Partial Differential Equations
AARMS Summer School 2003

James C. Robinson

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Contents

Introduction

In this course we will study the Navier-Stokes equations that govern fluid flow. Having derived the equations we will introduce various tools from functional analysis that we need in order to treat them mathematically, and then look in some detail at the mathematical questions of existence and uniqueness of solutions. That done we will concentrate of the two-dimensional equations, viewing them as a dynamical system, and draw some physical conclusions about 'two-dimensional turbulence'.

The main reference throughout is '[R]', which also contains many additional references:

J.C. Robinson (2001) *Infinite-Dimensional Dynamical Systems* (Cambridge University Press)

Other useful books are

P. Constantin & C. Foias (1988) *Navier-Stokes Equations* (University of Chicago Press)

C.R. Doering & J.D. Gibbon (1995) *Applied Analysis of the Navier-Stokes Equation* (Cambridge University Press)

M. Renardy & R.C. Rogers (1992) *An Introduction to Partial Differential Equations*, Texts in Applied Mathematics Volume 13 (Springer Verlag, New York).

Chapter 1

Physical derivation of the Navier-Stokes equations

In this chapter we will give a brief summary of the physical derivation of the Navier-Stokes equations, mainly to indicate that they are not an *ad hoc* model, but derived from sound fundamental principles. We will try to derive a model for a fluid moving in some region $D \subset \mathbb{R}^d$ (here we take $d = 3$, although we will also consider $d = 2$ later). For another treatment of the physical background leading to these equations, see Doering & Gibbon (1995).

1.1 The continuum hypothesis

The main simplification made in deriving the Navier-Stokes equations, and other models of solids or 'non-Newtonian' fluids, is that we consider the fluid to occupy space continuously, rather than being made up of a huge (but finite) number of interacting particles.

We want to be able to talk about the 'fluid velocity', the 'fluid density', etc., *at a point*, and in reality these concepts do not really make sense. Physically we have to think of the 'density at a point x' as meaning the average density over some small volume which is significantly larger than the molecular scale ($\sim 10^{-8}$ m), but significantly smaller than the lengths associated with the 'large-scale motion' of the fluid (e.g. 1 cm).

For example, Figure 1.1 shows schematically the result of measuring the average density over volumes whose sides have differing lengths. For a range of L away from the smallest and largest scales, these quantities appear to be essentially constant and well-defined.

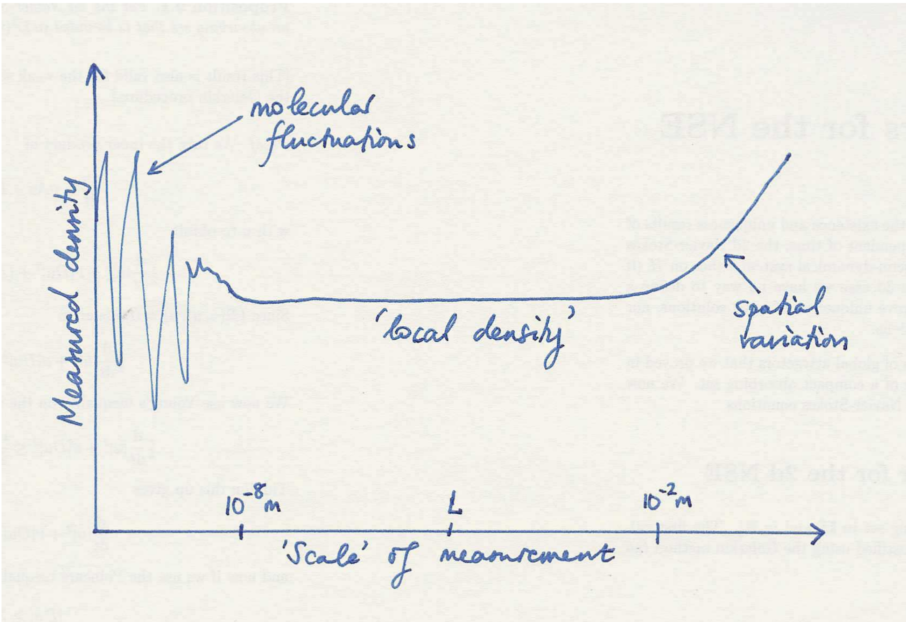

Figure 1.1: 'Local averages' of the density appear well-defined over a range of length scales. (After a figure in G.K. Batchelor's *Fluid Dynamics*)

1.2 Langrangian & Eulerian pictures

If we are prepared to accept this continuum hypothesis there are now two ways to think of the fluid.

In the Lagrangian picture we think of the fluid as a collection of 'fluid particles'. Suppose that we label each particle with its initial position $\mathbf{a} \in D$. As the fluid moves, each fluid particle will move, with some velocity $\mathbf{v}(\mathbf{a}, t)$;

we can find the position $\mathbf{x}(\mathbf{a},t)$ of 'particle $\mathbf{a}$' at time t by solving the ordinary differential equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}(\mathbf{a},t).$$

If we assume that the particles all retain their own distinct identity then we can also invert $\mathbf{a} \mapsto \mathbf{x}(\mathbf{a},t)$ if we want to find the initial position of the fluid particle that is now at $\mathbf{x}$, we denote this by $\mathbf{a}(\mathbf{x},t)$.

An alternative point of view, which is more useful for us, is to consider quantities referred to coordinates that are fixed in the domain D. This approach, 'the Eulerian picture', was pioneered by Euler and led to the first derivation of the equations of motion for fluids. We will denote by $\mathbf{u}(\mathbf{x},t)$ the fluid velocity at the point $\mathbf{x}$, i.e.

$$\mathbf{u}(\mathbf{x},t) = \mathbf{v}(\mathbf{a}(\mathbf{x},t),t).$$

Suppose that $\mathbf{f}(\mathbf{x},t)$ is any function of time and the Eulerian space coordinates. If we look how $\mathbf{f}$ changes as we move with a fluid particle then we need to consider

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{\mathbf{a}} \mathbf{f}(\mathbf{x}(\mathbf{a},t),t) &= \frac{\partial \mathbf{f}}{\partial t}\Big|_{\mathbf{x}} + \sum_{j=1}^{d} \frac{\partial \mathbf{f}}{\partial x_j}\frac{\partial x_j}{\partial t} \\ &= \frac{\partial \mathbf{f}}{\partial t} + \sum_{j=1}^{d} u_j \frac{\partial \mathbf{f}}{\partial x_j} \\ &= \frac{\partial \mathbf{f}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{f} \end{aligned}$$

The final expression here is the 'convective derivative' of $\mathbf{f}$,

$$\frac{\mathrm{D}\mathbf{f}}{\mathrm{D}t} := \frac{\partial \mathbf{f}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{f}, \tag{1.1}$$

reflecting both how $\mathbf{f}$ changes in time, and how it changes because the fluid is in motion.

1.3 Conservation of mass

The first equation we need embodies the conservation of mass – as the fluid flows around, mass in neither created nor destroyed. Suppose that V is a fixed volume in the domain D. Then the amount of mass in V can only change as the fluid flows into and out of V:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(\mathbf{x},t)\,\mathrm{d}V = -\int_{\partial V} \rho\,\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}\,\mathrm{d}S$$

Using the divergence theorem we can rewrite the right-hand side of this as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(\mathbf{x},t)\,\mathrm{d}V = -\int_V \nabla\cdot(\rho\,\mathbf{u}(\mathbf{x},t))\,\mathrm{d}V,$$

and so

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{u})\right]\,\mathrm{d}V = 0.$$

Since this equation has to hold for *any* volume V, we can remove the integral, and deduce the equation of mass conservation,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{u}) = 0. \tag{1.2}$$

Exercise 1.1. *Show that if f is continuous and*

$$\int_V f(\mathbf{x})\,\mathrm{d}V = 0$$

for any volume V, then $f = 0$.

We will concentrate in all that follows on *incompressible fluids*, in which the density does not change, i.e. for which $\mathrm{D}\rho/\mathrm{D}t = 0$, where D is the convective derivative from (1.1). If we rewrite (1.2) we have

$$\frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho + \rho(\nabla\cdot\mathbf{u}) = \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\nabla\cdot\mathbf{u}) = 0,$$

and since we are assuming that $\mathrm{D}\rho/\mathrm{D}t = 0$ this equation reduces to the *incompressibility condition*

$$\nabla\cdot\mathbf{u} = 0. \tag{1.3}$$

1.4 Conservation of momentum

Now we will consider a small volume V_t that moves with the fluid. Momentum conservation says that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} \rho\mathbf{u}\,\mathrm{d}V = \text{forces on } V_t. \tag{1.4}$$

1.4.1 The stress tensor

The forces on V_t arise due to the influence of the surrounding fluid, and will act on the boundary of the volume V_t. We now show that these forces can be derived from a symmetric stress tensor $\sigma_{ij}(\mathbf{x}, t)$, so that the force on an area A is given by

$$\mathbf{f} = \int_A \sigma \cdot \mathbf{n} \, \mathrm{d}A,$$

i.e.

$$f_i = \int_A \sigma_{ij} n_j \, \mathrm{d}A.$$

To see this, consider a small tetrahedral volume in the fluid, with three of its faces aligned with the coordinate axes, as in Figure 1.2, with sides that have length $O(l)$.

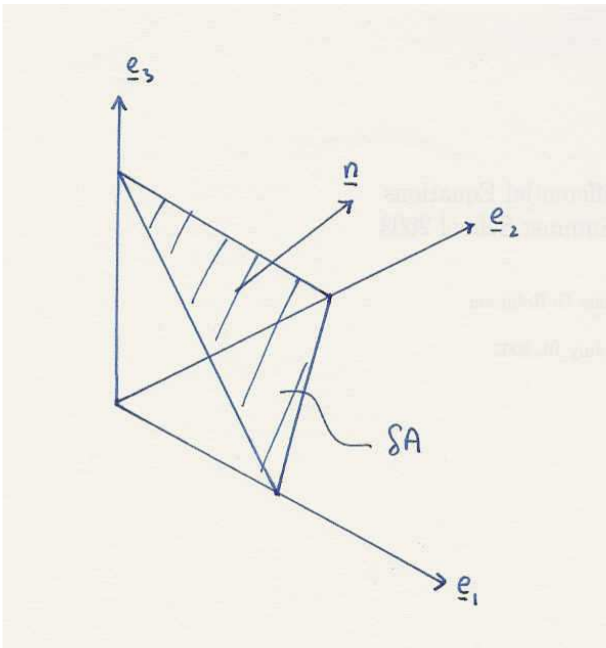

Figure 1.2: A small tetrahedral element in the fluid, and the forces on it

The mass of this little volume will be $O(l^3)$, while the area of the faces will be $O(l^2)$. Since the magnitude of the forces is proportional to the area of the faces, if the acceleration is not to become infinite as $l \to 0$, the forces on this volume must cancel. So we should have

$$\mathbf{f}(\mathbf{n})\delta A + \mathbf{f}(-\mathbf{e}_1)\delta A_1 + \mathbf{f}(-\mathbf{e}_2)\delta A_2 + \mathbf{f}(-\mathbf{e}_3)\delta A_3 = 0$$

However, since $\delta A_j = n_j\, \delta A$, and using Newton's third law ($\mathbf{f}(\mathbf{n}) = -\mathbf{f}(-\mathbf{n})$) it follows that

$$\mathbf{f}(\mathbf{n}) = \mathbf{f}(\mathbf{e}_1)n_1 + \mathbf{f}(\mathbf{e}_2)n_2 + \mathbf{f}(\mathbf{e}_3)n_3,$$

or

$$f_i(\mathbf{n}) = \sigma_{ij}n_j$$

for some appropriate stress tensor σ_{ij}.

If we now consider a small cube in the fluid with sides of $O(l)$ then the couple on the cube must be zero as $l \to 0$ to prevent infinite torque: this implies that $\sigma_{ij} = \sigma_{ji}$, i.e. that σ is symmetric. See Figure 1.3

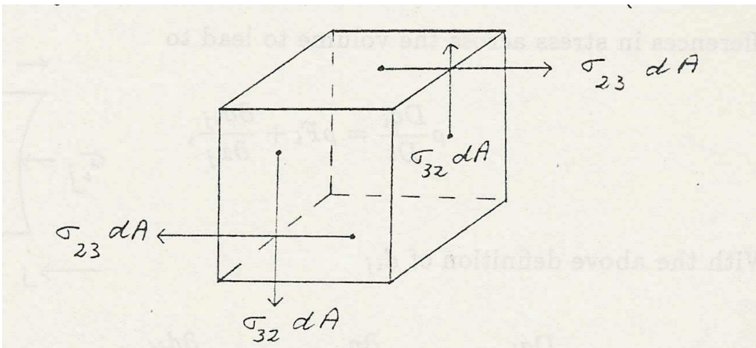

Figure 1.3: The couple on a tiny cube must be zero.

1.4.2 The form of the stress tensor

We write the stress tensor in the form

$$\sigma_{ij} = -p\,\delta_{ij} + T_{ij},$$

taking out the isotropic (direction independent) part to define the pressure p, and leaving the 'deviatoric stress' T_{ij}.

We now make the assumption that the fluid is Newtonian: that the deviatoric stress depends in a linear way on the 'rate of strain' tensor, an object that measures how much the movement of the fluid differs from a rigid body motion (translation or rotation). The rate of change of a small line element $\delta\mathbf{x}$ in the fluid is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\delta\mathbf{x}|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{x}\cdot\delta\mathbf{x}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{d}\delta x_j\,\delta x_j = 2\sum_{j=1}^{d}\frac{\mathrm{d}\delta x_j}{\mathrm{d}t}\,\delta x_j \\
&= 2\sum_{j=1}^{d}[u_j(\mathbf{x}+\delta\mathbf{x}) - u_j(\mathbf{x})]\,\delta x_j \\
&= 2\sum_{i,j=1}^{d}\frac{\partial u_j}{\partial x_i}\,\delta x_i\,\delta x_j \\
&= \sum_{i,j=1}^{d}\delta x_i\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\delta x_j \\
&= \sum_{i,j=1}^{d}\delta x_i\,E_{ij}\,\delta x_j,
\end{aligned}$$

where E_{ij} is the rate of strain tensor,

$$E_{ij} := \frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}.$$

The most general linear isotropic (direction independent) relationship that gives a symmetric tensor T from a tensor E is

$$T = \mu E + \beta\,\mathrm{Tr}(E)I,$$

where $\mathrm{Tr}(E)$ denotes the trace of E, i.e. the sum of the diagonal elements,

$$\mathrm{Tr}(E) = E_{11} + E_{22} + E_{33}.$$

However, when E is the rate of strain tensor we have

$$\mathrm{Tr}(E) = 2\left[\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right] = 2(\nabla\cdot\mathbf{u}),$$

and when the fluid is incompressible this is just zero. So for an incompressible fluid, the stress tensor is just a term due to the pressure, plus a scalar multiple of the rate of strain tensor:

$$\sigma = -pI + \mu E.$$

1.4.3 The equation of motion

We are now in a position to derive the equation of motion. Returning to (1.4) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_t} \rho \mathbf{u} \, \mathrm{d}V = \int_{\partial V_t} \sigma \cdot \mathbf{n} \, \mathrm{d}S.$$

Taking the time derivative under the integral on the left-hand side gives us the convective derivative, while using the divergence theorem on the right-hand side will give as a volume integral:

$$\int_{V_t} \frac{\mathrm{D}}{\mathrm{D}t} (\rho \mathbf{u}) \, \mathrm{d}V = \int_{V_t} \nabla \cdot \sigma \, \mathrm{d}V.$$

Now, since $\sigma = \mu E$ we have

$$\begin{aligned}
[\nabla \cdot \sigma]_i &= \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} \\
&= -\frac{\partial p}{\partial x_i} + \mu \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \\
&= [-\nabla p]_i + \mu \sum_{j=1}^{d} \frac{\partial^2 u_i}{\partial x_j^2} + \mu \frac{\partial}{\partial x_i} \sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j} \\
&= [-\nabla p]_i + \mu \, \Delta u_i + \mu \frac{\partial}{\partial x_i} [\nabla \cdot u] \\
&= [-\nabla p + \mu \, \Delta \mathbf{u}]_i,
\end{aligned}$$

using the incompressibility condition ($\nabla \cdot u = 0$) again, and defining the Laplacian of a vector $\mathbf{u}$ by

$$[\Delta \mathbf{u}]_i := \Delta u_i := \sum_{j=1}^{d} \frac{\partial^2 u_i}{\partial x_j^2}.$$

We therefore have

$$\int_{V_t} \left[\frac{\mathrm{D}}{\mathrm{D}t}(\rho \mathbf{u}) - \mu\, \Delta \mathbf{u} + \nabla p \right] \mathrm{d}V = 0,$$

and as before, since this should hold for any volume in the fluid we obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho \mathbf{u}) - \mu\, \Delta \mathbf{u} + \nabla p = 0. \tag{1.5}$$

Finally, since we are assuming that the fluid is incompressible we have

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho \mathbf{u}) = \rho \left[\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right],$$

and so we obtain the momentum equation

$$\rho \left[\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] - \mu\, \Delta \mathbf{u} + \nabla p = 0. \tag{1.6}$$

1.5 The Navier-Stokes equations

We have now derived the Navier-Stokes equations for an incompressible, Newtonian fluid:

$$\rho \left[\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] - \mu\, \Delta \mathbf{u} + \nabla p = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{u} = 0. \tag{1.7}$$

We will assume in what follows that ρ is constant.

The unknowns are the velocity $\mathbf{u}(\mathbf{x}, t)$, a d-component vector and the pressure $p(\mathbf{x}, t)$. We can specify the density ρ, and the viscosity μ. There are $d + 1$ equations in (1.7), so we can at least hope that our model is solvable. In what follows we will investigate when we can prove that this model has physically meaningful solutions that are valid for all $t \geq 0$.

First, however, we consider the effect of the various terms in the equation. Note that it is very nearly a linear equation. Indeed, without expanding the convective derivative (1.5) appears to linear. All our analytical problems will arise because of the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

The Laplacian term is our friend. It has the effect of dissipating energy – it reflects the viscous effects in fluid motion, which serves to slow things down. The pressure term (∇p) allows us to maintain the divergence-free condition as the fluid moves. The presence of the pressure term also causes problems, although we will be able to cope with these to some extent.

1.6 Boundary conditions

In physical settings our fluid lies in some spatial domain D with boundaries, and almost all physical flows arise from forcing at the boundaries. 'Physical boundary conditions' are Dirichlet, or non-slip boundary conditions: $\mathbf{u} = 0$ on ∂D, i.e. fluids stick to boundaries.

