

Thus we obtain, by induction, a projection as in the statement. □

7.3 Finite-dimensional dynamics?

Since the attractor can be parametrised by a finite number of parameters, and (equivalently) can be 'faithfully represented' in a finite-dimensional space $\mathbb{R}^{2d+1}$, it is a natural question whether one can construct a finite-dimensional dynamical system which has an attractor on which the dynamics are 'the same' as those on $\mathcal{A}$. In this sense, the question is whether the dynamics are in some sense 'asymptotically finite-dimensional'. We can make this precise in the following rather wordy definition.

Definition 7.5. *The dynamics of $S(t)$ are* asymptotically finite-dimensional *if for some k, comparable to $d_f(\mathcal{A})$, there exists a map $\varphi : H \to \mathbb{R}^k$ which is injective on $\mathcal{A}$, and a smooth ordinary differential equation on $\mathbb{R}^k$ with corresponding solution operator $T(t)$ and global attractor X, such that the dynamics on $\mathcal{A}$ and X are conjugate under φ, i.e.*

$$T(t)|_X = \varphi \circ S(t) \circ \varphi^{-1}.$$

An ordinary differential equation is here understood to be 'smooth' if its solutions are unique and depend continuously on the initial conditions (e.g. $\dot{x} = f(x)$ with f locally Lipschitz). [This question was first raised explicitly in Eden et al. (1994). Some partial results can be found in Robinson (1999) and Robinson (2001). Romanov (2000) makes a similar definition, dropping the requirement that X is the attractor for $T(t)$, but requiring φ to be bi-Lipschitz. He obtains some nice general results about such systems.]

In general, such a result is unknown.

7.3.1 Inertial manifolds

One situation in which it is guaranteed is when the equation admits an inertial manifold. An inertial manifold $\mathcal{M}$ for a general evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f(u)$$

is a finite-dimensional, positively invariant Lipschitz smooth manifold which exponentially attracts all trajectories (Foias *et al.*, 1985 & 1988), so that

$$S(t)\mathcal{M} \subset \mathcal{M} \qquad \mathrm{dist}(S(t)u_0, \mathcal{M}) \le C(u_0)e^{-kt}.$$

In order to describe the results further, we assume that A is a positive, linear, self-adjoint operator with compact inverse, and f is a Lipschitz function from $D(A^\alpha)$ (the domain of A^α) into H, for some $0 \le \alpha < 1$. Since A has a compact inverse, there is a set of orthonormal eigenfunctions $\{w_n\}$ of A with corresponding eigenvectors λ_n, which one can order such that

$$Aw_n = \lambda_n w_n \qquad \lambda_{n+1} \ge \lambda_n \qquad \lambda_n \to \infty,$$

see Renardy & Rogers (1992), for example. One can define the finite-dimensional projection operators P_n and their orthogonal complements Q_n by

$$P_n u = \sum_1^n (u, w_j) w_j \qquad Q_n u = \sum_{n+1}^{\infty} (u, w_j) w_j,$$

where $(\cdot,\cdot)$ is the scalar product in H.

All current existence proofs give the manifold as a Lipschitz (or smoother) graph over one of the finite-dimensional subspaces $P_n H$, i.e.

$$\mathcal{M} = \{p + \phi(p) : p \in P_n H\}. \tag{7.3}$$

For a general evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f(u), \tag{7.4}$$

restricting the flow from (7.4) to the manifold given by (7.3) immediately yields the set of ordinary differential equations for $p = Pu$,

$$\frac{\mathrm{d}p}{\mathrm{d}t} + Ap = Pf(p + \phi(p)). \tag{7.5}$$

Since $p \in P_n H \simeq \mathbb{R}^n$ and ϕ is Lipschitz, it follows that (4.3) has unique solutions (see Hartman (1964), chapter II, theorem 1.1, for example). Clearly the solutions of (4.3) on $P_n\mathcal{A}$ are precisely those projected down from $\mathcal{A}$, i.e.

$$p(t) = P_n S(t)[p(0) + \phi(p(0))],$$

and since $\mathcal{M}$ is an invariant manifold in H, $P_n\mathcal{A}$ is the global attractor for (4.3). This is exactly a system of the kind specified in (2.3), since $P_n^{-1}x$ is given by $x + \phi(x)$.

However, there are two problems with the inertial manifold approach. Outstandingly, the conditions known to be sufficient to prove the existence of such an object are restrictive: essentially a large gap is required in the spectrum of the linear operator A,

$$\lambda_{n+1} - \lambda_n > C\lambda_{n+1}^{\alpha}.$$

Although this is satisfied for some interesting examples (e.g. the Kuramoto-Sivashinsky equation (Temam, 1988; Foias et al., 1988b; Temam & Wang, 1994), the Ginzburg-Landau equation (Temam, 1988), and reaction-diffusion equations in space dimension 1 (Temam, 1988) (and some special domains in dimension 2 and 3, Mallet-Paret & Sell (1988)), there are many situations in which one can prove the existence of a finite-dimensional global attractor but not (at present) of an inertial manifold - of greatest interest, perhaps, are the 2d Navier Stokes equations.

Secondly, the dimension of the inertial manifold, and hence of the differential system (4.3) can be much greater than that of the attractor. For example, for the Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \qquad u(x + L, t) = u(x, t)$$

the best estimate of the dimension of the attractor is $d_f(X) \sim L^{1.275}$ (Temam, 1988; Collet *et al.*, 1993), whereas the best estimate of the dimension of the inertial manifold is $\dim \mathcal{M} \sim L^{1.64}(\ln L)^{0.2}$ (Temam & Wang, 1994).

7.3.2 The Foias-Temam conjecture

The existence of an inertial manifold for the 2d Navier-Stokes evolution equation is an important open problem, and related to the following conjecture due to Foias & Temam:

Conjecture 7.6. *For some $N < 0$, solutions on the attractor of the 2d Navier-Stokes equations are determined by their first N Fourier modes, i.e. if*

$$P_N u = P_N v \qquad \text{with} \qquad u, v \in \mathcal{A}$$

then in fact $u = v$.

We note here the following simple result.

Proposition 7.7. *Suppose that A is Lipschitz continuous on the attractor,*

$$|Au - Av| \leq L|u - v| \qquad \text{for all} \qquad u, v \in \mathcal{A}$$

for some $L > 0$. Then conjecture 7.6 is true. In fact the attractor is a subset of a Lipschitz manifold given as a graph over $P_N H$ for some N.

Proof. Write $w = u - v$ for $u, v \in \mathcal{A}$. If A is Lipschitz continuous from $\mathcal{A}$ into H then

$$|Aw| \leq L|w|$$

for some L. Now split $w = P_n w + Q_n w$, and observe that we have both

$$|Aw|^2 = |A(P_n w + Q_n w)|^2 = |A(P_n w)|^2 + |A(Q_n w)|^2 \geq \lambda_{n+1}^2 |Q_n w|^2$$

and

$$|Aw|^2 \leq L^2|w|^2 \leq L^2|P_n w|^2 + L^2|Q_n w|^2.$$

Since $\lambda_n \to \infty$ as $n \to \infty$, we can choose n large enough that $\lambda_{n+1} > L$, and then write

$$\lambda_{n+1}^2 - L^2|Q_n w|^2 \leq L^2|P_n w|^2,$$

i.e.

$$|Q_n w| \leq \left(\frac{L^2}{\lambda_{n+1}^2 - L^2} \right)^{1/2} |P_n w|.$$

It follows that we can define $\Phi(P_n u) = Q_n u$ uniquely for each $u \in \mathcal{A}$, and then

$$|\Phi(p_1) - \Phi(p_2)| \leq \left(\frac{L^2}{\lambda_{n+1}^2 - L^2} \right)^{1/2} |p_1 - p_2|,$$

so that the attractor is a subset of a Lipschitz graph over $P_n H$. □

The smoothness of A on $\mathcal{A}$ can be related to the regularity of functions in $\mathcal{A}$, as the following simple result shows: if $\mathcal{A}$ is bounded in $D(A^{1+r})$ then A is Hölder continuous on $\mathcal{A}$.

Lemma 7.8. *If $\mathcal{A}$ is bounded in $D(A^{1+r})$,*

$$\sup_{u\in\mathcal{A}} |A^{1+r}u| \leq K$$

then

$$|A(u-v)| \leq (2K)^{1/(1+r)}|u-v|^{r/(1+r)}. \tag{7.6}$$

Proof. Setting $w = u - v$ write $w = \sum c_n w_n$, where $\{w_n\}$ are the eigenfunctions of A, and then

$$|Aw|^2 = \sum_{n=1}^{\infty} \lambda_n^2 |c_n|^2.$$

Now use the Hölder inequality with $p = (1+r)/r$ and $q = 1+r$, so that

$$\begin{aligned} |Aw|^2 \quad &\leq \left(\sum_{n=1}^{\infty} |c_n|^2)^{r/(1+r)}\right)\left(\sum_{n=1}^{\infty} |\lambda_n|^{2(1+r)}|c_n|^2\right)^{1/(1+r)} \\ &= |w|^{2r/(1+r)}|A^{1+r}w|^{2/(1+r)}, \end{aligned}$$

which gives (7.6). □

In the light of proposition 7.7 the following result is extremely frustrating. It says that if the attractor is bounded in a Gevrey class then A is Lipschitz to with logarithmic corrections.

Lemma 7.9. *Suppose that $\mathcal{A}$ is bounded in $D(A^{1/2}\mathrm{e}^{\tau A^{1/2}})$. Then A is almost Lipschitz on $\mathcal{A}$: for some $c > 0$,*

$$|Au - Av| \leq c|u-v|(\log|u-v|)^2.$$

[A similar result holds if $\mathcal{A}$ is bounded in $D(\mathrm{e}^{\tau A^{1/2}})$ with slightly more messy algebra.]

Proof. If $u = \sum u_j w_j \in D(A^{1/2}\mathrm{e}^{\tau A^{1/2}})$ then

$$\begin{aligned} |A^{1/2}\mathrm{e}^{\tau A^{1/2}}u|^2 &= \sum_{j=0}^{\infty} \lambda_j \mathrm{e}^{2\tau\lambda_j^{1/2}}|u_j|^2 \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \lambda_j \frac{(2\tau\lambda_j^{1/2})^k}{k!}|u_j|^2 \\ &= \sum_{k=0}^{\infty} \frac{(2\tau)^k}{k!}\sum_{j=0}^{\infty} \lambda_j^{1+(k/2)}|u_j|^2 \\ &= \sum_{k=0}^{\infty} \frac{(2\tau)^k}{k!}|A^{(k/4)+\frac{1}{2}}u|^2 < +\infty. \end{aligned}$$

In particular we must have

$$|A^k u|^2 \leq \frac{(4k)!}{(2\tau)^{4k}}.$$

It therefore follows from lemma 7.8 that

$$|Au - Av| \leq \left(\frac{(4j)!}{(2\tau)^{4j}} \right)^{1/2j} |u - v|^{1-(1/j)}.$$

Certainly $(4j)! < (4j)^{4j}$ and so we have

$$|Au - Av| \leq Cj^2 |u - v|^{1-(1/j)}.$$

Now if we try to minimise the right-hand side over all possible choices of j we get $j = -\frac{1}{2} \log |u - v|$, and so

$$|Au - Av| \leq \tilde{C} |u - v| (\log |u - v|)^2$$

as claimed. □

7.4 Partial results

Here we assume that the PDE can be rewritten in the very abstract form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{F}(u).$$

If we assume that the 2d NSE attractor is bounded in $D(A^{3/2})$, for example (which is true if $f \in D(A)$) then A is Hölder on the attractor (by Lemma 7.8).

7.4.1 The construction of Eden et al.

In their monograph on exponential attractors, Eden *et al.* (1994) obtain a differential system

$$\dot{x} = F(x) = -\alpha(x - \nu(x)) + f(\nu(x)) \tag{7.7}$$

for which $X = L\mathcal{A}$ is an exponential attractor (attracting at an exponential rate α), and on which the dynamics agree with those projected from $\mathcal{A}$.