However, in this course we will consider a problem that is mathematically a little simpler, and allows us to avoid any consideration of boundaries. We suppose that the fluid is in a region that has 'periodic boundary conditions': we choose a box $Q = [0, L]^d$, and insist that

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j, t) = \mathbf{u}(\mathbf{x}, t) \qquad \text{for} \qquad j = 1, 2, 3, \tag{1.8}$$

where $\mathbf{e}_j$ in the unit vector in the direction of the j^{th} coordinate axis. Since there are now no boundaries at which we can apply forces to the fluid we make the fluid move by imposing a 'body force' $\mathbf{f}(\mathbf{x}, t)$ that acts, as if by magic, throughout the fluid.

We will therefore study the equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\, \Delta \mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}, t) \qquad \text{and} \qquad \nabla \cdot \mathbf{u} = 0, \tag{1.9}$$

where we have set $\nu = \mu/\rho$ and renamed p. To further simplify matters we will assume that the initial velocity and the forcing have zero average over Q:

$$\int_Q \mathbf{u}(\mathbf{x}, 0)\, \mathrm{d}V = 0 \qquad \text{and} \qquad \int_Q \mathbf{f}(\mathbf{x}, t)\, \mathrm{d}V = 0. \tag{1.10}$$

Physically this means that the forcing $\mathbf{f}$ does not change the total momentum of the fluid, and since this momentum is initially zero (by assumption) it remains zero.

Exercise 1.2. *Show that if (1.10) holds then* $\mathbf{u}(\mathbf{x}, t) = 0$ *for all* $t \geq 0$.

1.7 Fourier representation

One great advantage of considering flows with periodic boundary conditions is that we can use Fourier series in order to represent the solutions of the equation. We can expand $\mathbf{u}(\mathbf{x},t)$ in terms of an infinite sum of complex exponentials,

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}\in\mathbb{Z}_L^d} \hat{\mathbf{u}}(\mathbf{k},t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{1.11}$$

where $\mathbb{Z}_L^d$ is the set of all $\mathbf{k}$ in the form

$$\left(\frac{2\pi n_1}{L},\dots,\frac{2\pi n_d}{L}\right) \qquad \text{with} \qquad n_j \in \mathbb{Z},$$

and the coefficients $\hat{\mathbf{u}}(\mathbf{k},t)$ are given by

$$\hat{\mathbf{u}}(\mathbf{k},t) = \frac{1}{L^d}\int_Q \mathbf{u}(\mathbf{x},t)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^d\mathbf{x}.$$

Exercise 1.3. *If you are unfamiliar with Fourier series, suppose that you can write $\mathbf{u}(\mathbf{x},t)$ as in (1.11). Multiply both sides by $\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ and integrate to find the above expression for $\hat{\mathbf{u}}(\mathbf{k},t)$.*

In order for $\mathbf{u}(\mathbf{x},t)$ to be real we must have

$$\hat{\mathbf{u}}(\mathbf{k},t) = \overline{\hat{\mathbf{u}}(-\mathbf{k},t)}.$$

If we insist that $\int_Q \mathbf{u}(\mathbf{x},t)\,\mathrm{d}^d\mathbf{x} = 0$ then we must have $\hat{\mathbf{u}}(0,t) = 0$ for all t, so in fact

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} \hat{\mathbf{u}}(\mathbf{k},t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{1.12}$$

where $\dot{\mathbb{Z}}_L^d = \mathbb{Z}_L^d \setminus 0$.

The derivatives of such a Fourier series are easy to calculate, assuming that we can differentiate term-by-term: we have

$$\frac{\partial}{\partial x_j}\mathbf{u}(\mathbf{x},t) = \mathrm{i}\sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} k_j\hat{\mathbf{u}}(\mathbf{k},t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

We can rewrite the Navier-Stokes equations in terms of the Fourier coefficients, assuming that

$$p(\mathbf{x},t)=\sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d}\hat{p}(\mathbf{k},t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\qquad\text{and}\qquad\mathbf{f}(\mathbf{x},t)=\sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d}\hat{\mathbf{f}}(\mathbf{k},t)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

The momentum balance equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}(\mathbf{k},t)+\mathrm{i}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[\hat{\mathbf{u}}(\mathbf{k}',t)\cdot\mathbf{k}'']\hat{\mathbf{u}}(\mathbf{k}'',t)+\nu|\mathbf{k}|^2\hat{\mathbf{k}}(\mathbf{k},t)+\mathrm{i}\mathbf{k}\hat{p}(\mathbf{k},t)=\hat{\mathbf{f}}(\mathbf{k},t)\tag{1.13}$$

(in the sum we also have $\mathbf{k}',\mathbf{k}''\in\dot{\mathbb{Z}}_L^d$) while the incompressibility condition is simply

$$\mathbf{k}\cdot\hat{\mathbf{u}}(\mathbf{k},t)=0.$$

If we take the dot product of the momentum equation with $\mathbf{k}$ then we can use the incompressibility condition to eliminate $\hat{p}_k$:

$$\mathrm{i}\mathbf{k}\cdot\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[\hat{\mathbf{u}}(\mathbf{k}',t)\cdot\mathbf{k}'']\hat{\mathbf{u}}(\mathbf{k}'',t)+\mathrm{i}|\mathbf{k}|^2\hat{p}(\mathbf{k},t)=\mathbf{k}\cdot\hat{\mathbf{f}}(\mathbf{k},t),$$

and so

$$\hat{p}(\mathbf{k},t)=\frac{\mathrm{i}}{|\mathbf{k}|^2}\left[\mathrm{i}\mathbf{k}\cdot\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[\hat{\mathbf{u}}(\mathbf{k}',t)\cdot\mathbf{k}'']\hat{\mathbf{u}}(\mathbf{k}'',t)-\mathbf{k}\cdot\hat{\mathbf{f}}(\mathbf{k},t)\right].$$

Note that if $\mathbf{f}$ is not divergence free then that part of $\mathbf{f}$ is absorbed by the pressure term. In much of what follows we will therefore take $\mathbf{f}$ to be divergence-free, i.e. $\mathbf{k}\cdot\hat{\mathbf{f}}(\mathbf{k},t)=0$. In this case the second term in the expression for $\hat{p}$ drops out, and the momentum equation is therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{u}}(\mathbf{k},t)+\mathrm{i}\left(\mathbf{I}-\frac{\mathbf{k}\mathbf{k}}{|\mathbf{k}|^2}\right)\cdot\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}[\hat{\mathbf{u}}(\mathbf{k}',t)\cdot\mathbf{k}'']\hat{\mathbf{u}}(\mathbf{k}'',t)+\nu|\mathbf{k}|^2\hat{\mathbf{u}}(\mathbf{k},t)=\hat{\mathbf{f}}(\mathbf{k},t)\tag{1.14}$$

Without the nonlinear term the equation would be very simple. The Laplacian term causes $\hat{\mathbf{u}}(\mathbf{k},t)$ to decay exponentially fast: with no forcing term this would decay to zero. Note that the rate of decay increases as $|\mathbf{k}|$ increases – high wavenumbers, corresponding to small lengthscales in the flow, decay more rapidly. However, the nonlinear term 'mixes' the Fourier components of the flow: all scales between 0 and $2|\mathbf{k}|$ can contribute to the $\mathbf{k}^{\text{th}}$ wavenumber component of the flow. Note that the rôle of the pressure is to enforce the divergence free condition.

Exercise 1.4. *For any choice of Fourier coefficients* $\mathbf{c_k}$ *show that*

$$\sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} \left[\mathbf{c_k} - \frac{\mathbf{k}\cdot\mathbf{c_k}}{|\mathbf{k}|^2}\,\mathbf{k}\right] \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

is divergence free.

Chapter 2

Solutions of ODEs

In this chapter we will consider how to prove the existence and uniqueness of solutions of ordinary differential equations, using the Contraction Mapping Theorem. First we recall the definition of a Banach space which we need to state this theorem precisely [cf. Chapter 1 of [R]].

2.1 Banach spaces

A *norm* on a vector space X is a 'length' function $\|\cdot\| : X \to [0, \infty)$ satisfying

(i) $\|x\| = 0$ iff $x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$, $\lambda \in \mathbb{R}$, and

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (the "triangle inequality").

A *normed space* consists of a vector space X and a norm $\|\cdot\|_X$: strictly a normed space is a pair $(X, \|\cdot\|_X)$, but most spaces have a "standard" norm and we tend to drop this slightly pedantic notation.

Example: $\mathbb{R}^n$ with the standard Euclidean norm $|\cdot|$ is a normed space.

A sequence $\{x_n\} \in X$ is *Cauchy* if for any $\epsilon > 0$ there exists an N such that

$$\|x_n - x_m\| < \epsilon \qquad \text{for all} \qquad n, m > N.$$

A space X is *complete* if every Cauchy sequence in X converges to another element of X. $\mathbb{R}^n$ is complete.

A *Banach space* is simply a complete normed space: $\mathbb{R}^n$ is the simplest example.

Two norms on X, $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be *equivalent* if there exist constants α_1 and α_2 with $0 < \alpha_1 < \alpha_2$ such that

$$\alpha_1\|x\|_1 \leq \|x\|_2 \leq \alpha_2\|x\|_1.$$

If a sequence converges in the $\|\cdot\|_1$ norm iff it converges in the $\|\cdot\|_2$ norm, hence the idea that they are "equivalent". (All norms on $\mathbb{R}^n$ are equivalent, see Theorem 1.1. in [R].)

2.1.1 Continuous functions

Suppose that $\Omega \subset \mathbb{R}^n$, and X is a normed space. Then a function $f : \Omega \to Y$ is continuous at x if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - \tilde{x}| < \delta \qquad \Rightarrow \qquad \|f(x) - f(\tilde{x})\|_X < \epsilon,$$

and f is continuous if it is continuous at every $x \in \Omega$.

The collection $C^0(\Omega, X)$ of all continuous functions from Ω to X has a standard norm,

$$\|f\|_\infty = \sup_{x\in\Omega} \|f(x)\|_X.$$

With this norm $C^0(\Omega, X)$ is complete: any Cauchy sequence converges, and its limit (the uniform limit of continuous functions) is also continuous. Thus $C^0(\Omega, X)$ is a Banach space.

2.2 Local Existence

We now consider an ordinary differential equation on $\mathbb{R}^d$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \qquad x(0) = x_0, \tag{2.1}$$

and investigate when we can prove the existence and uniqueness of solutions. Chapter 2 of [R] has more details.

The first step is to change this into a problem which is easier to analyse.

Lemma 2.1. *Assume that f is continuous from $\mathbb{R}^d \to \mathbb{R}^d$. Then solutions of the differential equation*

$$\dot{x} = f(x) \qquad x(0) = x_0 \tag{2.2}$$

are equivalent to solutions of the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))\,\mathrm{d}s \tag{2.3}$$

for which $x(t)$ is continuous.

Proof. That solutions of (2.2) give solutions of (2.3) is obvious by integration from 0 to t. If $x(t)$ is continuous then $f(x(t))$ is continuous, and then

$$\int_0^t f(x(s))\,\mathrm{d}s$$

is C^1 and has derivative $f(x(t))$. Clearly (2.3) satisfies the initial condition. □

2.3 The contraction mapping theorem

A solution of (2.3) is a fixed point of the operator $\mathcal{J}$:

$$(\mathcal{J}[x])(t) = x_0 + \int_0^t f(x(s))\,\mathrm{d}s. \tag{2.4}$$

To guarantee the existence of such a fixed point we will use the Contraction Mapping Theorem.

Theorem 2.2. *(The Contraction Mapping Theorem). Let X be a closed subset of a complete normed space $(Y, \|\cdot\|)$, and $h : X \to X$ a function satisfying*

$$\|h(x) - h(y)\| \leq k\,\|x - y\| \qquad \text{for all} \qquad x, y \in X, \tag{2.5}$$

where $k < 1$ (we say that h is a contraction, or contraction mapping, on X). Then h has a unique fixed point x^* in X, i.e. there is a unique $x^* \in X$ with $h(x^*) = x^*$.

See [R] for a proof (Theorem 2.2).

We now assume that f is locally Lipschitz, i.e.

$$|f(x) - f(y)| \leq L(x, y)|x - y|,$$

where $L(x, y)$ can be chosen uniformly for all x, y in any bounded set B:

$$|f(x) - f(y)| \leq L(B)|x - y| \qquad \text{for all } x, y \in B. \tag{2.6}$$

We will show that equation (2.1) has a unique solution, at least on a small time interval $[0, T]$.

We will look for a solution that is an element of $C^0([0, T], \mathbb{R}^d)$, i.e. a continuous function from $[0, T]$ into $\mathbb{R}^d$. We will use the notation

$$\|x\|_\infty = \max_{0 \leq t \leq T} |x(t)|$$

to denote the maximum value of $|x(t)|$ on the interval $[0, T]$.

Theorem 2.1. *Suppose that f is locally Lipschitz as in (2.6). Then there exists a T, which depends on x_0, such that*

$$\mathrm{d}x/\mathrm{dt} = f(x) \qquad \text{with} \qquad x(0) = x_0$$

has a solution on $[0, T]$.

Proof. If $f(x_0) = 0$ then

$$x(t) = x_0 \qquad \text{for all} \qquad t \in \mathbb{R}$$

is clearly a solution, and we can take any $T > 0$. So assume that $f(x_0) \neq 0$.

We will choose T to ensure that the solution cannot move too far from its initial condition. Let $B = B_1(x_0)$, the ball of radius 1 centred on x_0,

$$B_1(x_0) = \{x : \ |x - x_0| \leq 1\}.$$

Since f is locally Lipschitz there exists a constant L_{x_0} such that

$$|f(x) - f(y)| \leq L_{x_0}|x - y| \qquad \text{for all} \qquad x, y \in B_1(x_0). \tag{2.7}$$

We apply the Contraction Mapping Theorem in the space

$$X = \{x \in C^0([0, T], \mathbb{R}^d) : \ \|x - x_0\|_\infty \leq 1\},$$

which is a closed subset of the Banach space $C^0([0, T], \mathbb{R}^d)$.

We have to show that $\mathcal{J}$ maps X into itself and is a contraction. It is clear that whatever choice we make for T, if $x \in C^0([0, T], \mathbb{R}^d)$ then so is $\mathcal{J}[x]$. Our first task is therefore to choose T such that

$$|\mathcal{J}[x](t) - x_0| \leq 1 \qquad \text{for all} \qquad 0 \leq t \leq T.$$

We have

$$\mathcal{J}[x](t) - x_0 = \int_0^t f(x(s))\,\mathrm{d}s,$$

so we want to make sure that

$$\left|\int_0^t f(x(s))\,\mathrm{d}s\right| \leq 1$$

if $\|x - x_0\|_\infty \leq 1$. We can write

$$\begin{aligned}
\left|\int_0^t f(x(s))\,\mathrm{d}s\right| &= \left|\int_0^t f(x(s)) - f(x_0) + f(x_0)\,\mathrm{d}s\right| \\
&\leq \int_0^t |f(x(s)) - f(x_0)| + |f(x_0)|\,\mathrm{d}s \\
&\leq \int_0^t L_{x_0}|x(s) - x_0|\,\mathrm{d}s + t|f(x_0)| \\
&\leq t[L_{x_0} + |f(x_0)|] \\
&\leq T[L_{x_0} + |f(x_0)|],
\end{aligned}$$

since $t < T$, and we have assumed that $x(t)$ remains in the set $B_1(x_0)$, where f has Lipschitz constant L_{x_0} [as in (2.7)].

So we will take

$$T = \frac{1}{L_{x_0} + |f(x_0)|}, \tag{2.8}$$

which ensures that $\|\mathcal{J}[x] - x_0\|_\infty \leq 1$ as we wanted.

Now we want to show that $\mathcal{J}$ is a contraction on X. To do this, take $x, y \in X$. Then we have

$$\begin{aligned}\mathcal{J}[x](t) - \mathcal{J}[y](t) &= \int_0^t f(x(s))\,\mathrm{d}s - \int_0^t f(y(s))\,\mathrm{d}s \\ &= \int_0^t f(x(s)) - f(y(s))\,\mathrm{d}s,\end{aligned}$$

and so, using the Lipschitz property of f on the ball $B_1(x_0)$ we have

$$\begin{aligned}|\mathcal{J}[x](t) - \mathcal{J}[y](t)| &\leq \int_0^t |f(x(s)) - f(y(s))|\,\mathrm{d}s \\ &\leq \int_0^t L_{x_0}|x(s) - y(s)|\,\mathrm{d}s \\ &\leq \int_0^t L_{x_0}\|x - y\|_\infty\,\mathrm{d}s \\ &\leq L_{x_0} t\|x - y\|_\infty \\ &\leq L_{x_0} T\|x - y\|_\infty.\end{aligned}$$

Since the bound on the right-hand side does not depend on t, we can deduce that

$$\|\mathcal{J}[x] - \mathcal{J}[y]\|_\infty \leq L_{x_0} T\|x - y\|_\infty. \tag{2.9}$$

Now, by our choice of T

$$k := L_{x_0} T = \frac{L_{x_0}}{|f(x_0)| + L_{x_0}} < 1.$$

It follows that $\mathcal{J}$ is a contraction on X, and hence has a unique fixed point in X.

There remains a subtle but significant point to deal with. We have shown that there is only one solution that remains in $B_1(x_0)$ for $t \in [0,T]$, but it is conceivable that there might be other solutions that do not remain in $B_1(x_0)$ for $t \in [0,T]$.

However, since any solution satisfies the integral equation (2.3), following the analysis above we must have

$$|x(t) - x_0| = \left| \int_0^t f(x(s))\,\mathrm{d}s \right| \le t\,[L_{x_0} + |f(x_0)|] \tag{2.10}$$

while $|x(t)-x_0| \le 1$. Suppose that $|x(t)-x_0| > 1$ for some $t \in [0,T]$. Since $x(t)$ is continuous there must exist a maximum $T^* < t$ such that $|x(t)-x_0| \le 1$ on $[0,T^*]$, and in particular $|x(T^*) - x_0| = 1$. However, since $|x(t) - x_0| \le 1$ for $t \in [0,T^*]$, (2.10) implies that

$$|x(T^*) - x_0| \le T^*\,[L_{x_0} + |f(x_0)|] = \frac{T^*}{T} < 1,$$

a contradiction.

So *all* solutions must stay in $B_1(x_0)$ for $t \in [0,T]$, and hence we have the unique solution of the equation. □

We note the following useful corollary.