The function f is defined for $y \in X$ by $f(y) = L\mathcal{F}(L^{-1}y)$, and $\nu(x)$ maps any point $x \in \mathbb{R}^k$ to one of the points $y \in X$ such that

$$|x - y| = \mathrm{dist}(x, X).$$

Thus the first term of (7.7) forces X to be attracting, and the second term reproduces the dynamics on $\mathcal{A}$.

In general such a function ν is only continuous at points where the "closest point" $\nu(x)$ is unique. Thus solutions of (7.7) have to be defined as solutions of the corresponding integral equation

$$x(t) = x_0 + \int_0^t F(x(s))\, ds,$$

and these cannot be guaranteed to be unique, even on X.

7.4.2 An ODE without uniqueness

Here we will need the following extension theorem. It guarantees that a continuous function defined on closed subsets of $\mathbb{R}^k$ has a continuous extension to the whole of $\mathbb{R}^k$, with essentially the same modulus of continuity. In particular, Lipschitz [Hölder] functions have Lipschitz [Hölder] extensions (set $\omega(r) = Cr^\theta$).

Theorem 7.10. *Let X be a compact subset of $\mathbb{R}^m$, and f a continuous function from X into $\mathbb{R}^k$ such that*

$$|f(x) - f(y)| \leq \omega(|x - y|), \tag{7.8}$$

where ω, the modulus of continuity of f, is convex, i.e.

$$\omega(r + s) \leq \omega(r) + \omega(s). \tag{7.9}$$

Then f has a continuous extension $F : \mathbb{R}^m \to \mathbb{R}^k$ which satisfies

$$|F(x) - F(y)| \leq (\sqrt{k})\omega(|x - y|). \tag{7.10}$$

One can always find such an ω for a continuous function defined on a compact set. The result of the theorem can be extended to cover a function defined on a closed subset of $\mathbb{R}^m$, provided that f satisfies (7.8) and is globally bounded ($\|f\|_\infty < \infty$).

Proof. It follows from (7.8) that

$$|f_j(x) - f_j(x)| \leq \omega(|x - y|) \tag{7.11}$$

for each component of f. We extend each component in turn preserving this property, and then combine them to give an extension of f itself. Set

$$F_j(y) = \sup_{x \in X} [f_j(x) - \omega(|x - y|)].$$

First, note that if $x, y \in X$ then

$$F_j(x) - F_j(y) + \omega(|x - y|) \leq |f_j(x) - f_j(y)| - \omega(|x - y|) \leq 0,$$

and so $F_j(y) = f_j(y)$. To show (7.10), we use (7.9),

$$\begin{aligned}
F_j(y) - F_j(z) &= \sup_{x \in X} [f_j(x) - \omega(|x - y|)] - \sup_{w \in X} [f_j(w) - \omega(|w - z|)] \\
&\leq \sup_{x \in X} [f_j(x) - \omega(|x - y|) - f_j(x) + \omega(|x - z|)] \\
&\leq \sup_{x \in X} [\omega(|x - z|) - \omega(|x - y|)] \\
&\leq \omega(|z - y|),
\end{aligned}$$

giving (7.11) for each F_j. Combining these inequalities yields (7.10) for F. □

We will show that there is a finite set of ordinary differential equations, with dimension comparable to that of the global attractor, which reproduces its dynamics. However, these ODEs do not have unique solutions, so we cannot really speak about the "dynamical system" they generate, nor the corresponding attractor (as required by our definition of 'asymptotically finite-dimensional').

Theorem 7.11. *Let $S(t)$ be the semigroup generated by the PDE*

$$\mathrm{d}u/\mathrm{d}t = \mathcal{F}(u), \tag{7.12}$$

where $\mathcal{F}(u)$ is [Hölder] continuous from $\mathcal{A}$ into H [with Hölder exponent α]. Suppose that $S(t)$ has a global attractor $\mathcal{A}$, with $d_f(\mathcal{A}) < d$. Then, for any $k \geq 2d+1$, there exists a system of ordinary differential equations in $\mathbb{R}^k$,

$$\mathrm{d}x/\mathrm{d}t = f(x), \tag{7.13}$$

where $f : \mathbb{R}^k \to \mathbb{R}^k$ is [Hölder] continuous, and a bounded linear map $L : H \to \mathbb{R}^k$, which is injective on $\mathcal{A}$, such that for every solution $u(t)$ of (7.12) with $u(t) \in \mathcal{A}$ there is a solution $x(t)$ of (7.13) such that

$$u(t) = L^{-1}[x(t)]. \tag{7.14}$$

Proof. Theorem 7.1 guarantees that there exists a bounded linear map L from H into $\mathbb{R}^k$, which is injective on $\mathcal{A}$ and has a continuous inverse on $L\mathcal{A}$. Now consider the ODE for $x \in L\mathcal{A}$ obtained from the equation on $\mathcal{A}$,

$$\dot{x} = L\mathcal{F}(L^{-1}x).$$

Now, the function $\tilde{f} : L\mathcal{A} \to \mathbb{R}^k$ given by

$$\tilde{f}(x) = L\mathcal{F}(L^{-1}x)$$

is certainly continuous. If we have assumed Hölder continuity of F, then $\tilde{f}$ is also Hölder, since we can use (7.2) to write

$$\begin{aligned} |\tilde{f}(x) - \tilde{f}(y)| &= |L\mathcal{F}(L^{-1}x) - L\mathcal{F}(L^{-1}y)| \\ &\leq \|L\|_{\mathrm{op}} |\mathcal{F}(L^{-1}x) - \mathcal{F}(L^{-1}y)| \\ &\leq K\|L\|_{\mathrm{op}} |L^{-1}x - L^{-1}y|^{\alpha} \\ &\leq CK\|L\|_{\mathrm{op}} |x-y|^{\theta\alpha}. \end{aligned}$$

One can then use theorem 7.10 to extend $\tilde{f}$ to a function $f : \mathbb{R}^k \to \mathbb{R}^k$, which is [Hölder] continuous and bounded. Then we have a system of ODEs

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^k, \tag{7.15}$$

with a [Hölder] continuous right-hand side, and solutions which exist for all time, since $f(x)$ is globally bounded.

However, the solutions of (7.15) *may not be unique*, as f is only [Hölder] continuous, not Lipschitz. Nevertheless, by construction, we have ensured that one of the solutions of (7.13) through $x_0 = Lu_0$ with $u_0 \in \mathcal{A}$ will be precisely

$$x(t) = Lu(t).$$

This guarantees (7.14). □

Note that we have that $L\mathcal{A}$ is "weakly invariant", in the sense that for any initial condition $x_0 \in L\mathcal{A}$ we cannot guarantee that $x(t)$ remains in $L\mathcal{A}$, but we do know that there is at least one solution $x(t; x_0)$ which remains in $L\mathcal{A}$ (see Bhatia & Szegö (1967) for some discussion of such sets).

7.4.3 A discrete time result

A more successful, but still partial, result is the following. Details are given in Robinson (1999 or 2001); the proof relies on topological properties of the attractor.

Theorem 7.12. *Let $\mathcal{A}$ have finite fractal dimension, $d_f(\mathcal{A}) < d$. Then given $T > 0$ and $\epsilon > 0$, for any $k \geq 2d + 1$ there exists a bounded linear map $L : \mathcal{A} \to \mathbb{R}^k$, injective on $\mathcal{A}$, and map f from $\mathbb{R}^k$ into itself, such if $X = L\mathcal{A}$ then*

(i) the dynamics on $\mathcal{A}$ and X are conjugate under L, i.e.

$$f|_X = L \circ S(T) \circ L^{-1},$$

and

(ii) the global attractor for the discrete dynamical system generated by f, X_f, satisfies

$$X_f \subset N(X, \epsilon),$$

and has X as an invariant subset.

Chapter 8

Determining nodes

Although the attractor is finite-dimensional we have seen that it is hard to give sense to the intuitive idea that the dynamics should be finite-dimensional. In a paper from 1967, Foias & Prodi showed that the dynamics are 'determined' by a finite number of Fourier modes, in that if $u(t)$ and $v(t)$ are two solutions of the Navier-Stokes equations, and for N sufficiently large

$$|P_N u(t) - P_N v(t)| \to \infty$$

then in fact $|u(t) - v(t)| \to \infty$.

It is as important to stress what such results do not say as what they do say. If you take two solutions of the full equations and know that the P modes are converging, you can deduce that the full solutions are converging. However, it does not say that the solutions of the P-mode Galerkin truncation determine the solution; nor that knowledge of the P modes at any instant will determine the full solution. Improving the bounds on the number of modes required to be determining has been an active area of research, and in a series of papers Jones & Titi reduced the bound to $N \sim G$ in the case of periodic boundary conditions [this bound is derived in Jones & Titi (1993)].

In this chapter we prove a similar result, but replacing the finite number of modes by a finite number of nodal values. This idea was introduced by Foias & Temam (1984), the bounds improved through a series of papers, Foias & Titi (1991), Jones & Titi (1992a), until the best current results which are due to Jones & Titi (1993) [this paper gives the best bounds for determining modes, nodes, and (see later) volume elements.]

We choose a finite set of points, or nodes, in $\Omega = [0, L]^2$, $\mathcal{N} = \{x_j\}$, and set

$$d(\mathcal{N}) = \sup_{x \in \Omega} \min_j |x - x_j|,$$

so that for every $x \in \Omega$ there is an x_j such that

$$|x - x_j| \leq d(\mathcal{N}). \tag{8.1}$$

We say that $\mathcal{N}$ is a set of determining nodes, if, whenever

$$\sup_j |u(x_j,t) - v(x_j,t)| \to 0$$

we have

$$\sup_{x\in\Omega} |u(x,t) - v(x,t)| \to 0. \tag{8.2}$$

Since, by Agmon's inequality in 2D,

$$\|u\|_\infty \le c|u||Au|,$$

and we know that there is an absorbing set in $D(A)$ (2.16), it suffices to show that

$$|u(t) - v(t)| \to 0. \tag{8.3}$$

In fact, we will show that

$$\|u(t) - v(t)\| \to 0,$$

which clearly gives (8.3) (and hence (8.2)) since $|u| \le \lambda^{-1/2}\|u\|$.

Fundamental to the proof is the following lemma, relating a bound on $w|_{\mathcal{N}}$ to a bound on $\|w\|_\infty$. The improvements to the result in Jones & Titi (1993) rely in part on better estimates of norms of w in terms of η and $d(\mathcal{N})$.

Lemma 8.1. *If $w \in D(A)$, and we set*

$$\eta(w) = \max_{x_j\in\mathcal{N}} |w(x_j)|,$$

then

$$\|w\|_\infty \le \eta(w) + cL^{1/2}d(\mathcal{N})^{1/2}|Aw|. \tag{8.4}$$

Proof. Recall the Sobolev embedding theorem $H^2(\Omega) \subset C^{0,1/2}(\bar\Omega)$, where $C^{0,1/2}(\bar\Omega)$ is the set of continuous functions on $\bar\Omega$ with Hölder exponent one half; since $D(A) \subset H^2$, and

$$\|u\|_{H^2} \le c|Au|$$

(this follows straightforwardly in the case of periodic boundary conditions) we have

$$|w(x) - w(y)| \le cL^{1/2}|Aw||x - y|^{1/2}.$$

The expression (8.4) follows immediately from this and the definition of $d(\mathcal{N})$ and $\eta(w)$. □

We now use this expression to study the time evolution of $\|w(t)\|$, where $w(t) = u(t) - v(t)$, the difference of two solutions, and prove:

Theorem 8.2. *(Determining nodes). There exists a $\delta > 0$ such that, if $d(\mathcal{N}) < \delta$, $\mathcal{N}$ are a set of determining nodes.*

In the proof we use the existence of an absorbing set in $D(A)$, which we have not proved in these notes.