Corollary 2.2. *The time of existence, T, in Theorem 2.1 can be taken to be uniform for initial conditions contained in any bounded subset $X \subset \mathbb{R}^d$.*

Proof. Without loss of generality assume that X is closed. A continuous function on a closed subset of $\mathbb{R}^d$ is bounded, so we must have $|f(x_0)| \le M$ for all $x_0 \in X$. Also, since f is locally Lipschitz, there exists an L such that

$$|f(x) - f(y)| \le L|x-y| \qquad \text{for all} \qquad x, y \in B_1(X),$$

where (in an abuse of notation) the bounded set $B_1(X)$ is defined by

$$B_1(X) = \{y : |y - x| \le 1 \text{ for some } x \in X.\}.$$

In particular, it follows that $L_{x_0} \le L$ for all $x_0 \in X$, and so

$$\frac{1}{|f(x_0)| + L_{x_0}|} \ge \frac{1}{M+L},$$

and so we can take $T = 1/(M+L)$ for all $x_0 \in X$. □

2.4 Gronwall's Lemma

We have proved the existence and uniqueness of a solution, at least on a short time interval. We now show that the solution depends continuously on its initial condition. In order to prove this we will use the following simple, but fundamental, result.

Lemma 2.3. *(Gronwall's Lemma). Let $y(t) \in \mathbb{R}^+$ satisfy*

$$\mathrm{d}y/\mathrm{d}t \leq ay + b \qquad y(0) \leq y_0. \tag{2.11}$$

Then

$$y(t) \leq \left(y_0 + \frac{b}{a}\right)\mathrm{e}^{at} - \frac{b}{a}.$$

Proof. Multiply both sides of (2.11) by the integrating factor e^{-at}

$$\frac{\mathrm{d}}{\mathrm{d}t}(y\mathrm{e}^{-at}) \leq b\mathrm{e}^{-at}$$

and integrate between 0 and t:

$$y(t)\mathrm{e}^{-at} \leq y(0) + b\int_0^t \mathrm{e}^{-as}\,\mathrm{d}s,$$

so that

$$y(t) \leq y_0\mathrm{e}^{at} + \frac{b}{a}(\mathrm{e}^{at} - 1),$$

which gives the result. □

Exercise 2.1. *By following the same integrating factor method from the proof above, show that if $x(t) \geq 0$ satisfies the differential inequality*

$$\frac{\mathrm{d}x}{\mathrm{d}t} \leq g(t)x + h(t)$$

then

$$x(t) \leq x(0)\exp[G(t)] + \int_0^t \exp[G(t) - G(s)]h(s)\,\mathrm{d}s,$$

where

$$G(t) = \int_0^t g(r)\,\mathrm{d}r.$$

The following observation is also useful. If $\mathrm{d}x/\mathrm{d}t = f(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}|x| \le |f(x(t))|. \tag{2.12}$$

Assuming that $|x(t)| \neq 0$ this is easy to see, since

$$\frac{\mathrm{d}}{\mathrm{d}t}|x|^2 = 2|x|\frac{\mathrm{d}}{\mathrm{d}t}|x|$$

and also

$$\frac{\mathrm{d}}{\mathrm{d}t}|x|^2 = \frac{\mathrm{d}}{\mathrm{d}t}(x \cdot x) = 2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Equating the right-hand sides of these two expressions gives

$$2|x|\frac{\mathrm{d}}{\mathrm{d}t}|x| = 2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t},$$

and provided that $|x(t)| \neq 0$ we can divide by $2|x|$ to give

$$\frac{\mathrm{d}}{\mathrm{d}t}|x| = \frac{x}{|x|} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \le \left|\frac{x}{|x|}\right| \left|\frac{\mathrm{d}x}{\mathrm{d}t}\right| = |f(x(t))|.$$

With a little care the problem that occurs when $|x(t)| = 0$ can be overcome, see Lemma 2.7 in [R].

Theorem 2.3. *The solution found in Theorem 2.1 depends continuously on the initial condition x_0. In particular, if the solution $x(t)$ of*

$$\mathrm{d}x/\mathrm{d}t = f(x) \qquad \text{with} \qquad x(0) = x_0$$

exists on $[0, T_x]$ and

$$\mathrm{d}y/\mathrm{d}t = f(y) \qquad \text{with} \qquad y(0) = y_0$$

exists on $[0, T_y]$ then there exists an L such that

$$|x(t) - y(t)| \le |x_0 - y_0|\mathrm{e}^{Lt} \qquad \text{for all} \qquad 0 \le t \le \min(T_x, T_y).$$

Note that this theorem gives an alternative proof of uniqueness, since if $x_0 = y_0$ we have $x(t) = y(t)$ on $[0, T]$.

Proof. Since $x(t)$ and $y(t)$ are continuous functions on bounded intervals they are bounded. In particular they lie within a bounded set B. Since f is locally Lipschitz, there exists an L_B such that

$$|f(x) - f(y)| \leq L_B|x - y| \qquad \text{for all} \qquad x, y \in B.$$

If we consider $z = x - y$ then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} = f(x) - f(y),$$

and so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|z| &\leq |f(x) - f(y)| \\ &\leq L_B|x - y|. \end{aligned}$$

Gronwall's Lemma implies that

$$|x(t) - y(t)| \leq \mathrm{e}^{L_B t}|x_1(0) - x_2(0)| \qquad \text{for all} \qquad t \in [0, T].$$

□

Exercise 2.2. *Suppose that f is globally Lipschitz,*

$$|f(x) - f(y)| \leq L|x - y| \qquad \text{for all} \qquad x, y \in \mathbb{R}^d.$$

If $x(t)$ is the solution of

$$\mathrm{d}x/\mathrm{d}t = f(x) \qquad x(0) = x_0$$

and $y(t)$ is any one of the solutions of

$$\mathrm{d}y/\mathrm{d}t = g(y) \qquad y(0) = x_0$$

(they may not be unique unless g is Lipschitz) use Gronwall's Lemma to show that

$$|x(t) - y(t)| \leq \frac{\|f - g\|_\infty}{L}(\mathrm{e}^{Lt} - 1).$$

2.5 Global Existence

The global existence of solutions (for all $t \geq 0$) is not automatic. Indeed, the solution of

$$\mathrm{d}x/\mathrm{d}t = x^2 \qquad x(0) = x_0 > 0$$

is

$$x(t) = \frac{x_0}{1 - x_0 t},$$

and so $x(t) \to \infty$ as $t \to x_0^{-1}$.

Definition 2.4. *We say that* $[0, T^*)$ *is the* maximal interval of existence *for a solution* $x(t)$ *if there is no solution* $y(t)$ *on a longer time interval* $[0, T^+)$*,* $T^+ > T^*$*, with* $x(t) = y(t)$ *on* $[0, T^*)$*. In other words, we cannot extend the solution* $x(t)$ *beyond the time* T^* *so that it remains a solution. If the solution exists for all time then its maximal interval of existence is* $[0, \infty)$*.*

To show existence for all time ("global existence"), it is enough to ensure that the solution does not "blow up" in finite time.

Lemma 2.5. *A solution* $x(t)$ *of Equation (2.1) has a finite maximal interval of existence* $[0, T^*)$ *iff* $|x(t)| \to \infty$ *as* $t \to T^*$*.*

Proof. Clearly, if $|x(t)| \to \infty$ as $t \to T^*$ then there can be no continuous extension of $x(t)$ to an interval containing T^*. Conversely, suppose that the solution is bounded on $[0, T^*)$, i.e. the solution $x(t)$ lies within a bounded set X for all $t \in [0, T^*)$.

Now use Corollary 2.2 to find a time T such that for any initial condition in X we can find a solution defined on $[0, T]$. If we use the existence result with initial condition $x(t)$ $(0 \leq t < T^*)$ this will give us a solution defined on $[0, t + T]$, which must agree (by uniqueness) with our previous solution of $[0, T^*)$. If we take $t > T^* - T$ we obtain a contradiction. □

Note that it does not follow that a solution which exists for all time must be bounded (e.g. the solution $x(t) = \mathrm{e}^t$ of $\mathrm{d}x/\mathrm{d}t = x$ with $x(0) = 1$).

Lemma 2.4. *If $f(x)$ is globally Lipschitz, i.e. if*

$$|f(x) - f(y)| \leq L|x - y| \qquad \text{for all} \qquad x, y \in \mathbb{R}^d,$$

then solutions are defined for all time.

Proof. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|x| &\leq |f(x(t))| \\ &\leq |f(x(t)) - f(0)| + |f(0)| \\ &\leq L|x(t)| + |f(0)|, \end{aligned}$$

and hence by Gronwall's Lemma

$$|x(t)| \leq \left(|x(0)| + \frac{|f(0)|}{L} \right) \mathrm{e}^{Lt},$$

which is always finite. □

Exercise 2.3. *Show that if $f(x)$ is bounded then solutions are defined for all time.*

2.6 A dynamical system

A dynamical system in continuous time consists of two elements,

$$(X, \{T(t)\}_{t \in \mathbb{R}}),$$

the phase space X and the flow $T(t)$. The phase space represents all possible configurations of the system (or some distinguished subclass of these), while $T(t)$ describes the dynamics of the system: if the state of the system at time 0 is x, then its state at time t is $T(t)x$. In particular, we must have $T(0) = \mathrm{id}$.

Requiring that solutions are unique – which they must be if we want to have a deterministic model – means that $T(t)$ must also satisfy

$$T(t + s) = T(t)T(s) = T(s)T(t).$$

In many examples, one can also show that the state of the system at time t depends continuously on its state at time 0, so that

$$T(t)x \quad \text{depends continuously on } x$$

for each t.

We have just seen that for globally Lipschitz ODEs, and for locally Lipschitz ODEs which do not blow up, we can use the solutions to define a dynamical system by setting $T(t)x_0 = x(t; x_0)$.

For globally Lipschitz ODEs, we can define $T(t)$ for all $t \in \mathbb{R}$, and we end up with a classical dynamical system.

In the case of locally Lipschitz f, is we know that solutions do not blow up for $t \geq 0$ we can, *a priori*, only define $T(t)$ for $t \geq 0$. This is similar to the situation we encounter in the PDE case, and gives rise only to a *semi-dynamical system*, where we must have $t \in \mathbb{R}^+$. (Later on we will use the notation $\{S(t)\}_{t\geq 0}$ to distinguish this case from a "full" dynamical system.)

2.7 Continuous f

When the nonlinearity $f(x)$ is continuous but not locally Lipschitz, our approach will be to approximate the problem

$$\mathrm{d}x/\mathrm{d}t = f(x) \qquad x(0) = x_0 \tag{2.13}$$

by a sequence of problems

$$\mathrm{d}x_n/\mathrm{d}t = f_n(x_n) \qquad x_n(0) = x_0,$$

where every $f_n(x)$ is Lipschitz. We can already find a unique solution x_n to each member of this sequence using Theorem 2.1. The idea is to show that some subsequence of these solutions converges to give a solution of (2.13). We will use a very similar method to construct solutions of PDEs.

2.7.1 Approximation by smooth functions

Mollification provides an explicit way of finding a smooth function which approximates a given less regular function. We choose an infinitely differentiable function $\rho(x)$ which satisfies

$$\rho(x) = 0 \qquad \text{for all} \qquad |x| \geq 1$$

and

$$\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1. \tag{2.14}$$

(One example of such a function is

$$\rho(x) = \begin{cases} c\exp\Big(\frac{1}{|x|^2-1}\Big) & |x| \leq 1 \\ 0 & |x| \geq 1, \end{cases}$$

where c is chosen so that (2.14) holds.)

The mollification of u, u_h, is

$$u_h(x) = h^{-m} \int_Q \rho\Big(\frac{x-y}{h}\Big) u(y)\,\mathrm{d}y. \tag{2.15}$$

Proposition 2.6. *Let $u \in C^0(\mathbb{R}^d)$ (i.e. let u be a continuous function on $\mathbb{R}^d$). Then $u_h \in C^\infty(\mathbb{R}^d)$, and $u_h \to u$ as $h \to 0$ uniformly on any compact subset of $\mathbb{R}^d$.*

For a proof see Proposition 1.6 in [R].

2.7.2 The Arzelà-Ascoli theorem

The result we will use to extract a convergent subsequence from the approximate solutions $\{x_n(t)\}$ is known as the Arzelà-Ascoli theorem. This characterises the compact subsets of $C^0(X, \mathbb{R}^m)$ as sets of bounded equicontinuous functions. If $\mathcal{S}$ is such a subset and $\{f_n\}$ is a sequence in $\mathcal{S}$, then there is a subsequence of $\{f_n\}$ which is uniformly convergent.

Theorem 2.7. *(Arzelà-Ascoli). Let X be a compact subset of $\mathbb{R}^{m_1}$, and $\{f_n\}$ a sequence of continuous functions from X into $\mathbb{R}^{m_2}$. If f_n is uniformly bounded, i.e. there exists an M such that*

$$\|f_n\|_\infty \le M \qquad \text{for all } n,$$

and equicontinuous ("uniformly uniformly continuous"), i.e. for every $\epsilon > 0$ there exists a $\delta > 0$, independent of n, such that

$$|x - y| \le \delta \qquad \Rightarrow \qquad |f_n(x) - f_n(y)| \le \epsilon,$$

then $\{f_n\}$ has a subsequence which converges uniformly on X.

See [R], Theorem 2.5, for a proof.

A prime example of a family of equicontinuous functions is the space of Lipschitz functions on some closed set K.

2.7.3 Local existence for continuous f

We now use this theorem to give an existence result for the case when we only know that f is continuous.

Theorem 2.8. *Let $f(x)$ be a bounded continuous function. Then for any $T > 0$ there is at least one solution of the equation*

$$\mathrm{d}x/\mathrm{d}t = f(x) \qquad x(0) = x_0 \tag{2.16}$$

on $[0, T]$.

Proof. We approximate a solution of (2.16) by a sequence of solutions of Lipschitz equations, for which Theorem 2.1 guarantees a solution. We set

$$f_n(x) = n^d \int_{\mathbb{R}^d} \rho(n(y - x)) f(y)\, \mathrm{d}y,$$

(this is the mollification from before, with $h = 1/n$), and note that

$$|f_n(x)| \le n^d \int_{\mathbb{R}^d} \rho(n(y - x)) \|f\|_\infty\, \mathrm{d}y,$$

and so $f_n(x)$ is also bounded, $\|f_n\|_\infty \leq \|f\|_\infty$. We also know from Proposition 2.6 that for any $R > 0$,

$$\sup_{|x-x_0|\leq R} \|f_n - f\|_\infty \to 0.$$

Now consider the solutions $x_n(t)$ of the equations

$$\mathrm{d}x_n/\mathrm{d}t = f_n(x_n) \qquad x_n(0) = x_0; \tag{2.17}$$

since f_n is (locally) Lipschitz, the solutions of (2.17) exist locally, by Theorem 2.1. Furthermore, we know that solutions cannot blow up, since, using the integral formulation from Lemma 2.1 we have

$$\sup_{t\in[0,T]} |x_n(t) - x_0| \leq T\|f\|_\infty.$$

So, from Lemma 2.5, all the solutions exist on $[0, T]$. In particular, this shows that $\{x_n\}$ is uniformly bounded,

$$\|x_n\|_\infty \leq |x_0| + T\|f\|_\infty.$$

We can also show similarly that

$$|x_n(s) - x_n(t)| \leq \|f\|_\infty |s - t|,$$

the $\{x_n\}$ are a uniformly bounded equicontinuous sequence of functions from $[0, T]$ into $\mathbb{R}^d$. As such, the Arzelà-Ascoli theorem guarantees that they have a uniformly convergent subsequence, $\{x_{n_j}\}$, with

$$\sup_{t\in[0,T]} |x_{n_j}(t) - x^*(t)| \to 0$$

as $j \to \infty$.

The function $x^*(t)$ is clearly a candidate for a solution of (2.16); to show that it is one, consider the integral form of (2.16) from lemma 2.1,

$$x_{n_j}(t) = x_0 + \int_0^t f_{n_j}(x_{n_j}(s))\,\mathrm{d}s.$$

Since x_{n_j} converges uniformly to x^* on $[0,T]$, and f_{n_j} converges uniformly to f on $|x - x_0| \leq T\|f\|_\infty$, all the terms converge uniformly on $[0,T]$, to give

$$x^*(t) = x_0 + \int_0^t f(x^*(s))\,\mathrm{d}s,$$

so that x^* is indeed a solution of (2.16) on $[0,T]$. □

The lack of uniqueness is not just an artefact of the proof, as the following simple example shows. The equation

$$\mathrm{d}x/\mathrm{d}t = x^{1/2} \qquad x(0) = 0$$

has, for $t \geq 0$, an infinite family of solutions which can be parametrised by $c \geq 0$,

$$x_c(t) = \begin{cases} 0 & 0 \leq t \leq c \\ (t-c)^2/4 & t > c. \end{cases}$$

Chapter 3

Global attractors

We now consider a general semidynamical system $(X, \{S(t)\}_{t\geq 0})$, where $S(t)$ is a semigroup,

$$S(0) = I$$
$$S(t)S(s) = S(s)S(t) = S(s+t)$$
$$S(t)x_0 \text{ is continuous in } x_0 \text{ and } t.$$

Definition 3.1. *The* global attractor $\mathcal{A}$ *is a compact invariant set*

$$S(t)\mathcal{A} = \mathcal{A} \qquad \text{for all } t \tag{3.1}$$

that attracts all bounded sets,

$$\text{dist}(S(t)B, \mathcal{A}) \to 0 \qquad \text{as } t \to \infty. \tag{3.2}$$

The distance in (3.2) is the distance between two sets,

$$\text{dist}(X, Y) = \sup_{x\in X} \inf_{y\in Y} |x - y|.$$

Equation (3.1) says that if you start in $\mathcal{A}$ you stay in $\mathcal{A}$, while (3.2) says that $\mathcal{A}$ attracts all orbits, at a rate uniform on any bounded set.

3.1 An existence result

The existence of a global attractor follows from the existence of a compact absorbing set.

Definition 3.2. *A set B is* absorbing *if for any bounded set X there exists a $t_0(X)$ such that*

$$S(t)X \subset B \text{ for all } t \geq t_0(X).$$

Definition 3.3. *The ω-limit set of a bounded set X, $\omega(X)$, consists of all the asymptotic limit points of the trajectory of X:*

$$\omega(X) := \{y : \exists\ t_n \to \infty,\ x_n \in X \text{ with } S(t_n)x_n \to y\},$$

or equivalently

$$\omega(X) := \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)X}.$$

Exercise 3.1. *Show that these two definitions of $\omega(X)$ are equivalent.*

Theorem 3.4. *If $S(t)$ has a compact absorbing set K then $\mathcal{A} = \omega(K)$ is the global attractor for $S(t)$.*

Proof. First we show that $\mathcal{A}$ is non-empty, compact, and invariant, and then we show that it attracts all bounded sets.

The sets $\overline{\cup_{s\geq t_0(B)} S(s)K}$ are a sequence of non-empty and compact sets decreasing as t increases, and so their intersection ($\mathcal{A}$) is non-empty and compact.