Proof. The equation for $w(t) = u(t) - v(t)$ is

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu Aw + B(u,w) + B(w,u) - B(w,w) = 0,$$

and taking the inner product of this with Aw and using $b(u,u,Au) = 0$ (1.6) and the three-term identity

$$b(u,w,Aw) + b(w,u,Aw) + b(w,w,Au) = 0$$

which follows by differentiating, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu|Aw|^2 = b(w,w,Au).$$

Using the bound on b given by (1.9), we can write

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu|Aw|^2 &\leq \|w\|_\infty\|w\||Au| \\ &\leq [\eta(w) + cL^{1/2}d(\mathcal{N})^{1/2}|Aw|]\|w\||Au| \\ &\leq \eta(w)\|w\||Au| + cL^{1/2}d(\mathcal{N})^{1/2}\lambda_1^{-1/2}|Aw|^2|Au|,\end{aligned}$$

so therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + [\nu - c\lambda_1^{-1/2}L^{1/2}d(\mathcal{N})^{1/2}|Au|]\lambda_1\|w\|^2 \leq \eta(w)\|w\||Au|.$$

Now, we know that we have absorbing sets in V and $D(A)$, so that for large enough t

$$|Au| \leq R_D \qquad \|w\| \leq 2R_V,$$

and therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + [\nu - c\lambda^{1/2}L^{1/2}R_Dd(\mathcal{N})^{1/2}]\lambda_1\|w\|^2 \leq 2R_VR_D\eta(w).$$

Now, choose δ such that

$$\mu = \nu - c\lambda_1^{1/2}L^{1/2}R_D\delta^{1/2} > 0. \tag{8.5}$$

Then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \mu\|w\|^2 \leq 2R_VR_D\eta(w).$$

By assumption, we know that $\eta(w) \to 0$, and we also know that for t large enough $\|w(t)\|^2 \leq 4R_V^2$.

So we need to analyse the differential inequality

$$\frac{\mathrm{d}X}{\mathrm{d}t} + aX \leq b(t), \tag{8.6}$$

where we know that $b(t) \to 0$ and that $X(t) \leq k$ for all $t \geq t_0$. We will show that under these conditions, $X(t) \to 0$. Indeed, choose $\epsilon > 0$. Then there exists a T such that $b(t) \leq \epsilon/2$ for all $t \geq T$. So for $t \geq T$,

$$\frac{\mathrm{d}X}{\mathrm{d}t} + aX \leq \epsilon/2.$$

By Gronwall's inequality,

$$X(T+t) \leq X(T)e^{-at} + \epsilon/2,$$

and so choosing τ large enough that

$$ke^{-a\tau} < \epsilon/2,$$

we have

$$X(t) \leq \epsilon \qquad \text{for all } t \geq T + \tau,$$

so that $X(t) \to 0$.

Thus $\|w(t)\|^2 \to 0$, and so $\|w(t)\|_\infty \to 0$ and the nodes $\mathcal{N}$ are determining. □

Here we have made no attempt to obtain the best estimate for the separation δ, and indeed, the estimate derived from (8.5) is very coarse. The best estimate, due to Jones & Titi (1993), is

$$\delta \leq cG^{-1/2};$$

compare this with the length scale from the Kraichnan theory discussed in section 6.5, where

$$(L_\chi/L) \sim G^{-1/3}.$$

Note that one can obtain a similar result by taking the average the solution over a subgrid of a set of smaller squares, termed "determining volume elements" (introduced by Foias & Titi, 1991; see also Jones & Titi, 1992b & 1993). So, we split Ω into N equal squares, of sides $L/\sqrt{N}$, and label them Q_j, $j = 1, \ldots, N$. We define the average of $u(x,t)$ over a square,

$$\langle u \rangle_{Q_j} = \frac{N}{L^2} \int_{Q_j} u(x)\, dx,$$

and show that if N is "large enough", then if

$$\sup_j |\langle u(x,t) - v(x,t) \rangle_{Q_j}| \to 0$$

then

$$\sup_{x \in \Omega} |u(x,t) - v(x,t)| \to 0.$$

A unified treatment of determining nodes, modes, volume elements, and more general 'determining functionals' is given in Cockburn et al. (1997).

Chapter 9

Nodal parametrisation

Foias & Temam (1984) conjectured that on the attractor one should be able to take a finite number of nodes and distinguish functions by their value at these nodes at one fixed time, i.e. for a set $\{x_j\}_{j=1}^k \in Q$, if $u, v \in \mathcal{A}$ then

$$u(x_j) = v(x_j) \qquad \text{for all} \qquad k = 1, \ldots, k \qquad \Rightarrow \qquad u = v.$$

Here we show that this is true, provided that the attractor consists of analytic functions, with $k \sim d_f(\mathcal{A})$, for almost every choice of k points in Q. The result, originally due to Friz & Robinson (2001) has been considerably refined, and the most powerful version is given in Kukavica & Robinson (2003). We give only a sketch of the argument here, but note that it makes use of the attractor, its finite-dimensionality, and the regularity of its elements.

We need a slightly weaker property than analyticity to state the theorem exactly. Despite being weaker than analyticity, this property enables the results to be proved much more simply.

If $u \in C^\infty(\Omega, \mathbb{R}^d)$ then the order of vanishing of u at x is the smallest integer k such that $\partial^\alpha u(x) \neq 0$ for some multi-index α with $|\alpha| = k$. We say that u has finite order of vanishing in Ω if the order of vanishing of u is finite at every $x \in \Omega$. [Note that while this definition does not require that the order of vanishing of u be uniformly bounded in Ω, nevertheless the order of vanishing of u is uniformly bounded on any compact subset K of Ω. Arguing by contradiction, suppose not: then there is a sequence $x_j \in K$ with the order of vanishing of u at x_j at least j. Since K is compact, x_j has a subsequence that converges to some $x^* \in K$: it follows that u vanishes to infinite order at x^*, a contradiction.]

Theorem 9.1. *Let $\mathcal{A}$ be a compact subset of $L^2(\Omega, \mathbb{R}^d)$ with finite dimension $d_f(\mathcal{A})$ that, for each $r \in \mathbb{N}$ and for every compact subset K of Ω, is a bounded subset of $C^r(K, \mathbb{R}^d)$. Assume also that $u - v$ has finite order of vanishing for all $u, v \in \mathcal{A}$ with $u \neq v$. Then for $k \geq 16 d_f(\mathcal{A}) + 1$ almost every set $\mathbf{x} = (x_1, \ldots, x_k)$ of k points in Ω makes the map $E_{\mathbf{x}}$, defined by*

$$E_{\mathbf{x}}[u] = (u(x_1), \ldots, u(x_k))$$

one-to-one between X and its image.

Furthermore the point values of u at $(x_1, \ldots, x_k)$ parametrise $\mathcal{A}$: the map $E_{\mathbf{x}}^{-1} : E_{\mathbf{x}}[\mathcal{A}] \to \mathcal{A}$ is continuous from $\mathbb{R}^{kd}$ into $L^2(\Omega, \mathbb{R}^d)$ and into $C^r(K, \mathbb{R}^d)$ for every $K \subset\subset \Omega$ and $r \in \mathbb{N}$.

We make here the trivial observation that the conditions on the order of vanishing is satisfied if all functions in $\mathcal{A}$ are real analytic. Note that parametrisation of the attractor by nodal values was addressed by Foias & Titi (1991), who showed that the result holds in any system that has an inertial manifold.

For the proof we need to use the Hausdorff as well as the fractal dimension. The Hausdorff dimension of a subset X of $\mathbb{R}^n$ is defined as

$$d_H(X) = \inf\{d \geq 0 : \mathcal{H}^d(X) = 0\},$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure,

$$\mathcal{H}^d(X) = \lim_{\epsilon \to 0} \inf \left\{ \sum_i r_i^d : X \subseteq \cup_i B(x_i, r_i) \text{ with } r_i \leq \epsilon \right\}$$

(here $B(x, r)$ is a ball centred at x with radius r; see Falconer (1985 or 1990) for further details). We will require the following four properties:

(1) If $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$ is a θ-Hölder function then the Hausdorff dimension of the graph of f,

$$G = \{(x, f(x)) : x \in X\} \subset \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m,$$

satisfies

$$d_H(G) \leq n + (1 - \theta)m. \tag{9.1}$$

(2) Hausdorff dimension is stable under countable unions,

$$d_H\left(\bigcup_{j=1}^{\infty} X_j\right) = \sup_j d_H(X_j). \tag{9.2}$$

(3) Hausdorff dimension does not increase under the application of bounded linear maps L,

$$d_H(LX) \leq d_H(X). \tag{9.3}$$

(4) A set in $\mathbb{R}^n$ with Hausdorff dimension strictly less than n has zero Lebesgue measure.

For (1) see Friz et al. (2001), for (2) and (3) see Falconer (1985 or 1990); (4) follows from the definition of Hausdorff dimension, since n-dimensional Hausdorff measure is proportional to n-dimensional Lebesgue measure (see theorem 1.12 in Falconer, 1985).

Proof. A sketchy version of the proof, skating over some subtleties, is as follows. If we choose a collection of points for which $E_{\mathbf{x}}$ is not one-to-one then there must exists to distinct elements $u, v \in \mathcal{A}$ such that

$$u(x_j) = v(x_j) \qquad \text{for all} \qquad j = 1, \ldots, k.$$

If we consider $w = u - v$ this says that $w(x_j) = 0$ for all $j = 1, \ldots, k$. So if we can control the size of the set of possible zeros of functions w in

$$W = \{u - v : \; u, v \in \mathcal{A}, \; u \neq v\}$$

we have a chance of proving the result.

Functions in W are as smooth as functions in $\mathcal{A}$, and by assumption have finite order of vanishing. Using properties of the fractal dimension, $d_f(W) \leq 2d_f(\mathcal{A})$.

Since the attractor consists of smooth functions so does W, and a variant of the simple result that $\tau(X) \leq d/k$ if X is bounded in H^k shows that $\tau(W) = 0$. It follows using the embedding result of Hunt & Kaloshin (theorem 7.1) that W can be viewed as a parametrised set of functions, where if we use D parameters (with $N > 4d_f(\mathcal{A})$) the parametrisation

$$w(x; p) \qquad p \in \mathbb{R}^N$$

is Hölder continuous with exponent $\theta < 1 - [4d_f(\mathcal{A})/N]$.

Because the function w has finite order of vanishing the size of its zero set is controlled. The set of zeros of $w(x, p)$, considered as a subset of $Q \times \mathbb{R}^D$, are contained in a countable collection of sets, each of which is the graph of a θ-Hölder function,

$$(x', x_j(x'; \varepsilon); \varepsilon),$$

where $x' = (x_1, x_{j-1}, x_{j+1}, x_m)$. Each of these manifolds has $(m - 1) + N$ free parameters.

It follows that collections of k such zeros (considered as a subset of $\Omega^k \times \mathbb{R}^N$) are contained in the product of k such manifolds. Since the coordinate p is common to each of these, they are the graphs of θ-Hölder functions from a subset of $\mathbb{R}^{N+(m-1)k}$ into $\mathbb{R}^k$. Equation (9.1) shows that each of these sets has Hausdorff dimension at most

$$N + (m - 1)k + k(1 - \theta),$$

and using (9.2) the same goes for the whole countable collection.

The projection of this collection onto Ω^k enjoys the same bound on its dimension (9.3), and so to make sure that these 'bad choices' do not cover $\Omega^k \subset \mathbb{R}^{mk}$ we need

$$N + (m - 1)k + k(1 - \theta) < mk.$$

This is certainly true if

$$k > \frac{N}{\theta}$$

and since θ can be chosen arbitrarily close to $1-(4d_f(\mathcal{A})/N)$, it follows that

$$k > \frac{N^2}{N-4d_f(\mathcal{A})}$$

will suffice. Choosing the integer value of N with $8d_f(\mathcal{A})-\frac{1}{2} \le N < 8d_f(\mathcal{A})+\frac{1}{2}$ shows that $k \ge 16d_f(\mathcal{A})+1$ suffices.