To show invariance, suppose that $x \in \mathcal{A}$. Then there exist $t_n \to \infty$ and $x_n \in K$ with

$$S(t_n)x_n \to x,$$

and so

$$S(t)S(t_n)x_n = S(t + t_n)x_n \to S(t)x,$$

since $S(t)$ is continuous. So $S(t)\mathcal{A} \subset \mathcal{A}$. To show equality, for $t_n \geq \max(t, t_0(K))$, the sequence $S(t_n - t)x_n$ is in K and so possesses a convergent subsequence

$$S(t_{n_j} - t)x_{n_j} \to y,$$

so $y \in \omega(K)$. But since $S(t)$ is continuous,

$$x = \lim_{j\to\infty} S(t)S(t_{n_j} - t)x_{n_j} = S(t)y,$$

so $\mathcal{A} \subset S(t)\mathcal{A}$. Thus $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$. If $S(t)$ is injective on $\mathcal{A}$ (as it is always for ODEs) then there is a backwards trajectory also and $S(t)\mathcal{A} = \mathcal{A}$ for every $t \in \mathbb{R}$.

Now suppose that $\mathcal{A}$ does not attract all bounded sets. Then there is a bounded set X, a $\delta > 0$ and $t_n \to \infty$ with

$$\operatorname{dist}(S(t_n)X, \mathcal{A}) \geq \delta,$$

and so there are $x_n \in X$ with

$$\operatorname{dist}(S(t_n)x_n, \mathcal{A}) \geq \delta/2. \tag{3.3}$$

As X is bounded, $S(t_n)x_n \in K$ for n large enough. As K is compact, there is a subsequence with

$$S(t_{n_j})x_{n_j} \to \beta \in K.$$

But

$$\begin{aligned} \beta &= \lim_{j\to\infty} S(t_{n_j})x_{n_j} \\ &= \lim_{j\to\infty} S(t_{n_j} - t_0(X))S(t_0(X))x_{n_j}, \end{aligned}$$

and setting $\beta_j = S(t_0(X))x_{n_j}$ notice that $\beta_j \in K$, and thus $\beta \in \mathcal{A}$, contradicting (3.3). □

Exercise 3.2. *Show that if Y is a compact invariant set then $Y \subseteq \mathcal{A}$, and that if Z attracts all bounded sets then $\mathcal{A} \subseteq Z$. [$\mathcal{A}$ is the 'maximal compact invariant set' and the 'minimal set that attracts bounded sets'. You can find $\mathcal{A}$ referred to as both the 'maximal attractor' and 'minimal attractor' in the literature.]*

Exercise 3.3. *Show that the global attractor is connected, i.e. there do not exist two disjoint open sets O_1 and O_2 such that*

$$O_j \cap \mathcal{A} \neq \emptyset \qquad \text{and} \qquad \mathcal{A} \subset O_1 \cup O_2.$$

Exercise 3.4. *Show that there exists a global attractor iff there is a compact attracting set, i.e. a compact set K such that*

$$\text{dist}(S(t)X, K) \to 0 \qquad \text{as} \qquad t \to \infty$$

for all bounded sets X. First show that $\omega(K)$ is non-empty, invariant, and attracts K, and then deduce that $\mathcal{A} = \omega(K)$. You may assume without proof that if K is compact and $\{x_n\}$ is a sequence such that $\text{dist}(x_n, K) \to 0$ *as $n \to \infty$ then there is a subsequence that converges to some $x^* \in K$.*

3.2 A global attractor for the Lorenz equations

As an example, it is simple to show that the Lorenz equations

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= -\sigma x + \sigma y \\ \mathrm{d}y/\mathrm{d}t &= rx - y - xz \\ \mathrm{d}z/\mathrm{d}t &= xy - bz, \end{aligned}$$

with σ, r, b all positive have a global attractor. By Theorem 3.4, all we need to show is the existence of a compact absorbing set. In a finite-dimensional space it is enough to show the existence of a bounded absorbing set: here we show that a large enough sphere centred on $(0, 0, r + \sigma)$ is absorbing.

Consider

$$V(x, y, z) = x^2 + y^2 + (z - r - \sigma)^2,$$

which satisfies

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= -2\sigma x^2 - 2y^2 - 2bz^2 + 2b(r + \sigma)z \\ &= -2\sigma x^2 - 2y^2 - b(z - r - \sigma)^2 - bz^2 + b(r + \sigma)^2 \\ &\le -\alpha V + b(r + \sigma)^2, \end{aligned}$$

where $\alpha = \min(2\sigma, 2, b)$. Then by the Gronwall lemma

$$V(t) \le \frac{2b(r + \sigma)^2}{\alpha}$$

when t is large enough. It follows that there is a bounded absorbing set, and hence a global attractor. This is shown in Figures 3.1 and 3.2.

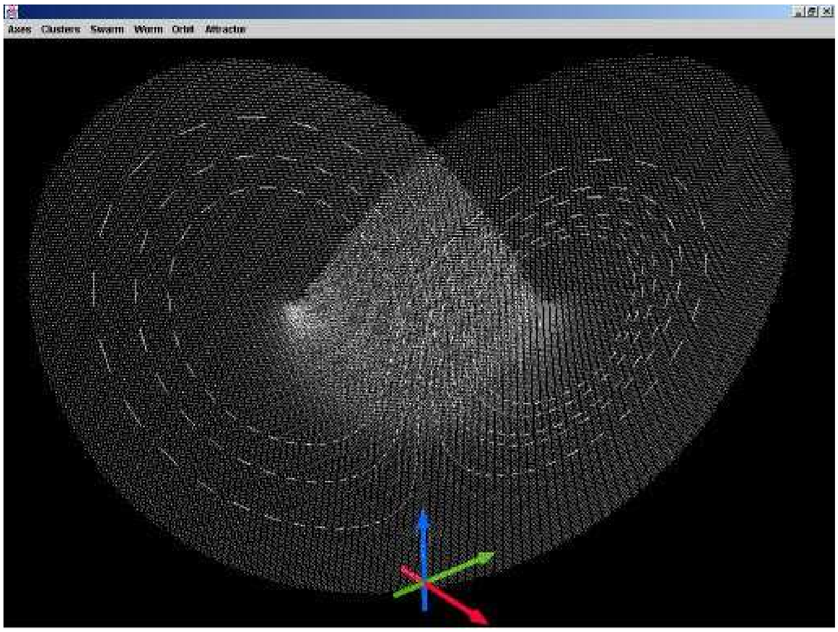

Figure 3.1: The global attractor for the Lorenz equations, computed by Oliver Tearne.

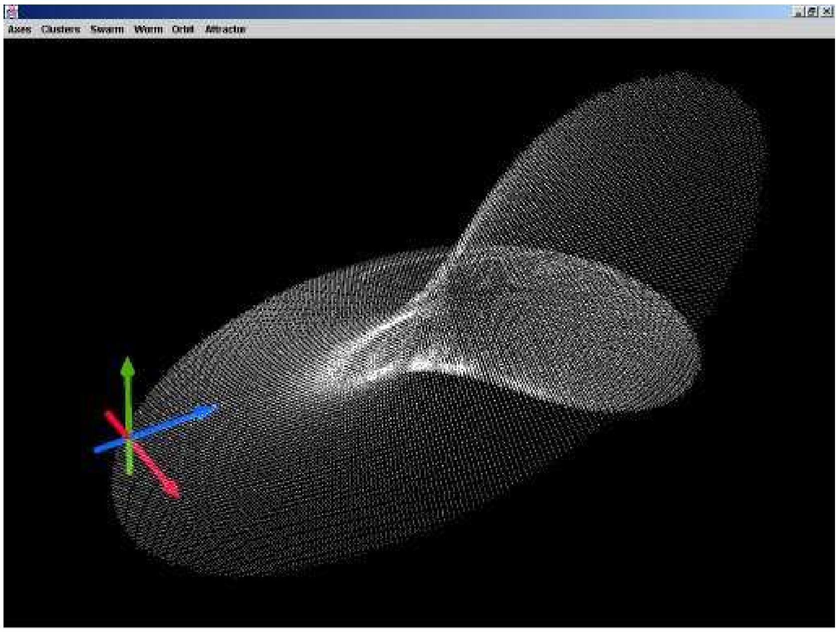

Figure 3.2: Another view of the Lorenz global attractor.

Chapter 4

Banach & Hilbert spaces

This is an abbreviated version of Chapter 1 of [R].

We have already met Banach spaces (complete normed spaces) at the beginning of the previous chapter. We now give some more examples, and cover some basic theory of Hilbert spaces.

4.1 Spaces of continuous functions

To simplify matters throughout this course we will study problems only on *periodic domains*, i.e. functions on d-dimensional "boxes" $[0, L]^d$, which are L-periodic in all directions as in (1.8). This enables us (i) to avoid any problems that arise from considering domains with boundaries (and there are many) and (ii) to use Fourier series.

Always we will use Q to denote the box $[0, L]^d$. The space $C^0(Q)$ consists of all continuous functions on Q, the space $C^0_{\rm p}(Q)$ consists of all continuous functions on Q which are L-periodic as in (1.8). Since functions in either of these two spaces are bounded, we can equip these spaces with the supremum ("sup") norm,

$$\|u\|_\infty = \sup_{x\in Q} |u(x)|. \tag{4.1}$$

$C^0(Q)$ and $C^0_{\rm p}(Q)$ are both Banach spaces (they are complete), and consist of functions which are uniformly continuous on Q.

4.1.1 Multi-index notation

Other spaces of continuous functions involve higher orders of differentiability. Multi-index notation is an elegant way to express mixed partial derivatives. We will use the notation D_j to denote $\partial/\partial x_j$, so that, for example,

$$|\nabla u|^2 = \sum_{j=1}^{d} |D_j u|^2. \tag{4.2}$$

A multi-index α is a vector consisting of m non-negative integers $(\alpha_1, \ldots, \alpha_m)$; we write $|\alpha|$ for the sum of the entries,

$$|\alpha| = \alpha_1 + \ldots + \alpha_m.$$

For any vector $\mathbf{k} = (k_1, \ldots, k_m)$, we define $\mathbf{k}^\alpha$ by

$$\mathbf{k}^\alpha = k_1^{\alpha_1} \ldots k_m^{\alpha_m}.$$

We also set

$$D^\alpha = D_1^{\alpha_1} \ldots D_m^{\alpha_m},$$

(as if D were the "vector" $(D_1, \ldots, D_m)$) so that we have

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}.$$

If we define

$$\alpha! = \alpha_1! \ldots \alpha_m!$$

then we can write, for example, Taylor's theorem as

$$f(\mathbf{x} + \mathbf{h}) = \sum_{|\alpha| \le k} \frac{D^\alpha f(\mathbf{x})}{\alpha!} \mathbf{h}^\alpha \ + \sum_{|\alpha| = k+1} \frac{D^\alpha u(\mathbf{x} + t\mathbf{h})}{\alpha!} \, \mathbf{h}^\alpha \qquad \text{some } t \in (0, 1).$$

4.1.2 Differentiable functions, $C^r(Q)$ and $C_{\mathrm{p}}^r(Q)$

$C^r(Q)$ consists of functions f all of whose derivatives up to and including order r are continuous,

$$C^r(Q) = \{f : \ D^\alpha f \in C^0(Q) \qquad \text{for all} \qquad |\alpha| \le r\}.$$

This space has norm

$$\|f\|_{C^r} = \sum_{|\alpha|\le r} \sup_{x\in Q} |D^\alpha f(x)|,$$

which makes it a Banach space.

$C^\infty(Q)$ consists of "smooth" functions which are infinitely differentiable on Q,

$$C^\infty(Q) = \bigcap_{r=0}^{\infty} C^r(Q).$$

There is no norm that makes $C^\infty(Q)$ into a Banach space.

$C_{\mathrm{p}}^r(Q)$ and $C_{\mathrm{p}}^\infty(Q)$ are defined as above, with the obvious addition of the p suffix on the right-hand side also.

4.2 The Lebesgue spaces $L^p(Q)$

In this section we introduce the Lebesgue spaces $L^p(Q)$ as the completion of spaces of continuous functions with finite L^p norm; this is the approach adopted in Renardy & Rogers' book.

4.2.1 The pseudo-Lebesgue spaces $\tilde{L}^p(Q)$ for $1 \le p < \infty$

The "energy norm"

$$\|u\|_{L^2} = \left(\int_Q |u(x)|^2 \, \mathrm{d}x\right)^{1/2}$$

is part of a whole family of integral norms, the "L^p norms", given by

$$\|f\|_{L^p} = \left(\int_Q |f(x)|^p \,\mathrm{d}x\right)^{1/p}. \tag{4.3}$$

Let us consider first the normed space "pseudo $L^p(Q)$",

$$\tilde{L}^p(Q) = (C^0(Q), \|\cdot\|_{L^p}),$$

i.e. the vector space $C^0(Q)$ equipped with the L^p norm[1]. First we want to show that (4.3) does indeed define a norm on $\tilde{L}^p(Q)$.

Axiom (i) ($\|f\| = 0$ iff $f = 0$) is straightforward; axiom (ii) (that $\|\lambda f\| = |\lambda|\|f\|$) is even simpler, but the triangle inequality (iii) requires some work. We will reach this via two other useful inequalities - Young's inequality and Hölder's inequality.

Lemma 4.1. *(Young's inequality). If $a, b \geq 0$, $p, q > 1$ with $p^{-1} + q^{-1} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{4.4}$$

Sometimes we will use "Young's inequality with ϵ",

$$ab \leq \epsilon a^p + \epsilon^{-q/p} b^q, \tag{4.5}$$

which follows from (4.4) applied to $ab = [(\epsilon)^{1/p} a][(\epsilon)^{-1/p} b]$ since $p, q > 1$.

Proof. Consider the function

$$f(t) = \frac{t^p}{p} + \frac{1}{q} - t.$$

Then for $t \geq 0$ this has the minimum value $f(t) = 0$ when $t = 1$. Setting $t = ab^{-q/p}$ it follows that

$$\frac{a^p b^{-q}}{p} + \frac{1}{q} - ab^{-q/p} \geq 0,$$

[1] As vector spaces, $\tilde{L}^p(Q) = C^0(Q)$, but as normed spaces they differ (since they have different norms).

which simplifies to give

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab^{-q/p}b^q = ab.$$

□

When p and q are as in the above lemma we say that they are "conjugate indices" (sometimes it will be convenient to write "(p, q) are conjugate indices").

Lemma 4.2. *(Hölder's inequality). Let (p, q) be conjugate indices with $1 < p < \infty$, and suppose that $f \in \tilde{L}^p(Q)$ and $g \in \tilde{L}^q(Q)$. Then $fg \in \tilde{L}^1(Q)$, with*

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

A more general version of this inequality is given in the problems.

Proof. Consider

$$\int_Q \frac{|f(x)|}{\|f\|_{L^p}} \frac{|g(x)|}{\|g\|_{L^q}} \, \mathrm{d}x,$$

so that Young's inequality (lemma 4.1) gives

$$\int_Q \frac{f(x)}{\|f\|_{L^p}} \frac{g(x)}{\|g\|_{L^q}} \, \mathrm{d}x \leq \int_Q \frac{1}{p} \frac{|f(x)|^p}{\|f\|_{L^p}^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_{L^q}^q} \, \mathrm{d}x = \frac{1}{p} + \frac{1}{q} = 1,$$

as required. □

Using Hölder's inequality, we can now establish the triangle inequality for the $\tilde{L}^p$ spaces, which is important enough that it has its own name.

Lemma 4.3. *(Minkowski's inequality). If $f, g \in \tilde{L}^p(Q)$, $1 \leq p < \infty$, then $f + g \in \tilde{L}^p(Q)$, with*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof. We integrate the inequality

$$|f(x)+g(x)|^p \le |f(x)+g(x)|^{p-1}|f(x)| + |f(x)+g(x)|^{p-1}|g(x)|$$

and use Hölder's inequality on each of the two terms on the right-hand side to obtain

$$\int_Q |f(x)+g(x)|^p \le \Big(\int_Q |f(x)+g(x)|^p\Big)^{1/q} (\|f\|_{L^p} + \|g\|_{L^p}),$$

which yields, on dividing, (since $1-q^{-1}=p^{-1}$)

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

□

So $(\tilde{L}^p, \|\cdot\|_{L^p})$ is a normed space. However, it is not complete, as the following example shows.

The sequence $f_n(x) : [-1,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} -1 & x < -1/n \\ nx & |x| \le 1/n \\ 1 & x > 1/n, \end{cases}$$

converges in the norm (4.3) to the discontinuous step function that is $\mathrm{sgn}(x)$. The mathematical solution is to add to $C^0(Q)$ all functions that can be obtained as the limits of convergent sequences under (4.3). First we need to say a little about the notion of a property which holds "almost everywhere".

4.2.2 Measure and integration

The measure of the cuboid

$$I = \prod_{j=1}^{d} [a_j, b_j]$$

in Q (or $\mathbb{R}^d$) is defined just as one would expect,

$$\mu(A) = \prod_{j=1}^{d} (b_j - a_j).$$

A general subset A of Q is said to have measure $\mu(A)$ if

1. for any countable collection of cuboids I_k such that

$$A \subset \bigcup_k I_k \tag{4.6}$$

   we have $\mu(A) \leq \sum_j \mu(I_j)$, and

2. given any $\epsilon > 0$ there exists a countable collection of cuboids I_k such that (4.6) holds and

$$\sum_j \mu(I_j) < \mu(A) + \epsilon.$$

In particular, since $\mu(I_j) > 0$, a set has *measure zero* if, for any $\epsilon > 0$, there exists a countable collection of cuboids I_k such that (4.6) holds and

$$\sum_j \mu(I_j) < \epsilon.$$

E.g. any countable collection of points $X = \{x_j\}_{j=1}^{\infty}$ has zero measure, since we can find a cuboid I_j of measure $\epsilon 2^{-j}$ which contains x_j; then

$$X \subset \bigcup_{j=1}^{\infty} I_j$$

and

$$\sum_{j=1}^{\infty} \mu(I_j) = \sum_{j=1}^{\infty} \epsilon 2^{-j} = \epsilon.$$

If $A \subset B$, a bounded set in $\mathbb{R}^d$, we say that A has full measure in B if $B \setminus A$ has zero measure

A property which holds at all points in Q except for a set which has measure zero is said to hold *almost everywhere.*

Exercise 4.1. *Suppose that each of the properties P_n, $n \in \mathbb{Z}$, holds almost everywhere in Q. Show that $\cap_{n\in\mathbb{Z}} P_n$ holds almost everywhere in Q (i.e. they hold simultaneously a.e. in Q).*

The Lebesgue integral

$$\int_Q f(x)\,\mathrm{d}x$$

is defined in a similar way, starting with "simple" functions that are piecewise constant on blocks I_j which are disjoint (except for their edges),

$$s(x) = \sum_{j=1}^{n} c_j \chi[I_j]$$

($\chi[A](x)$ is the indicator function of A, 1 if $x \in A$ and zero otherwise). There can be little argument that a sensible definition of the integral of s is

$$\int_Q s(x)\,\mathrm{d}x = \sum_{j=1}^{n} c_j \mu(I_j).$$

Integrals for more general functions f are then defined using a limit process.

As such, it is perhaps little surprise that if $f(x) = g(x)$ almost everywhere, then their integrals are equal,

$$\int_Q f(x)\,\mathrm{d}x = \int_Q g(x)\,\mathrm{d}x.$$

4.2.3 The Lebesgue spaces $L^p(Q)$, $1 \le p < \infty$

A fundamental result from the theory of Lebesgue integration guarantees that if $\{u_n\}$ is a Cauchy sequence in $\tilde{L}^p$ (and in fact in L^p itself) then a subsequence of the u_n converges almost everywhere to a function $u(x)$, and $\|u_n - u\|_{L^p} \to 0$. We now define $L^p(Q)$ as the set of all such functions which can be arrived at as the limit in the L^p norm of Cauchy sequences in $\tilde{L}^p(Q)$. Since $C^0(Q) = \tilde{L}^p(Q)$ as a vector space (they contain the same functions) we make the following definition.

Definition 4.4. *$L^p(Q)$ is the completion of $C^0(Q)$ with respect to the norm (4.3)*[2]*.*

[2]Strictly speaking, elements of $\mathbb{X}$, the completion of a space X with respect to a norm

A corollary of the definition is that $C^0(Q)$ is *dense* in L^p: we say that subset Y of a Banach space X is dense in X if every element of X can be approximated arbitrarily closely by an element of Y, so that for any $\epsilon > 0$ there exists a $y \in Y$ such that $\|x - y\|_X < \epsilon$. In particular it follows that if $x \in X$ we can find a sequence $y_n \in Y$ such that

$$y_n \to x \in X \qquad \text{as} \qquad n \to \infty,$$

i.e. such that $\|y_n - x\|_X \to 0$ as $n \to \infty$.

Since $\|f - g\|_{L^p} = 0$ if $f = g$ almost everywhere, the L^p norm does not satisfy $\|f\| = 0$ iff $f = 0$ (and so is not a *bona fide* norm) on $L^p(Q)$, unless we identify functions which agree almost everywhere. Thus strictly we have to view each element of L^p not as a function, but as an equivalence class of functions which agree almost everywhere. In practice this distinction causes few problems, but we must always bear in mind that "$u = 0$ in $L^p(Q)$" means only that $u = 0$ *almost everywhere* in Q.

We will see later (using Fourier series) that $C_{\rm p}^\infty(Q)$ is dense in $L^2(Q)$ (a result which is in fact valid for all the $L^p(Q)$ spaces with $1 \le p < \infty$).

Exercise 4.2. *Use Hölder's inequality to show that if $r > s$ then $L^r(Q) \subset L^s(Q)$, and*

$$\|f\|_{L^s} \le L^{m(r-s)/rs} \|f\|_{L^r}. \tag{4.7}$$

4.2.4 The Lebesgue space $L^\infty(Q)$

$f \in L^\infty(Q)$ if its essential supremum, given by

$$\|f\|_\infty = \text{ess sup}_Q |f(x)|, \tag{4.8}$$

is finite. The right-hand side of (4.8) denotes the smallest value which bounds f almost everywhere,

$$\text{ess sup}_Q |f(x)| = \inf\{\sup_{x \in S} |f(x)| : \ S \subset Q, \ \text{with } Q \setminus S \text{ of measure zero}\}.$$

$\|\cdot\|$ are equivalence classes of Cauchy sequences in X, where

$$\{x_n\} \simeq \{y_n\} \qquad \text{if} \qquad \|x_n - y_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

You can see that every element x of X has a natural representative in $\mathbb{X}$, namely the constant sequence $x_n = x$, so that by this identification $X \subset \mathbb{X}$.

In particular $|f(x)| \leq \|f\|_\infty$ almost everywhere, and it follows that if $f \in C^0(Q)$ then the essential supremum of f is the same as its supremum (which explains the notation in (4.1)).

E.g. consider the function $f : [0,1] \to \mathbb{R}+$ defined by

$$f(x) = \begin{cases} q & x = p/q \text{ (a rational given in its lowest terms)} \\ x & x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Then $f(x) = x$ almost everywhere in $[0,1]$, and $\|f\|_\infty = 1$.

Hölder's inequality can be extended to cover the case $p = 1$, $q = \infty$ (these indices are clearly still conjugate):

$$\begin{aligned} \left| \int_Q f(x)g(x)\,\mathrm{d}x \right| &\leq \int_Q |f(x)||g(x)|\,\mathrm{d}x \\ &\leq \|f\|_{L^\infty} \int_Q |g(x)|\,\mathrm{d}x \\ &= \|f\|_{L^\infty}\|g\|_{L^1}. \end{aligned}$$

However, note that $C^0(Q)$ is not a dense subset of $L^\infty(Q)$, since $C^0(Q)$ is complete in the L^∞ norm (which is the same as the sup norm).

Exercise 4.3. *Use the standard Hölder inequality together with induction on n to prove the generalised Hölder inequality: if $p_1, \ldots, p_n$ are such that*

$$\sum_{j=1}^{n} \frac{1}{p_j} = 1$$

and $f_j \in L^{p_j}(Q)$ then $f_1 \ldots f_n \in L^1(Q)$, with

$$\int_Q f_1(x) \ldots f_n(x)\,\mathrm{d}x \leq \|f_1\|_{L^{p_1}} \ldots \|f_n\|_{L^{p_n}}. \tag{4.9}$$

4.3 Hilbert Spaces

$\mathbb{R}^d$ not only has a notion of length, but also a notion of the angle between two elements, derived from the dot product $\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{d} x_j y_j$. This is the most familiar example of an inner product.

An inner product on a general vector space X is a map $(\cdot,\cdot) : X\times X \to \mathbb{C}$, such that[3]

(i) $(\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z)$,

(ii) $(y,x) = \overline{(x,y)}$ for all $x, y \in X$, and

(iii) $(x,\bar{x}) \geq 0$ for all $x \in X$, with equality iff $x = 0$.

Note that there is a natural norm associated with the inner product,

$$\|x\| = (x,\bar{x})^{1/2}. \tag{4.10}$$

The Cauchy-Schwarz inequality,

$$|(x,y)| \leq \|x\|\|y\|,$$

follows almost immediately from the axioms, and provides the triangle inequality for the norm defined by (4.10).

Exercise 4.4. *(i) By considering $\|x+\lambda y\|^2$ show that the Cauchy-Schwarz inequality holds, and deduce the triangle inequality.*

(ii) Show that the parallelogram law

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

holds.

A complete inner product space is called a *Hilbert space.*

The canonical example of an infinite-dimensional Hilbert space is l^2, the space of square summable real sequences,

$$l^2 = \Big\{\mathbf{x} = \{x_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\Big\}.$$

[3]We will always be considering spaces of real functions, so the inner product will be a map into $\mathbb{R}$ rather than into $\mathbb{C}$. However, we will be using a complex Fourier basis for $L^2(Q)$ (complex exponentials which combine the sine and cosine functions), and inner products of individual terms in the series may be complex.

An element $\mathbf{x}$ of l^2 is essentially an infinite-dimensional "vector", and indeed the inner product and norm are just like those on $\mathbb{R}^d$,

$$(\mathbf{x},\mathbf{y})_{l^2} = \sum_{n=1}^{\infty} x_n \overline{y_n} \qquad \|\mathbf{x}\|_{l^2} = \Big(\sum_{n=1}^{\infty} |x_n|^2\Big)^{1/2}.$$

Exercise 4.5. *Prove that l^2 is complete, i.e. that any Cauchy sequence $x^{(n)}$ of elements in l^2 converges to some element $x \in l^2$. That $x^{(n)}$ is a Cauchy sequence in l^2 means that given $\epsilon > 0$ there exists an N such that for all $n, m \geq N$,*

$$\|x^{(n)} - x^{(m)}\|_{l^p} \leq \epsilon,$$

i.e.

$$\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^2 \leq \epsilon^2.$$

Deduce that for each fixed index i, $x_i^{(n)} \to x_i$ for some x_i. Now show that $x \in L^2$, and that $\|x^{(n)} - x\|_{l^2} \to 0$ as $n \to \infty$.

The example that we will make most use of, however, is the Lebesgue space $L^2(Q)$, for which the inner product is given by the integral

$$(f,g)_{L^2} = \int_Q f(x)\overline{g}(x)\,\mathrm{d}x.$$

Note that, for $p = q = 2$, Hölder's inequality (lemma 4.2) becomes the Cauchy-Schwarz inequality in L^2,

$$|(f,g)_{L^2}| \leq \|f\|_{L^2}\|g\|_{L^2}.$$

We will write (f,g) and $|f|$ for $(f,g)_{L^2}$ and $\|f\|_{L^2}$ in what follows, only being specific when care is needed to distinguish this norm from the modulus. Context should usually make it clear which is meant, since generally we will write $|f|$ for the L^2 norm of the function f, and $|f(x)|$ for the modulus of $f(x)$.

4.3.1 Bases in Hilbert spaces

In $\mathbb{R}^d$ we can write any vector in terms of a finite set of basis elements, for example the coordinate basis

$$\{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,\ldots,0,1)\}.$$

We now want to investigate when we can find a countable basis for an infinite-dimensional Hilbert space, in terms of which to expand any point in H. A set of vectors is *orthogonal* (in H) if

$$(e_i, e_j) = 0 \qquad i \neq j,$$

and *orthonormal* (in H) if

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Definition 4.5. *An orthonormal set* $\{e_j\}$ *is a* basis *for* H *if*

$$u = \sum_{j=1}^{\infty} (u, e_j) e_j \tag{4.11}$$

for every $u \in H$.

In particular, this means that we can find the norm of u in H using the coefficients in the expansion:

Lemma 4.6. *Let* $\{e_j\}$ *be an orthonormal basis for* H*. Then*

$$\|u\|^2 = \sum_{j=1}^{\infty} |(u, e_j)|^2 \qquad \text{for all} \qquad u \in H. \tag{4.12}$$

Proof. If $\{e_j\}$ form a basis then for any finite n

$$\left\| \sum_{j=1}^{n} (u, e_j) e_j \right\|^2 = \sum_{j=1}^{n} |(u, e_j)|^2,$$

and (4.12) follows by taking limits, since the sum (4.11) converges in H. □

4.3.2 A Fourier basis for $L^2(Q)$

Fundamental for the analysis that is to come is the following simple result from the theory of Fourier series. For a proof see Renardy & Rogers.

Theorem 4.1. *The complex exponentials*

$$L^{-d/2} \mathbf{e}_j \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \qquad \mathbf{k} \in \mathbb{Z}^d_L, \ j = 1, \ldots, d$$

form an orthonormal basis for $L^2(Q)$.

4.3.3 The orthogonal projection onto a linear subspace

If M is a subset of H, then the *orthogonal complement* of H, $M^\perp$, is given by

$$M^\perp = \{u \in H : \ (u, v) = 0 \text{ for all } v \in M\}.$$

When M is a closed linear subspace of H we can decompose any vector uniquely into an element of M plus an element of its orthogonal complement.

Proposition 4.7. *If M is a closed linear subspace of H, then every $x \in H$ has a unique decomposition as*

$$x = u + v \qquad u \in M, \ v \in M^\perp.$$

For the proof see [R], Proposition 1.21. The idea is to choose u to be the point in M that minimises the distance between x and u.

One can use this proposition to define the *orthogonal projection of x onto M*, P_M, by

$$P_M x = u.$$

Clearly $P_M^2 = P_M$, and it follows from the definition of u that

$$\|x\|^2 = \|u\|^2 + \|x - u\|^2,$$

thus ensuring that

$$\|P_M x\| \leq \|x\|,$$

i.e. the projection can only decrease the norm.

4.3.4 Non-compactness of the unit ball

Since l^2 shares many properties in common with $\mathbb{R}^d$, it is tempting to think of it as "$\mathbb{R}^\infty$". However, there is one very large caveat. Unlike $\mathbb{R}^d$, closed bounded sets are not compact. Recall that in a metric space compactness is equivalent to sequential compactness: X is (sequentially) compact if every sequence $\{x_n\}_{n=1}^\infty$ in X has a convergent subsequence.

Proposition 4.8. *The unit ball in an infinite-dimensional Hilbert space is not compact.*

The same result is true, with a harder proof, for general infinite-dimensional Banach spaces.

Proof. We give the proof in a Hilbert space which admits a countable (orthonormal) basis. For any sequence $\{w_n\}_{n=1}^{\infty}$ of orthonormal elements we have

$$\|w_n - w_m\|^2 = 2$$

if $n \neq m$. It follows that no subsequence of $\{w_n\}$ is Cauchy and thus the unit ball cannot be compact. $\square$

The compactness of bounded sets in $\mathbb{R}^d$ is extremely useful, and we lose this in infinite-dimensional spaces. However, there is a weaker property, "weak compactness", which we can often use instead. We introduce this in the next chapter.

Chapter 5

Linear Functionals & Dual Spaces

Linear maps from a normed space X into $\mathbb{R}$ are called linear functionals on X. In this chapter we study the space of bounded linear functionals on X, the *dual space* of X. This essentially a watered-down version of Chapter 4 of [R].

A linear functional on X is bounded if

$$|f(x)| \leq k\|x\|_X \qquad \text{for all} \qquad x \in X, \tag{5.1}$$

and the norm of f, $\|f\|_{X^*}$, is the smallest value of k for which (5.1) holds.

Proposition 5.1. *Let $f : X \to \mathbb{R}$ be a linear functional. Then f is continuous iff it is bounded.*

Proof. If f is bounded then

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq \|f\|_{X^*}\|x_n - x\|_X,$$

which gives continuity. If f is continuous but unbounded then for every n there exists a y_n such that $|f(y_n)| > n^2\|y_n\|_X$. Then

$$x_n = y_n/(n\|y_n\|_X) \to 0$$

but $|f(x_n)| > n$, so that f is not continuous at the origin, a contradiction. Thus continuity implies boundedness. □

5.1 Dual Spaces

Definition 5.2. *Let X be a real normed space. The space of all linear functionals on X, is denoted X^* and is called the* dual space *of X.*

Theorem 5.3. *X^* is a Banach space.*

Exercise 5.1. *Prove Theorem 5.2:*

(i) *Write down exactly what "$\{A_n\}$ is a Cauchy sequence in X^*" means*

(ii) *Show that for every fixed $x \in X$ the sequence $\{A_n x\}$ is a Cauchy sequence in $\mathbb{R}$.*

(iii) *Define the operator A by*

$$Ax = \lim_{n\to\infty} A_n x.$$

Show that A is linear.

(iv) *By taking appropriate limits in what you wrote for (i), show that $A \in X^*$ and that $A_n \to A$ in X^*.*

When no confusion can arise, the simpler notation $\|f\|_*$ will sometimes be used for $\|f\|_{X^*}$.

Theorem 5.4. *(Hahn-Banach Theorem). Let X be a Banach space and M a linear subspace of X. Suppose that f is a linear functional on M, such that*

$$|f(x)| \leq k\|x\| \qquad \text{for all } x \in M. \tag{5.2}$$

Then there exists an extension F of f to all of X, such that

$$|F(x)| \leq k\|x\| \qquad \text{for all } x \in X. \tag{5.3}$$

We give the proof only for the case when X is a Hilbert space. The Banach space proof is much more involved; it can be found in [R] (Theorem 4.3).

Proof. For $u \in X$ write $u = m + d$, where $m \in M$ and $d \in M^\perp$, and define

$$F(u) = f(m).$$

Then clearly F extends f, and

$$|F(u)| = |f(m)| \leq k\|m\| \leq k\|u\|,$$

since

$$\|u\|^2 = \|m\|^2 + \|d\|^2.$$

□

As an indication of why studying dual spaces can be expected to yield interesting results, we apply the Hahn-Banach theorem to show that if $f(x) = f(y)$ for all $f \in X^*$ then $x = y$.

Lemma 5.5. *Let X be a Banach space. If $x, y \in X$ and $f(x) = f(y)$ for every $f \in X^*$ then $x = y$.*

Proof. If $x = y = 0$ then we are done, so without loss of generality assume that $x \neq 0$. Let Z be the linear space spanned by x and y. If x and y are linearly dependent then set

$$f(\alpha x) = \alpha\|x\|; \tag{5.4}$$

otherwise set

$$f(\alpha x + \beta y) = \alpha\|x\| \qquad \text{for all} \quad \beta \in \mathbb{R}. \tag{5.5}$$

In both cases $f(x) \neq f(y)$. Now extend f to an element $F \in X^*$ using the Hahn-Banach theorem, and clearly $F(x) \neq F(y)$. □

The proof of this lemma included the additional factor of $\|x\|$ in the definitions (5.4) and (5.5) so as to provide the following immediate corollary (f in the statement below is just F from the proof of Lemma 5.5).

Corollary 5.6. *Let $x \in X$, a non-trivial Banach space. Then there exists an $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\|_{X^*} = 1$.*

Exercise 5.2. *Show that if Y is a proper linear subspace of a Banach space X then there exists a non-zero linear functional in X^* which vanishes on Y.*

5.2 Examples of dual spaces

Before we turn to how dual spaces can be used in analysis, we will first give several examples.

5.2.1 The dual space of $\mathbb{R}^d$

We now characterise the dual space of $\mathbb{R}^d$. Standard linear algebra shows that any linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$ can be expressed as a $1 \times m$ matrix, i.e. for every $L \in (\mathbb{R}^d)^*$, there exists an $l \in \mathbb{R}^d$ such that

$$Lx = \begin{pmatrix} l_1 & l_2 & \dots & l_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = (l, x).$$

We now show that the norm of this element of $\mathbb{R}^d$ is the same as the norm of L in $(\mathbb{R}^d)^*$.

First, observe that

$$|Lx| = |(l, x)| \leq |l||x|$$

using the standard Cauchy-Schwarz inequality in $\mathbb{R}^d$. It follows from the definition of $\|L\|_{(\mathbb{R}^d)^*}$ that

$$\|L\|_{(\mathbb{R}^d)^*} \leq |l|.$$

Now, if we set $x = l$ then in fact we have

$$|Ll| = |l|^2,$$

and so no constant $k < |l|$ will work in

$$|Ll| \leq k|l|.$$

Thus

$$\|L\|_{(\mathbb{R}^d)^*} = |l|.$$

Since it is clear that any $l \in \mathbb{R}^d$ gives rise to an element of $(\mathbb{R}^d)^*$ by defining

$$Lx = (l, x)$$

we have shown that $(\mathbb{R}^d)^*$ is isometrically isomorphic to $\mathbb{R}^d$: i.e. they are isomorphic, and this isomorphism is actually an isometry, i.e. it preserves the norm. We write $(\mathbb{R}^d)^* \simeq \mathbb{R}^d$ (and you will often see "$\mathbb{R}^d$ is the dual of $\mathbb{R}^d$).