Since the collection of 'bad choices' is a subset of $\mathbb{R}^{km}$ with Hausdorff dimension less than km it follows from fact (4) about the Hausdorff dimension (above) that almost every choice (wrt Lebesgue measure) makes $E_{\mathbf{x}}$ one-to-one.

The continuity of $E_{\mathbf{x}}^{-1}$ follows since $\mathcal{A}$ is compact and $E_{\mathbf{x}}$ is one-to-one between $\mathcal{A}$ and its image. □

We note here that for the 2d Navier-Stokes equations, if we space our nodes evenly over the domain, then the separation required by our theorem is of the order of $G^{-1/3}$, with logarithmic corrections, confirming the Kraichnan length scale by analytically rigorous means. Note also that this is an entirely natural way to produce a length-scale from the equations, and ties in with the heuristic argument that one would expect that

$$d_f(\mathcal{A}) \sim \left(\frac{L}{L_\chi}\right)^2,$$

cf. section 6.5.

9.1 Extensions

The result can be extended in various ways, all of which are a corollary of the following more general formulation, whose proof is essentially the same as that of theorem 9.1. In the statement we let $\Omega_1 \subset \mathbb{R}^{m_1}$ and $\Omega_2 \subset \mathbb{R}^{m_2}$ be two open connected sets.

Theorem 9.2. *Let X be a compact subset of $L^2(\Omega_1,\mathbb{R}^{d_1})$ with finite fractal dimension, $d_f(X)$. Let Y be a subset of $L^2(\Omega_2,\mathbb{R}^{d_2})$ that is bounded in $C^r(K,\mathbb{R}^{d_2})$ for each $r \in \mathbb{N}$ and each compact subset K of Ω_2. Assume that $u-v$ has finite order of vanishing for every $u,v \in Y$ such that $u \not\equiv v$. Also, assume that there exists a one-to-one map $\Sigma\colon X \to Y$ that is Lipschitz from $L^2(\Omega_1,\mathbb{R}^{d_1})$ into $L^2(\Omega_2,\mathbb{R}^{d_2})$. Then for every $k \ge 16d_f(X)+1$ almost every set $\mathbf{y}=(y_1,\ldots,y_k)$ of k points in Ω_2 makes the map*

$$u \mapsto \Big((\Sigma u)(y_1),\ldots,(\Sigma u)(y_k)\Big)$$

one-to-one between X and its image.

We now give various applications of this result, only pointing out the appropriate choice of X, Σ, and Y. Note that the result allows observations to be take at points in a different domain (Ω_2) from the natural domain of definition of functions in X (which is Ω_1).

9.1.1 Almost every collection of points in space-time

Suppose that the set X consists of solutions of a partial differential equation that we write in an (extremely) abstract form as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u); \tag{9.4}$$

we assume that F is a local C^∞ function of u and its derivatives, i.e. that for each $k \in \mathbb{N}$ and every compact set $K \subset \Omega$ there exists a $k' \in \mathbb{N}$ and a compact set $K' \subset \Omega$ such that

$$\max_{|\alpha|\le k} \sup_{x\in K} |D^\alpha F(u)| \le \max_{|\beta|\le k'} \sup_{x\in K'} |D^\beta u|. \tag{9.5}$$

Denote by $S(t)u_0$ the solution at time t of (9.4) with initial condition u_0. The next result appears to have many conditions, but they are readily satisfied by many well-known examples of partial differential equations.

Corollary 9.3. *Suppose that $\mathcal{A}$ is an invariant set under the dynamics of a PDE (9.4) satisfying (9.5). Assume also that for each $t > 0$ the solution operator $S(t)$ restricted to $\mathcal{A}$ is (i) Lipschitz from $L^2(\Omega; , \mathbb{R}^d)$ into itself,*

$$|S(t)u_0 - S(t)v_0| \le L(t)|u_0 - v_0| \qquad \textit{with} \qquad L(t) \in L^2(0, T), \tag{9.6}$$

and (ii) injective, i.e. if $S(t)u_0 = S(t)v_0$ then $u_0 = v_0$. Then provided that the assumptions of theorem 9.1 hold and that $k > 16d_f(\mathcal{A}) + 1$, for any $T > 0$ almost every collection of k points $\{(x_j, t_j)\}_{j=1}^k$ from $\Omega \times [0, T]$ make the map

$$u \mapsto \big(u(x_1, t_1), \ldots, u(x_k, t_k)\big)$$

one-to-one between $\mathcal{A}$ and its image.

(The notation $u(x, t)$ above is shorthand for $[S(t)u](x)$.)

Proof. Let $X = \mathcal{A}$, and define $\Sigma : L^2(\Omega, \mathbb{R}^d) \to L^2(\Omega \times [0, T], \mathbb{R}^d)$ by

$$[\Sigma u](t) = S(t)u \qquad \text{for every } t \in [0, T];$$

set $Y = \Sigma(X)$. □

9.1.2 A Takens type result for the Ginzburg-Landau equation

Takens' celebrated time delay embedding theorem (Takens, 1980, see also Sauer et al., 1993) guarantees, under various genericity assumptions, that a finite number of repeated observations at equally spaced time intervals[1] are sufficient

[1] In fact these time intervals need not be equally spaced, see remark 2.9 in Sauer et al. (1993), but it is much easier to reconstruct the underlying dynamics if they are by using a simple shift on the time series.

to distinguish between different elements of the attractor of a finite-dimensional dynamical system: if the attractor has dimension d then for a prevalent set of Lipschitz functions $h : \mathbb{R}^n \to \mathbb{R}$ and all T sufficiently small, the map

$$u \mapsto H[u] = \big(h[u(0)], h[u(T)], h[u(2T)], \ldots, h[u(2dT)]\big)$$

is one to one on the attractor (this formulation is from Hunt & Kaloshin, 1999).