5.2.2 The dual space of L^p, $1 < p < \infty$

We first consider the Lebesgue L^p spaces, with $1 < p < \infty$, and observe that if $f \in L^q(Q)$, with (p, q) conjugate indices, we can define a linear functional L_f on L^p via

$$L_f(g) = \int_Q f(x)g(x)\,\mathrm{d}x, \tag{5.6}$$

since Hölder's inequality gives

$$|L_f(g)| \le \|f\|_{L^q}\|g\|_{L^p}.$$

Certainly, then, $\|L_f\|_{(L^p)^*} \le \|f\|_{L^q}$, and if we consider

$$g(x) = |f(x)|^{q-2}f(x)$$

then we have

$$\|g\|_{L^p} = \left(\int_Q |f(x)|^{(q-1)p}\,\mathrm{d}x\right)^{1/p} = \left(\int_Q |f(x)|^q\,\mathrm{d}x\right)^{1/p} = \|f\|_{L^q}^{q/p},$$

and

$$|L_f(g)| = \left|\int_Q |f(x)|^q\,\mathrm{d}x\right| = \|f\|_{L^q}^q.$$

In this case, therefore,

$$|L_f(g)| = \|f\|_{L^q}\|g\|_{L^p},$$

showing that in fact

$$\|L_f\|_{(L^p)^*} = \|f\|_{L^q}.$$

This shows that the map $f \mapsto L_f$ is an isometry from L^q *into* $(L^p)^*$.

In fact this map is onto, i.e. every element of $(L^p)^*$ can be realised as L_f for some $f \in L^q$. Thus L^q and $(L^p)^*$ are isometrically isomorphic, which we will write as $L^q \simeq (L^p)^*$. Since it is therefore natural to identify $(L^p)^*$ with L^q via (5.6), you will often see $L^q = (L^p)^*$ (a convenient abuse of notation).

5.2.3 The dual spaces of L^1 and L^∞

We are still left with the spaces $L^1(Q)$ and $L^\infty(Q)$. It is easy to see that every element of L^∞ can be used to define an element of $(L^1)^*$ via (5.6), and one can show that

$$\|L_f\|_{(L^1)^*} = \|f\|_{L^\infty}.$$

In this case, as with the L^p spaces above, every element of the dual space of $L^1(Q)$ can be obtained, via (5.6), from an element of $L^\infty(Q)$. (Since $L^\infty \simeq L^1$, one way of defining L^∞ is as the dual space of L^1.)

However, $L^1(Q)$ is *not* the whole dual space of $L^\infty(Q)$, which is much more complicated to characterise (see Yosida (1980) for example).

Exercise 5.3. *(i) For $f \in L^\infty(Q)$ define L_f by*

$$L_f(g) = \int_Q f(x)g(x)\,\mathrm{d}x.$$

Show using Hölder's inequality that

$$\|L_f\|_{(L^1)^*} \leq \|f\|_{L^\infty}.$$

(ii) Set $g(x) = |f(x)|^{p-2}f(x)$: by finding $\|g\|_{L^1}$ and $|L_f(g)|$ deduce that

$$\|L_f\|_{(L^1)^*} \geq \frac{\|f\|_{L^p}^p}{\|f\|_{L^{p-1}}^{p-1}}.$$

(iii) Now use the result that

$$\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}$$

to deduce that

$$\|L_f\|_{(L^1)^*} \geq \|f\|_{L^\infty},$$

and hence that $\|L_f\|_{(L^1)^*} = \|f\|_{L^\infty}$.

5.3 Dual spaces of Hilbert spaces

Since $(L^p)^* \simeq L^q$, if $p = q = 2$ this shows that the dual of L^2 is identifiable with L^2 itself. This is a particular case of a more general result valid for any Hilbert space, the Riesz representation theorem. This shows that any linear functional l on H can be represented as an inner product with some appropriate element x_l of H itself (5.7), and that the norm of l (in H^*) and of x_l (in H) are the same.

Theorem 5.7. *(Riesz Representation Theorem). For any Hilbert space H, $H^* \simeq H$. In particular, for every $x \in H$*

$$l_x(y) \equiv (x, y) \tag{5.7}$$

is bounded and has norm $\|l_x\|_{H^*} = \|x\|$. Furthermore, for every bounded linear functional $l \in H^*$ there exists a unique $x_l \in H$ such that

$$l(y) = (x_l, y) \qquad \text{for all } y \in H,$$

and $\|x_l\|_H = \|l\|_{H^*}$. It follows that $l \mapsto x_l$ is continuous.

Proof. The first part of the theorem is straightforward, since (5.7) provides a linear functional l_x associated with x. The Cauchy Schwarz inequality gives

$$|l_x(y)| = |(x, y)| \leq \|x\|\|y\|$$

so that $l_x \in H^*$ with $\|l_x\|_{H^*} \leq \|x\|$, and the choice $y = x$ shows that in fact $\|l_x\|_{H^*} = \|x\|$.

The second part of the theorem involves a little more work. Suppose that $l \in H^*$. Then the kernel of l, $K = \{y \in H : l(y) = 0\}$, is a closed subspace of H. The subspace $K^\perp$ of vectors orthogonal to K is a one-dimensional subspace of H, since if $u, v \in K^\perp$ we have

$$l[l(u)v - l(v)u] = 0,$$

and so $l(u)v - l(v)u \in K$. Since u and v are orthogonal to all elements of K so is $l(u)v - l(v)u$, which implies that $l(u)v - l(v)u = 0$, and so u and v are proportional.

Now choose a unit vector z in $K^\perp$. We can decompose every $y \in H$ as $y = (z, y)z + w$, where $w \in K$. Then $l(y) = (z, y)l(z)$, so if we set $x_l = l(z)z$, we have

$$(x_l, y) = (l(z)z, y) = l(y).$$

It is immediate that $\|l\|_{H^*} \leq \|x_l\|$, and if we take $y = x_l$ then $\|x_l\|^2 \leq \|l\|_{H^*}\|x_l\|$, giving equality of the norms. Uniqueness follows since if

$$(x, y) = l(y) \qquad \text{and} \qquad (\tilde{x}, y) = l(y) \qquad \text{for all} \qquad y \in H$$

then $(x - \tilde{x}, y) = 0$ for all $y \in H$, which implies that $x = \tilde{x}$. □

5.4 Reflexive spaces

If H is a Hilbert space the Riesz Theorem shows that H^* is isometrically isomorphic to H. The situation in a general Banach space X is somewhat more complicated. For an element $x \in X$ one can define a linear functional G_x on X^* (i.e. an element of $X^{**} \equiv (X^*)^*$) by

$$G_x(f) = f(x) \qquad \text{for all} \qquad f \in X^*. \tag{5.8}$$

Noting that

$$|G_x(f)| \leq \|f\|_{X^*}\|x\|,$$

one has

$$\|G_x\|_{X^{**}} \leq \|x\|.$$

We saw in corollary 5.6 that given an $x \in X$ there is an $f \in X^*$ with $\|f\|_{X*} = 1$ and $f(x) = |x|$; this shows that in fact

$$\|G_x\|_{X^{**}} = \|x\|, \tag{5.9}$$

and so A is an isometry from X onto a subspace of X^{**}. When this isometry is onto, i.e. when $X \simeq X^{**}$, then X is called *reflexive*.

The Riesz Representation Theorem shows that all Hilbert spaces are reflexive; the Lebesgue spaces $L^p(Q)$, with $1 < p < \infty$ are also reflexive. However, the Lebesgue spaces $L^1(Q)$ and $L^\infty(Q)$ are not reflexive: $(L^\infty)^* \not\simeq L^1$.

5.5 Notions of Weak Convergence

As the final topic of this chapter we introduce the extremely powerful notion of weak convergence, which will enable us – to some extent – to circumvent the problem that the unit ball is not compact in an infinite-dimensional space.

5.5.1 Weak convergence

Since we know from Lemma 5.5 that linear functionals can distinguish elements of X, we will define a notion of convergence based on the application of linear functionals.

Definition 5.8. *Let X be a Banach space. A sequence $x_n \in X$ converges weakly to x (in X)*

$$x_n \rightharpoonup x \qquad \text{in} \qquad X,$$

if $f(x_n) \to f(x)$ for every $f \in X^*$.

First we will motivate the terminology. Another terminology for the standard convergence in the norm of the space ($x_n \to x$) is strong convergence of x_n to x – we will show that this "strong" convergence implies weak convergence, but not vice versa.

Lemma 5.9. *If $x_n \to x$ (strong convergence) then $x_n \rightharpoonup x$ (weak convergence).*

For some more properties of weak limits, see the problems.

Proof. Every element of X^* is a bounded linear functional and hence continuous, thus $f(x_n) \to f(x)$ for every $f \in X^*$, which is precisely $x_n \rightharpoonup x$. □

However, there are simple examples of weakly convergent subsequences which do not converge strongly. For example, let $\{e_j\}$ be an orthonormal basis in a separable Hilbert space. Then we know that the $\{e_j\}$ have no convergent subsequence (Proposition 4.8), but we can show that $e_j \rightharpoonup 0$. Indeed, using the Riesz Representation Theorem (Theorem 5.7) every element $l \in H^*$ has a representation as $(x_l, \cdot)$ for some $x_l \in H$, and so it suffices to consider sequences (x_l, e_j) for each $x_l \in H$. Since $\{e_j\}$ is an orthonormal basis,

$$\|x_l\|^2 = \sum_{j=1}^{\infty} |(x_l, e_j)|^2,$$

and so $|(x_l, e_j)| \to 0$ as $j \to \infty$. But this is exactly $l(e_j) \to 0$ for every $l \in H^*$, and so $e_j \rightharpoonup 0$.

For another example consider the space of continuous functions $C^0(Q)$. Then if $g_n \rightharpoonup f$ in $C^0(Q)$ it must converge pointwise to g, since the map

$$\delta_x : g \mapsto g(x)$$

is a bounded linear functional for each $x \in Q$, and so we must have

$$\delta_x(g_n) \to \delta_x(g) \qquad \text{i.e.} \qquad g_n(x) \to g(x).$$

[The linear functional here is δ_x and it acts on a $g \in C^0(Q)$. Do not confuse $f(x)$ which we use above to mean the result of applying $f \in X^*$ to $x \in X$ with $g(x)$ here, which is the value of the function g at x.]

The following fundamental result is a consequence of lemma 5.5, and so ultimately of the Hahn-Banach theorem.

Proposition 5.10. *Weak limits are unique.*

Proof. If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$ then from the definition we must have $f(x) = f(y)$ for all $f \in X^*$. Lemma 5.5 shows that this implies that $x = y$. $\square$

The example of the orthonormal basis $\{e_j\}$ with $|e_j| = 1$ for all j but $e_j \rightharpoonup 0$ shows that taking weak limits can decrease the norm. However, the following result shows that it can never increase it.

Lemma 5.11. *If $x_n \rightharpoonup x$ in X then*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|. \tag{5.10}$$

Proof. If $x_n \rightharpoonup x$, choose $f \in X^*$ such that $f(x) = \|x\|$ and $\|f\|_* = 1$ (see corollary 5.6). Then

$$\|x\| = \|f(x)\| = \lim_{n\to\infty} \|f(x_n)\| \le \liminf_{n\to\infty} \|f\|_* \|x_n\| = \liminf_{n\to\infty} \|x_n\|,$$

and (5.10) follows. $\square$

Exercise 5.4. *Suppose that M is a linear subspace of a Banach space X, and that $\{x_n\}$ is a sequence of elements of M which converges weakly to x in X.*

(i) *Show that if $f(x) = 0$ for every $f \in X^*$ which is zero on M then $x \in M$.*

(ii) *Deduce that* $x \in M$.

(iii) *Deduce further that there exist coefficients* $\{c_j\}$ *such that*

$$x = \sum_{j=1}^{\infty} c_j x_j.$$

Exercise 5.5. *Let* H *be a Hilbert space. Show that if* $x_n \rightharpoonup x$ *in* H*, and* $\|x_n\| \to \|x\|$ *then* $x_n \to x$ *in* H.

5.5.2 Weak star convergence

We could also apply the definition of weak convergence to elements of X^*. By definition, $f_n \rightharpoonup f$ in X^* provided that

$$G(f_n) \to G(f)$$

for every element $G \in X^{**}$. However, a more useful type of weak convergence for elements of X^* is weak-* convergence, which treats their effect on elements of X.

Definition 5.12. *A sequence* $f_n \in X^*$ *converges weakly-* to* f*, written*

$$f_n \overset{*}{\rightharpoonup} f,$$

if $f_n(x) \to f(x)$ *for every* $x \in X$.

We now prove that weak-∗ limits are unique. One of the problems discusses the relationship between weak and weak-* convergence.

Lemma 5.13. *Weak-* limits are unique,*

Proof. Uniqueness for weak-* limits follows immediately from the definition, for if $f, g \in X^*$ with $f(x) = g(x)$ for all $x \in X$ then $f = g$. □

Exercise 5.6. *(i) Show that* $f_n \rightharpoonup f$ *in* X^* *implies that* $f_n \overset{*}{\rightharpoonup} f$ *in* X^*. *[Recall that for each* $x \in X$ *you can define an element* $G_x \in X^{**}$ *by*

$$G_x(f) = f(x)$$

for all $f \in X^*$.*]*

(ii) Show that $f_n \overset{*}{\rightharpoonup} f$ in X^* implies that $f_n \rightharpoonup f$ if X is reflexive. [Recall that if X is reflexive then any element $G \in X^{**}$ can be written as

$$G(f) = f(x) \qquad \text{for all} \qquad f \in X^*$$

for some $x \in X$.]

5.6 The Alaoglu compactness theorem

The real power of the theory of dual spaces and weak convergence comes from the following compactness theorem. Recall that the unit ball in an infinite-dimensional space is not compact (Proposition 1.25). However, the following result and its corollary show that these balls are weakly compact – in particular, we will see that a bounded sequence in a reflexive Banach space has a *weakly* convergent subsequence.

A space is *separable* if it contains a countable dense subset. (The L^p spaces with $1 \leq p < \infty$ are separable, as are the spaces $C^r(Q)$.)

Theorem 5.14. *(Alaoglu weak-* compactness). Let X be a separable Banach space and let f_n be a bounded sequence in X^*. Then f_n has a weakly-* convergent subsequence, i.e. a subsequence f_{n_j} such that*

$$f_{n_j}(x) \quad \textit{converges for each} \quad x \in X.$$

See [R] (Theorem 4.18) for a proof. We give a direct proof of the corollary on weak convergence for Hilbert spaces below.

As an example, $L^\infty \simeq (L^1)^*$. So a bounded sequence $f_n \in L^\infty(Q)$ has a weakly-* convergent subsequence f_{n_j}: there exists an $f \in L^\infty(Q)$ such that

$$\int_Q f_{n_j}(x) g(x) \, \mathrm{d}x \to \int_Q f(x) g(x) \, \mathrm{d}x$$

for each $g \in L^1(Q)$.

The following corollary of Theorem 5.14 will be extremely useful.

Theorem 5.1. *(Reflexive weak compactness). Let X be a reflexive Banach space and x_n a bounded sequence in X. Then x_n has a subsequence which converges weakly in X.*

Exercise 5.7. *Prove the above theorem as a corollary of of Theorem 5.14 under the assumption that X^* is separable.*

We will give a direct proof of Theorem 5.1 when $X = H$ is a separable Hilbert space (recall that all Hilbert spaces are reflexive), since it contains at least elements of the idea behind the Alaoglu Theorem and its corollary.

Proof. Take a bounded sequence $\{x_n\} \in H$, and assume that $|x_n| \leq M$. Let $\{e_j\}_{j=1}^{\infty}$ denote a countable basis for H. Since H is a Hilbert space, any linear functional $f \in H^*$ has a representative $u_f \in H$ such that

$$\langle f, x \rangle = (u_f, x),$$

so in order to show that $x_{n_j} \rightharpoonup x$ it is enough to show that

$$(u, x_{n_j}) \to (u, x) \qquad \text{for all} \qquad u \in H.$$

Now consider (x_n, e_1): since

$$|(x_n, e_1)| \leq M$$

this is a bounded sequence in $\mathbb{R}$ and so has a convergent subsequence, $x_{n_{1,j}}$. Now consider $(x_{n_{1,j}}, e_2)$. Again, this is bounded and so has a convergent subsequence $x_{n_{2,j}}$ such that $(x_{n_{2,j}}, e_2)$ converges, as well as $(x_{n_{2,j}}, e_3)$. Continuing in this way we obtain subsequences $x_{n_{k,j}}$ such that

$$(x_{n_{k,j}}, e_s) \qquad \text{converges for all} \quad s = 1, 2, \ldots, k.$$

Finally, set $n_j = n_{j,j}$, so that (x_{n_j}, e_k) converges for every k.

Now set

$$x_k = \lim_{j \to \infty} (x_{n_j}, e_k).$$

We want to define

$$x = \sum_{k=1}^{\infty} x_k e_k,$$

for then

$$(x, e_k) = \lim_{j \to \infty} (x_{n_j}, e_k)$$

by definition. First we have to check that $\sum |x_k|^2 < \infty$ so that this definition of x makes sense. But this is easy: for each $N < \infty$,

$$\sum_{k=1}^{N} (x_{n_j}, e_k)^2 \leq |x_{n_j}|^2 \leq M^2,$$

and letting $j \to \infty$ gives

$$\sum_{k=1}^{N} |x_k|^2 \leq M^2.$$

Now we can let $N \to \infty$ to deduce that $\sum |x_k|^2 \leq M^2$, and so $|x| \leq M$.

We now deduce that (x_{n_j}, u) converges for any $u \in H$. If

$$u = \sum_{k=1}^{\infty} c_k e_k$$

then choose N large enough that

$$\left\| u - \sum_{j=1}^{n} c_j e_j \right\| < \epsilon/4M$$

for all $n \geq N$. It follows that

$$\begin{aligned} |(x_{n_j}, u) - (x, u)| &\leq |(x_{n_j} - x, \sum_{j=1}^{n} c_j e_j)| + |(x_{n_j} - x, u - \sum_{j=1}^{n} c_j e_j| \\ &\leq \sum_{j=1}^{n} c_j |(x_{n_j} - x, e_j)| + \epsilon/2, \end{aligned}$$

and since $(x_{n_j}, e_k) \to (x, e_k)$ for each k, it follows that for j large enough

$$|(x_{n_j}, u) - (x, u)| < \epsilon.$$

□

Exercise 5.8. *Show that a Hilbert space is separable iff it has a countable basis. [Hint: use the Gram-Schmidt process to construct an orthonormal basis from the countable dense subset.]*

Chapter 6

Sobolev spaces

Sobolev spaces are a family of spaces of functions akin to $C^r(Q)$, in that they consist of functions with conditions on their derivatives. Rather than requiring the derivatives to be in $C^0(Q)$, we require them to be elements of $L^2(Q)$. This means that the Sobolev spaces are all Hilbert spaces, which makes them much easier to use in our analysis.

6.1 The weak derivative

We first introduce the "weak derivative", which has to be an element of $L^1(Q)$. Suppose that u *is* differentiable ($u \in C^1_{\mathrm{p}}(Q)$). Then for any "test function" $\phi \in C^\infty_{\mathrm{p}}(Q)$, the identity

$$\int_Q \frac{\partial u}{\partial x_j}\,\phi\,\mathrm{d}x = -\int_Q u\,\frac{\partial \phi}{\partial x_j}\,\mathrm{d}x, \tag{6.1}$$

holds, after an integration by parts in the x_j variable (the boundary terms vanish since the integrand is periodic). Repeating this process $|\alpha|$ times, we have similarly

$$\int_Q D^\alpha u\,\phi\,\mathrm{d}x = (-1)^{|\alpha|}\int_Q u\,D^\alpha\phi\,\mathrm{d}x, \tag{6.2}$$

for any multi-index α.

The weak derivative is defined by analogy with (6.1): for a function $u \in L^1(Q)$, we say that v is the *weak derivative* of u with respect to x_j, written $v = D_j u$, if $v \in L^1(Q)$ and

$$\int_Q v\,\phi\,\mathrm{d}x = -\int_Q u\,\frac{\partial\phi}{\partial x_j}\,\mathrm{d}x \tag{6.3}$$

for all $\phi \in C_{\mathrm{p}}^{\infty}(Q)$. The requirement that u and v are elements of $L^1(Q)$ is natural, since $u, v \in L^1(Q)$ ensures that both integrals make sense.

We can also define, inductively, higher order weak derivatives: if $u, v \in L^1(Q)$ then v is the αth weak derivative of u, $v = D^{\alpha}u$, if

$$\int_Q v\,\phi\,\mathrm{d}x = (-1)^{|\alpha|}\int_Q u\,D^{\alpha}\phi\,\mathrm{d}x.$$

As an example, consider the $L^1(-2, 2)$ function

$$u(x) = \begin{cases} -x/2 & -2 < x \le 0 \\ x & 0 < x \le 1 \\ 1 & 1 < x < 2. \end{cases} \tag{6.4}$$

For any $\phi \in C_{\mathrm{p}}^{\infty}(-2, 2)$ we have

$$\begin{aligned} -\int_{-2}^{2} u\,\phi'\,\mathrm{d}x &= \int_{-2}^{0} \tfrac{1}{2}x\phi'\,\mathrm{d}x - \int_{0}^{1} x\phi'\,\mathrm{d}x - \int_{1}^{2} \phi'\,\mathrm{d}x \\ &= \Big[\tfrac{1}{2}x\phi\Big]_{-2}^{0} - \int_{-2}^{0} \tfrac{1}{2}\phi\,\mathrm{d}x - \Big[x\phi\Big]_{0}^{1} + \int_{0}^{1} \phi\,dx - \phi(2) + \phi(1) \\ &= -\tfrac{1}{2}\int_{-2}^{0} \phi\,\mathrm{d}x + \int_{0}^{1} \phi\,\mathrm{d}x, \end{aligned}$$

(since $\phi(2) = \phi(-2)$) and so u has weak derivative

$$v(x) = \begin{cases} \frac{1}{2} & -2 < x < 0 \\ 1 & 0 < x \le 1 \\ 0 & 1 < x < 2. \end{cases} \tag{6.5}$$

6.2 Sobolev spaces of periodic functions

6.2.1 $H^s(Q)$

Roughly, the Sobolev space $H^s_{\mathrm{p}}(Q)$ consists of periodic functions whose generalised derivatives up to (and including) order s lie in $L^2(Q)$ (the "roughly" is because it is not clear how to characterise "periodic functions" when they may not have point values). The norm is given by the sum of the L^2 norms of all these derivatives,

$$\|u\|_{H^s} = \Big(\sum_{|\alpha|\le s} |D^\alpha u|^2\Big)^{1/2}.$$

To give a proper definition we proceed by analogy with our definition of $L^2(Q)$ spaces, and define $H^s_{\mathrm{p}}(Q)$ as the completion of $C^\infty_{\mathrm{p}}(Q)$ in the H^s norm. Since we will always deal with spaces of periodic functions from now on, we will drop the p suffix on Sobolev spaces.

Definition 6.1. *$H^s(Q)$ is the completion of $C^\infty_{\mathrm{p}}(Q)$ in the H^s norm.*

$H^s(Q)$ is a Hilbert space when equipped with the inner product

$$((u,v))_{H^s} = \sum_{|\alpha|\le s} (D^\alpha u, D^\alpha v).$$

For a more useful characterisation we enforce the periodicity of the function u by writing it as a formal Fourier series,

$$u = \sum_{\mathbf{k}\in\mathbb{Z}^d_L} \hat{u}_k \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \qquad \text{with} \qquad \hat{u}_{-\mathbf{k}} = \overline{\hat{u}_{\mathbf{k}}}.$$

For any function $u \in C^\infty_{\mathrm{p}}(Q)$, this Fourier series is uniformly convergent, and the derivatives of u are given by

$$D^\alpha u(x) = \mathrm{i}^{|\alpha|} \sum_{\mathbf{k}} \hat{u}_{\mathbf{k}} \mathbf{k}^\alpha \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

It follows that

$$|D^\alpha u|^2 = L^d \sum_{\mathbf{k}\in\mathbb{Z}_L^d} |\hat{u}_{\mathbf{k}}|^2 \mathbf{k}^{2\alpha}. \tag{6.6}$$

Proposition 6.2. *There exist constants $C_1(s,d)$ and $C_2(s,d)$ such that*

$$C_1\|u\|_{H^s} \leq \Big(\sum_{k\in\mathbb{Z}_L^d}(1+|\mathbf{k}|^{2s})|\hat{u}_{\mathbf{k}}|^2\Big)^{1/2} \leq C_2\|u\|_{H^s}$$

for all $u \in H^s(Q)$, i.e. the H^s norm and

$$\Big(\sum_{k\in\mathbb{Z}_L^d}(1+|\mathbf{k}|^{2s})|\hat{u}_{\mathbf{k}}|^2\Big)^{1/2}$$

are equivalent.

See Appendix A of [R] for the proof.

The completion of $C_{\mathrm{p}}^\infty(Q)$ in the H^s norm is therefore the same as its completion in the norm

$$\|u\|_{H_{\mathrm{f}}^s} = \Big(\sum_{\mathbf{k}\in\mathbb{Z}_L^d}(1+|\mathbf{k}|^{2s})|\hat{u}_{\mathbf{k}}|^2\Big)^{1/2},$$

and we can identify $H^s(Q)$ with the collection of all formal Fourier series such that the norm H_{f}^s is finite.

Proposition 6.3. *The Sobolev space of periodic functions $H^s(Q)$ is the same as*

$$\Big\{u : u = \sum_{\mathbf{k}\in\mathbb{Z}_L^d} \hat{u}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},\ \hat{u}_{\mathbf{k}} = \overline{\hat{u}_{-\mathbf{k}}},\ \sum_{\mathbf{k}\in\mathbb{Z}^d} |\mathbf{k}|^{2s}|\hat{u}_{\mathbf{k}}|^2 < \infty\Big\}. \tag{6.7}$$

Corollary 6.4. *$C_{\mathrm{p}}^\infty(Q)$ is dense in $H^s(Q)$.*

6.2.2 $H_0^s(Q)$

Sometimes we will be able to restrict our attention to functions which have zero average over Q,

$$\int_Q u(x)\,\mathrm{d}x = 0$$

(in terms of Fourier series this reduces simply to $\hat{u}_{\mathbf{0}} = 0$). The Sobolev spaces of periodic functions with this condition we label $H_0^s(Q)$. These can be particularly useful, primarily because of the following result, known as Poincaré's inequality.

We write Du for the vector of first partial derivatives, so that

$$|Du|^2 = \sum_{|\alpha|=1} |D^\alpha u|^2. \tag{6.8}$$

Lemma 6.5. *(Poincaré's inequality). If $u \in H_0^1(Q)$ then*

$$|u| \leq \left(\frac{L}{2\pi}\right)|Du|. \tag{6.9}$$

Proof. Since

$$u = \sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} \hat{u}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

we have

$$Du = \sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} \mathrm{i}\mathbf{k}\hat{u}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

It follows that

$$|u|^2 = L^d \sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} |\hat{u}_{\mathbf{k}}|^2 \qquad \text{and} \qquad |Du|^2 = L^d \sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} |\mathbf{k}|^2 |\hat{u}_{\mathbf{k}}|^2,$$

and so

$$|u| \leq \left(\frac{L}{2\pi}\right)|Du|$$

as claimed. □

In this situation $|Du|$ is a norm equivalent to the standard H^1 norm. This follows since

$$|Du|^2 \leq \|u\|_{H^1}^2 = |u|^2 + |Du|^2 \leq (1+C)|Du|^2.$$

Because of this, we think of $|Du|$ as the standard norm on $H_0^1(Q)$, and write

$$\|u\|_{H_0^1} = |Du|.$$

This norm can be derived from the H_0^1 inner product

$$((u, v))_{H_0^1} = \sum_{|\alpha|=1} (D^\alpha u, D^\alpha v). \tag{6.10}$$

Note that following the argument of Proposition 6.2 we can show that in this case

$$\left(\sum_{\mathbf{k} \in \dot{\mathbb{Z}}_L^d} |\mathbf{k}|^2 |\hat{u}_\mathbf{k}|^2 \right)^{1/2}$$

is an equivalent norm.

The situation is similar in the higher order spaces $H_0^s(Q)$: an induction argument shows that

$$\|u\|_{H^s}^2 \le C \sum_{|\alpha|=s} |D^\alpha u|^2 \qquad \text{for all} \qquad u \in H_0^s(Q).$$

6.2.3 $H^{-s}(Q)$

Finally we introduce Sobolev spaces H^s with negative s:

Definition 6.6. *The space $H^{-s}(Q)$ is the dual space of $H_0^s(Q)$.*

We can characterise $H^{-s}(Q)$ in just the same way as $H^s(Q)$.

Proposition 6.7.

$$H^{-s}(Q) = \left\{ \phi = \sum_{\mathbf{k} \in \dot{\mathbb{Z}}_L^d} \hat{\phi}_\mathbf{k} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} : \sum_{\mathbf{k} \in \dot{\mathbb{Z}}_L^d} |\mathbf{k}|^{-2s} |\hat{\phi}_\mathbf{k}|^2 < \infty \right\}.$$

With this characterisation we understand the result $\langle \phi, u \rangle$ of applying ϕ to u as being (ϕ, u), i.e. the L^2 inner product understood in terms of the Fourier coefficients of ϕ and u.

To prevent too clumsy a notation we assume in what follows that all sums over $\mathbf{k}$ are for $\mathbf{k} \in \dot{\mathbb{Z}}_L^d$.

Proof. We first show that such a ϕ is an element of $H^{-s}(Q)$. Take $u \in H_0^s$, and then

$$\begin{aligned}
\langle \phi, u \rangle &= \sum_{\mathbf{k}} \hat{\phi}_{\mathbf{k}}\, c_{\mathbf{k}} \\
&\le \sum_{\mathbf{k}} |\hat{\phi}_{\mathbf{k}}||c_{\mathbf{k}}| \\
&\le \left(\sum_{\mathbf{k}} |\mathbf{k}|^{-2s} |\hat{\phi}_{\mathbf{k}}|^2 \right)^{1/2} \left(\sum_{\mathbf{k}} |\mathbf{k}|^{2s} |c_{\mathbf{k}}|^2 \right) \\
&\le C \left(\sum_{\mathbf{k}} |\mathbf{k}|^{-2s} |\hat{\phi}_{\mathbf{k}}|^2 \right)^{1/2} \|u\|_{H^s}.
\end{aligned}$$

Now, given a $\phi \in H^{-s}(Q)$, the Riesz representation theorem shows that there exists a $v \in H_0^s(Q)$ such that

$$((v, u))_{H_0^s} = \langle \phi, u \rangle.$$

If we use the Fourier version of the inner product on H_0^s we have

$$((v, u))_{H_0^s} = \sum_{\mathbf{k}} |\mathbf{k}|^{2s} v_{\mathbf{k}} u_{\mathbf{k}} = \langle \phi, u \rangle,$$

where

$$\phi = \sum_{\mathbf{k}} (|\mathbf{k}|^{2s} v_{\mathbf{k}}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \equiv \sum_{\mathbf{k}} \hat{\phi}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

Since $v \in H_0^s(Q)$ it follows that

$$\sum |\mathbf{k}|^{-2s} |\hat{\phi}_{\mathbf{k}}|^2 = \sum |\mathbf{k}|^{2s} |v_{\mathbf{k}}|^2 < \infty$$

as claimed. □

6.3 The Sobolev embedding theorem

In this section we investigate the relationship between the spaces $H^s(Q)$, $C^r(Q)$, and $L^p(Q)$. We first find conditions to ensure that a function in $H^s(Q)$ is in fact continuous, and then investigate its integrability properties.

6.3.1 Conditions for $H^s(Q) \subset C^0_{\mathrm{p}}(Q)$

We will show that if $s > d/2$ then $H^s(Q) \subset C^0_{\mathrm{p}}(Q)$, i.e. that if a function is in H^s then it is in fact continuous[1].

Theorem 6.8. *If $u \in H^s(Q)$ with $s > d/2$, then $u \in C^0_{\mathrm{p}}(Q)$ and*

$$\|u\|_\infty \le C_s \|u\|_{H^s}.$$

Proof. Write

$$u = \sum_{\mathbf{k} \in \mathbb{Z}^d_L} \hat{u}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},$$

and then

$$\begin{aligned}
\|u\|_\infty &\le \sum_{\mathbf{k} \in \mathbb{Z}^d_L} |\hat{u}_{\mathbf{k}}| \\
&\le \sum_{\mathbf{k} \in \mathbb{Z}^d_L} \frac{1}{(1+|\mathbf{k}|^{2s})^{1/2}} (1+|\mathbf{k}|^{2s})^{1/2} |\hat{u}_{\mathbf{k}}| \\
&\le \left(\sum_{\mathbf{k} \in \mathbb{Z}^d_L} (1+|\mathbf{k}|^{2s}) |\hat{u}_{\mathbf{k}}|^2 \right)^{1/2} \sum_{\mathbf{k} \in \mathbb{Z}^d_L} \frac{1}{(1+|\mathbf{k}|^{2s})}.
\end{aligned}$$

Now,

$$\sum_{\mathbf{k} \in \mathbb{Z}^d_L} \frac{1}{(1+|\mathbf{k}|^{2s})} = C_s < \infty$$

provided that $s > d/2$. Therefore

$$\|u\|_\infty \le C_s \|u\|_{H^s},$$

and the absolute convergence of the coefficients yields the uniform convergence of the Fourier series and hence continuity of u. □

Exercise 6.1. *Prove that if $s > d/2 + j$ and $u \in H^s(Q)$ then $u \in C^j_{\mathrm{p}}(Q)$ and*

$$\|u\|_{C^j} \le C \|u\|_{H^s}.$$

[1]Strictly, if $u \in H^s(Q)$ with $s > d/2$ then u "has a representative which is equal to a continuous function". Since the $H^s(Q)$ norm is defined by an integral, functions which agree almost everywhere are "the same" in H^s.

Exercise 6.2. *Use the Fourier expansion to show that if* $u \in H^2_{\rm p}(Q)$*, where* $Q = [0, L]^2$*, then*

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq c\|u\|_{H^2}|\mathbf{x} - \mathbf{y}|^{1/2}.$$

[You may assume that

$$|\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} - \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}}| \leq C|\mathbf{k}|^{\lambda}|\mathbf{x} - \mathbf{y}|^{1/2}.$$

You can prove this easily by considering the two cases $|\mathbf{k}||\mathbf{x}-\mathbf{y}| \leq 1$ *and* $|\mathbf{k}||\mathbf{x}-\mathbf{y}| > 1$ *separately.]*

6.3.2 Integrability properties of functions in H^s

The summability properties of the Fourier coefficients of u and its integrability properties are related in the following way. We say that $\mathbf{c} = \{c_{\mathbf{k}}\} \in l^p$ if

$$\|\mathbf{c}\|_{l^p} = \Big(\sum_{\mathbf{k}} |c_{\mathbf{k}}|^p\Big)^{1/p} < \infty.$$

Then, for $1 \leq p \leq 2$,

$$u = \sum_{\mathbf{k}} c_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \in L^q,$$

(p and q conjugate) with

$$\|u\|_{L^q} \leq \alpha\|\mathbf{c}\|_{l^p}. \tag{6.11}$$

The proof is not straightforward, and relies on complex variable methods (cf. Rudin (1974), Theorem 12.11). (To make this seem at least reasonable, observe that

$$\|u\|_{L^2} \leq \alpha\|\mathbf{c}\|_{l^2} \qquad \text{and} \qquad \|u\|_{L^\infty} \leq \alpha\|\mathbf{c}\|_{l^1},$$

so that (6.11) holds at the two "extreme" values of p.)

Theorem 6.9. *If* $u \in H^s(Q)$ *with* $s < d/2$ *then* $u \in L^p(Q)$*, with*

$$p \in \Big[2, \frac{d}{(d/2) - s}\Big).$$

If $s = d/2$ *then the embedding holds for all* $p < \infty$ *(but not* $p = \infty$*). In particular if* $s = 1$ *or* 2 *then* $H^1(Q) \subset L^p(Q)$ *for every* $1 \leq p < \infty$*.*

We show the result for the half-open interval, but in fact it holds for the closed interval. This requires other methods.