However, this time-delay embedding theorem has only been proved for finite-dimensional systems. Here we apply theorem 9.2 to deduce Takens-type theorems for the Ginzburg-Landau equation

$$u_t - (1 + i\nu)u_{xx} + (1 + i\mu)|u|^2 u - au = 0 \tag{9.7}$$

subject to periodic boundary conditions on $[0, 1]$. The proof is based on the following result due to Kukavica (1992): there exists a number $\delta_0 > 0$ such that if x_1, x_2 is an arbitrary pair of different points with $|x_1 - x_2| \leq \delta_0$, then for any two solutions u_1 and u_2 belonging to the global attractor $\mathcal{A}$,

$$u_1(x_j, t) = u_2(x_j, t), \qquad j = 1, 2, \qquad \text{for every } t \geq 0$$

implies that

$$u_1(x, t) = u_2(x, t), \qquad x \in \Omega, \qquad \text{for every } t \geq 0.$$

We say that x_1 and x_2 are a set of “determining nodes”. (The constant δ_0 can be explicitly computed in terms of μ, ν, and a.) By combining this with theorem 9.2 we obtain the following result.

Theorem 9.4. *Let x_1 and x_2 be two points with $|x_1 - x_2| \leq \delta_0$ (δ_0 as above), choose $T_0 > 0$, and let $k \geq 16d_f(\mathcal{A}) + 1$. Then for almost every set of k times*

$$\mathbf{t} = (t_1, t_2, \ldots, t_k)$$

where $t_1, \ldots, t_k \in [0, T_0]$ the mapping

$$E_{\mathbf{t}} \colon \mathcal{A} \to \mathbb{R}^{2k}$$

defined by

$$E_{\mathbf{t}}(u) = \Big([S(t_1)u](x_1), \ldots, [S(t_k)u](x_1), [S(t_1)u](x_2), \ldots, \big[S(t_k)u](x_2)\Big)$$

is one-to-one between $\mathcal{A}$ and its image.

This means, in particular, that there exist $0 \leq t_1 < t_2 < \cdots < t_k$ such that if $u_1(x, t)$ and $u_2(x, t)$ are two solutions belonging to the global attractor $\mathcal{A}$ with

$$u_1(x_1, t_j) = u_2(x_1, t_j), \qquad j = 1, \ldots, k$$

and

$$u_1(x_2, t_j) = u_2(x_2, t_j), \qquad j = 1, \ldots, k$$

then $u_1 \equiv u_2$.

Note that by the invariance of the global attractor $\mathcal{A}$, we may replace the interval $[0, T_0]$ with any $[a, b]$ where $-\infty < a < b < \infty$. Since $\mathbb{R} = \cup_{n \in \mathbb{N}}[-n, n]$, Theorem 9.4 holds also if we replace $[0, T_0]$ by $\mathbb{R}$.

Proof of theorem 9.4. We will apply theorem 9.2 with $X = \mathcal{A}$ and with Y and Σ chosen as follows. Let $\Omega_1 = (-1, 1)$ (any other open interval containing $[-1/2, 1/2]$ would do) and $\Omega_2 = (1, T_0 + 1)$. We choose $d_1 = 1$ and $d_2 = 2$. We define

$$\Sigma : X \to C(\Omega_2, \mathbb{R}^2)$$

by

$$\Big(\Sigma(u_0)\Big)(t) = \Big([S(t)u_0](x_1), [S(t)u_0](x_2)\Big) \tag{9.8}$$

for $t \in \Omega_2 = (1, T_0 + 1)$ (note that $S(t)u_0$ is a (joint) analytic function of $x \in \mathbb{R}$ and $t > 0$ and can thus be evaluated at $x = x_1$ and $x = x_2$). □

Although the use of two nodes is necessary in the case of periodic boundary conditions, only one spatial node is required in the case of Dirichlet boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t \geq 0.$$

In this case, we can choose any $x_1 \in (0, \delta_0]$ and $x_2 = 0$. Since all solutions automatically agree at $x_2 = 0$, we get the following statement.

Theorem 9.5. *Take an arbitrary point x_1 with $0 < x_1 \leq \delta_0$, choose $T_0 > 0$, and let $k \geq 16 d_f(\mathcal{A}) + 1$. Then for almost every set of points*

$$\mathbf{t} = (t_1, t_2, \ldots, t_k), \qquad t_j \in [0, T_0]$$

the mapping

$$E_{\mathbf{t}} : \mathcal{A} \to \mathbb{R}^{2k}$$

defined by

$$E_{\mathbf{t}}[u] = \Big(\big(S(t_1)u\big)(x_1), \ldots, \big(S(t_k)u\big)(x_1), \Big)$$

is one-to-one between $\mathcal{A}$ and its image.

Proof. The result is simply a corollary of theorem 9.4 since solutions of the CGLE with Dirichlet boundary conditions can be extended to odd periodic solutions of the CGLE. □

9.1.3 Determining modes

In chapter 8 we mentioned briefly the result of Foias & Prodi that a finite number of modes are determining for the Navier-Stokes equations, in the sense that if N is large enough then

$$|P_N u(t) - P_N v(t)| \to 0 \qquad \Rightarrow \qquad |u(t) - v(t)| \to 0. \tag{9.9}$$

Here we prove a result on the attractor that is reminiscent of the Foias-Temam conjecture of section 7.3.2, but requires a finite number of measurements of the first N modes.

Theorem 9.6. *Suppose that the first N modes are determining in the sense of (9.9), that the attractor fulfils the conditions of theorem 9.1, and that solutions are analytic functions of time. Then for any $T > 0$, for almost every set of k times in $[0, T]$ (with $k > 16d_f(\mathcal{A})$) if $u, v \in \mathcal{A}$ and*

$$P_N u(t_j) = P_N v(t_j) \qquad \text{for all} \qquad j = 1, \ldots, k$$

then $u = v$.

Proof. Define $\Sigma : L^2(\Omega_1, \mathbb{R}^{d_1}) \to L^2([0,T], \mathbb{R}^{Nd_1})$ by $\Sigma[u] = (P_N u)(t)$. □

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