Proof. It is immediate that $H^s(Q) \subset L^2(Q)$ with $|u| \leq \|u\|_{H^s}$, so we prove the embedding only for $p > 2$. Then, using (6.11),

$$\begin{aligned}
\|u\|_{L^p} &\leq \alpha\Big(\sum_{\mathbf{k}\in\mathbb{Z}_L^d} |\hat{u}_\mathbf{k}|^q\Big)^{1/q} \\
&\leq \alpha\Big(\sum_{\mathbf{k}\in\mathbb{Z}_L^d} (1+|\mathbf{k}|^{2s})^{q/2}(1+|\mathbf{k}|^{2s})^{-q/2}|\hat{u}_\mathbf{k}|^q\Big)^{1/q} \\
&\leq \alpha\Big[\Big(\sum_{\mathbf{k}\in\mathbb{Z}_L^d} (1+|\mathbf{k}|^{2s})|\hat{u}_\mathbf{k}|^2\Big)^{q/2}\Big(\sum_{\mathbf{k}\in\mathbb{Z}_L^d} (1+|\mathbf{k}|^{2s})^{-q/(2-q)}\Big)^{(2-q)/2}\Big]^{1/q} \\
&\leq \alpha\|u\|_{H^s}\Big(\sum_{\mathbf{k}\in\mathbb{Z}_L^d} (1+|\mathbf{k}|^{2s})^{-q/(2-q)}\Big)^{(2-q)/2q}.
\end{aligned}$$

The sum in this final expression is finite provided that $qs/(2-q) > d/2$, i.e. provided that $q > 2d/(2s+d)$. Since $p^{-1} + q^{-1} = 1$, this requires

$$p < \frac{d}{\frac{d}{2} - s},$$

as in the statement of the theorem. □

6.4 The Rellich-Kondrachov theorem

Theorem 6.10. *$H^1(Q)$ is compactly embedded in $L^2(Q)$.*

Proof. Consider a sequence $\{u_n\}$,

$$u_n = \sum_{\mathbf{k}\in\mathbb{Z}_L^d} \hat{u}_{n\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

bounded in H^1. Then, for some M,

$$\sum_{\mathbf{k}\in\mathbb{Z}_L^d} (1+|\mathbf{k}|^2)|\hat{u}_{n\mathbf{k}}|^2 \leq M. \tag{6.12}$$

In particular, each Fourier component is uniformly bounded in n, so one can extract a subsequence $u_{n_{1j}}$ such that $\hat{u}_{n_{1j}\mathbf{k}_1}$ converges. From this take a subsequence $u_{n_{2j}}$ such that $\hat{u}_{n_{2j}\mathbf{k}_1}$ *and* $\hat{u}_{n_{2j}\mathbf{k}_2}$ converge. Continue in this way with subsequences $u_{n_{lj}}$ such that $\hat{u}_{n_{lj}\mathbf{k}_1}, \ldots, \hat{u}_{n_{lj}\mathbf{k}_l}$ all converge. Now take the diagonal sequence $\tilde{u}_j = u_{n_{jj}}$, and write $\hat{u}_{j\mathbf{k}}$ for the corresponding Fourier coefficients. Then, for each k, $\hat{u}_{j\mathbf{k}}$ converges to a limit $\hat{u}^*_{\mathbf{k}}$. From (6.12) we know that

$$\sum_{\mathbf{k}} (1 + |\mathbf{k}|^2)|\hat{u}^*_{\mathbf{k}}|^2 \leq M.$$

Thus

$$\sum_{\mathbf{k}} (1 + |\mathbf{k}|^2)|\hat{u}_{j\mathbf{k}} - \hat{u}^*_{\mathbf{k}}|^2 \leq 2M.$$

Now,

$$\begin{aligned} |\tilde{u}_j - u^*|^2 &= \sum_{\mathbf{k}} |\hat{u}_{j\mathbf{k}} - \hat{u}^*_{\mathbf{k}}|^2 \\ &= \sum_{|\mathbf{k}| \leq K} |\hat{u}_{j\mathbf{k}} - \hat{u}^*_{\mathbf{k}}|^2 + \frac{1}{K^2} \sum_{|\mathbf{k}| \geq K} |\hat{u}_{j\mathbf{k}} - \hat{u}^*_{\mathbf{k}}|^2 |\mathbf{k}|^2 \\ &\leq \sum_{|\mathbf{k}| \leq K} |\hat{u}_{j\mathbf{k}} - \hat{u}^*_{\mathbf{k}}|^2 + \frac{2M}{K^2}. \end{aligned}$$

Given $\epsilon > 0$, choose K large enough that the second term is $\leq \epsilon/2$, and j large enough that the first term is too. Then $u_j \to u^*$ in L^2. □

Corollary 6.11. *$H^{s+1}(Q)$ is compactly embedded in $H^s(Q)$.*

Exercise 6.3. *Using Theorem 6.10 prove this corollary: show that if u_n is bounded in $H^{s+1}(Q)$ then there is a subsequence u_{n_j} such that*

$$D^\alpha u_{n_j} \to u_\alpha$$

in L^2, for some $u_\alpha \in L^2$. Writing u for the limit in L^2 of u_{n_j}, by considering

$$\int_Q D^\alpha u \, \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_Q u D^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \lim_{j \to \infty} \int_Q u_{n_j} D^\alpha \phi \, \mathrm{d}x$$

show that in fact $u_\alpha = D^\alpha u$ and deduce that $u_{n_j} \to u$ in $H^s(Q)$.

Chapter 7

The Stokes Operator

This chapter contains material from Chapters 6 and 9 of [R].

At last we return to the Navier-Stokes equations, or at least a simplified version of them. If we only keep the linear terms we end up with

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{and} \qquad \nabla \cdot \mathbf{u} = 0. \tag{7.1}$$

The problem still consists of two coupled equations, one involving the velocity $\mathbf{u}$ and the pressure p, and the other (the incompressibility condition) involving only the velocity. We will restrict to $\mathbf{f}$, $\mathbf{u}$, and p with zero average over $Q = [0, L]^d$.

Since we are considering the case of periodic boundary conditions, we can use Fourier series, bearing in mind that the function u is not a scalar, but a vector valued function.

We first define a space of smooth functions which incorporate the periodicity and the divergence-free condition,

$$\mathbb{V} = \{\mathbf{u} \in [\dot{C}_{\mathrm{p}}^{\infty}(Q)]^d : \ \nabla \cdot \mathbf{u} = 0\}, \tag{7.2}$$

i.e. $\mathbb{V}$ consists of d-component divergence-free vectors, each component of which is in $\dot{C}_{\mathrm{p}}^{\infty}(Q)$. Since we will be using spaces like $\mathbb{V}$ consisting of d-vectors throughout the next few chapters, we will use the notation

$$\mathbb{L}^2(Q) = [L^2(Q)]^d \qquad \text{and} \qquad \mathbb{H}_{\mathrm{p}}^k(Q) = [H_{\mathrm{p}}^k(Q)]^d,$$

where, for example, the norm of $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{L}^2(Q)$ is

$$\|\mathbf{u}\|^2_{\mathbb{L}^2(Q)} = \sum_{j=1}^{d} |u_j|^2_{L^2(Q)}.$$

We now take the inner product of the first equation in (7.1) with an element $\mathbf{v}$ of $\mathbb{V}$, to obtain

$$-\nu \int_Q [\Delta \mathbf{u}(\mathbf{x})]\mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_Q \nabla p \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} = \int_Q \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

If we integrate the first term by parts and use the fact that $\mathbf{u}$ and $\mathbf{v}$ are periodic we get

$$-\int_Q \sum_{i,j=1}^{d} [D_i^2 u_j(\mathbf{x})] v_j(\mathbf{x})\,\mathrm{d}\mathbf{x} = \sum_{i,j=1}^{d} \int_Q D_i u_j(x) D_i v_j(x)\,\mathrm{d}\mathbf{x} = \int_Q \nabla \mathbf{u} : \nabla \mathbf{v}\,\mathrm{d}\mathbf{x}, \tag{7.3}$$

where the colon means that we sum over all derivatives and all components (as in the central term of the above equality). Now, if we integrate the p term by parts we obtain

$$\int_Q \nabla p \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} = \int_Q p\,(\nabla \cdot \mathbf{v})\,\mathrm{d}\mathbf{x} = 0, \tag{7.4}$$

since $\mathbf{v} \in \mathbb{V}$ and so $\nabla \cdot \mathbf{v} = 0$. The pressure term has dropped out, and we are left with

$$\int_Q \nabla \mathbf{u} : \nabla \mathbf{v}\,\mathrm{d}\mathbf{x} = \int_Q \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{7.5}$$

Exercise 7.1. *Consider Poisson's equation*

$$-\Delta u = f(x)$$

where now u and f are scalar functions of $\mathbf{x} \in Q$. Impose the addition conditions that $\int_Q u = \int_Q f = 0$. Show that

$$\int_Q \nabla u \cdot \nabla v\,\mathrm{d}\mathbf{x} = \int_Q f(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

for all $v \in C^\infty_{\mathrm{p}}(Q)$.

If $\mathbf{u} \in C^2_{\rm p}(Q)$ is divergence free ($\nabla \cdot \mathbf{u} = 0$) and $\mathbf{f} \in C^0_{\rm p}(Q)$ one can show that (7.5) implies that u is a classical solution of (7.1).

Exercise 7.2. *For the simpler example of Poisson's equation in Exercise 7.1 show that if $u \in C^2_{\rm p}(Q)$ and $f \in C^0_{\rm p}(Q)$, and*

$$\int_Q \nabla u \cdot \nabla v \,\mathrm{d}x = \int_Q f v \,\mathrm{d}x$$

for all $v \in C^\infty_{\rm p}(Q)$ then in fact $-\Delta u = f$.

However, (7.5) as it stands only requires $\mathbf{u}$ to be a C^1 function. We will see that this can be weakened further. We choose to write the left-hand side of (7.5) as $a(\mathbf{u}, \mathbf{v})$, i.e. we define

$$a(\mathbf{u}, \mathbf{v}) = \int_Q \nabla \mathbf{u} : \nabla \mathbf{v} \,\mathrm{d}\mathbf{x},$$

and then we have

$$\nu\, a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \text{for all} \qquad \mathbf{v} \in \mathbb{V}. \tag{7.6}$$

Now, note that $a : C^\infty_{\rm p}(Q) \times C^\infty_{\rm p}(Q) \to \mathbb{R}$ is a bilinear form on $C^\infty_{\rm p}(Q) \times C^\infty_{\rm p}(Q)$ that satisfies (using the Cauchy-Schwarz inequality)

$$\begin{aligned}
|a(\mathbf{u}, \mathbf{v})| &= \left| \sum_{i,j=1}^d \int_Q (D_i u_j)(D_i v_j) \,\mathrm{d}x \right| \\
&\leq \sum_{i,j=1}^d |D_i u_j| |D_i v_j| \\
&\leq \sum_{j=1}^d \left[\left(\sum_{i=1}^d |D_i u_j|^2 \right)^{1/2} \left(\sum_{i=1}^d |D_i v_j|^2 \right)^{1/2} \right] \\
&= \sum_{j=1}^d |D u_j| |D v_j| \\
&\leq \left(\sum_{j=1}^d |D u_j|^2 \right)^{1/2} \left(\sum_{j=1}^d |D v_j|^2 \right)^{1/2} \\
&= |D\mathbf{u}| |D\mathbf{v}|.
\end{aligned}$$

We have shown that

$$|a(\mathbf{u},\mathbf{v})| \leq |D\mathbf{u}||D\mathbf{v}|, \tag{7.7}$$

and so a is continuous in the $\mathbb{H}_0^1(Q)$ norm.

We denote by V the completion of $\mathbb{V}$ in the $\mathbb{H}_0^1(Q)$ norm,

$$V = \{\mathbf{u} \in \mathbb{H}_0^1(Q) : \ \nabla \cdot \mathbf{u} = 0\}. \tag{7.8}$$

V is a Hilbert space with the $\mathbb{H}_0^1$ inner product

$$(D\mathbf{u}, D\mathbf{v}) = \sum_{|\alpha|=1} (D^\alpha \mathbf{u}, D^\alpha \mathbf{v})$$

and corresponding norm

$$\sum_{|\alpha|=1} |D^\alpha \mathbf{u}|^2 \equiv |D\mathbf{u}|^2.$$

It follows that $C_p^\infty(Q)$ is dense in V, and that a can be extended to a continuous bilinear form from $V \times V$ into $\mathbb{R}$.

Exercise 7.3. *Suppose that $\mathbb{X}$ is a Banach space (with norm $\|\cdot\|$), and X is a dense subspace of $\mathbb{X}$. Let f be a linear functional from X into $\mathbb{R}$ which satisfies*

$$|f(x)| \leq M\|x\| \qquad \text{for all} \qquad x \in X.$$

Show that f has a unique extension to a continuous function

$$F : \mathbb{X} \to \mathbb{R}$$

(i.e. $F(x) = f(x)$ for all $x \in X$), and that F satisfies

$$|F(x)| \leq M\|x\| \qquad \text{for all} \qquad x \in \mathbb{X}.$$

Exercise 7.4. *Show that the bilinear form*

$$a(u,v) = \int_Q \nabla u \cdot \nabla v \, \mathrm{d}x$$

from Exercise 7.2 can be extended to a bilinear form on $H_0^1(Q) \times H_0^1(Q)$.

In fact, (7.6) makes sense for all $\mathbf{v} \in V$ whenever $\mathbf{u} \in \mathbb{H}_0^1(Q)$ and $\mathbf{f}$ is contained in the dual of V, $V^* = \mathbb{H}^{-1}(Q)$, provided that we understand the derivatives in a weak sense. So we can re-pose the original problem in its *weak form*: given $\mathbf{f} \in V^*$ find $\mathbf{u} \in V$ such that

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \text{for all} \qquad \mathbf{v} \in V, \tag{7.9}$$

where $\langle \mathbf{f}, \mathbf{v} \rangle$ denotes the pairing between $\mathbf{f} \in V^*$ and $\mathbf{v} \in V$.

Note that $a(\mathbf{u}, \mathbf{v})$ is just (6.10), the inner product on $\mathbb{H}_0^1(Q)$ obtained using Poincaré's inequality. The problem is therefore equivalent to finding a $\mathbf{u} \in V$ such that

$$((\mathbf{u}, \mathbf{v}))_V = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \text{for all} \qquad \mathbf{v} \in V. \tag{7.10}$$

Now, the Riesz Representation Theorem (Theorem 4.9) immediately tells is that there exists a unique $\mathbf{u} \in V$ such that (7.10) holds, and that the map $\mathbf{f} \mapsto \mathbf{u}$ is continuous.

Theorem 7.1. *For $\mathbf{f} \in V^*$, the weak form of the Stokes problem (7.6) has a unique solution $\mathbf{u} \in V$, and*

$$|D\mathbf{u}| = \|\mathbf{f}\|_*.$$

Exercise 7.5. *Write down the weak form of Poisson's equation (from the previous three exercises) and deduce that this equation has a unique weak solution.*

There is another way to consider equation (7.10), which will become more useful later. Observe that for each fixed $\mathbf{u} \in V$ the map

$$\mathbf{v} \mapsto a(\mathbf{u}, \mathbf{v})$$

is a linear functional on V. It follows that we can define a linear operator $A : V \to V^*$ by

$$\langle A\mathbf{u}, \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}) \qquad \text{for all} \qquad \mathbf{v} \in V. \tag{7.11}$$

Observe that (7.7) shows that A is a bounded operator from V into V^*. We can now write the equation $a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle$ as an equation which has to hold in V^*:

$$\nu A\mathbf{u} = \mathbf{f}.$$

7.1 Higher regularity

To investigate further regularity for the Stokes problem we want to consider the smoothness of u when $f \in \dot{\mathbb{L}}^2(Q)$. We will use the Fourier expansion. We take $\nu = 1$ and solve

$$A\mathbf{u} = \mathbf{f},$$

expanding $\mathbf{f}$ as

$$\mathbf{f} = \sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{f}}_{\mathbf{k}},$$

where now each $\hat{\mathbf{f}}_{\mathbf{k}} \in \mathbb{R}^d$, and we have $\hat{\mathbf{f}}_{\mathbf{0}} = 0$ and $\sum_{\mathbf{k}} |\hat{\mathbf{f}}_{\mathbf{k}}|^2 < \infty$ (since $\mathbf{f} \in \dot{\mathbb{L}}^2(Q)$). If we similarly expand $\mathbf{u}$ as

$$\mathbf{u} = \sum_{\mathbf{k}\in\dot{\mathbb{Z}}_L^d} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{u}}_{\mathbf{k}},$$

(with $\mathbf{u}_{\mathbf{k}} \in \mathbb{R}^d$) and p as

$$p = \sum_{\mathbf{k}\in\mathbb{Z}_L^d} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\hat{p}_k,$$

(note that p is a scalar, so $\hat{p}_k \in \mathbb{R}$), we have

$$-\Delta u = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}|^2\hat{\mathbf{u}}_k$$

and

$$\nabla p = \mathrm{i}\sum_{\mathbf{k}\in\mathbb{Z}_L^d} \mathbf{k}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\hat{p}_{\mathbf{k}}.$$

Equating coefficients in (7.1) now gives

$$|\mathbf{k}|^2\hat{\mathbf{u}}_{\mathbf{k}} - \mathrm{i}\mathbf{k}\hat{p}_{\mathbf{k}} = \hat{\mathbf{f}}_{\mathbf{k}} \tag{7.12}$$

and the divergence free condition ($\nabla \cdot u = 0$) becomes

$$\mathbf{k} \cdot \hat{\mathbf{u}}_{\mathbf{k}} = 0. \tag{7.13}$$

Taking the scalar product of (7.12) with $\mathbf{k}$ gives an expression for $\hat{p}_{\mathbf{k}}$ when $\mathbf{k} \neq 0$,

$$\hat{p}_{\mathbf{k}} = -\mathrm{i}\frac{(\hat{\mathbf{f}}_{\mathbf{k}} \cdot \mathbf{k})}{|\mathbf{k}|^2}, \tag{7.14}$$

and so one obtains the following expression for $\hat{\mathbf{u}}_{\mathbf{k}}$ when $\mathbf{k} \neq 0$,

$$\hat{\mathbf{u}}_{\mathbf{k}} = \frac{1}{|\mathbf{k}|^2}\Big(\hat{\mathbf{f}}_{\mathbf{k}} - \frac{\mathbf{k}(\mathbf{k}\cdot\hat{\mathbf{f}}_{\mathbf{k}})}{|\mathbf{k}|^2}\Big).$$

To fix $\mathbf{u}$ and p we set $\hat{\mathbf{u}}_{\mathbf{0}} = \mathbf{0}$ and $\hat{p}_{\mathbf{0}} = 0$, which is fine since we required $\hat{\mathbf{f}}_{\mathbf{0}} = \mathbf{0}$ in the first place (and we want $u_0 \in V$).

It follows that

$$\|\mathbf{u}\|^2_{\mathbb{H}^{s+2}} \leq C\|\mathbf{f}\|^2_{\mathbb{H}^s}, \tag{7.15}$$

and so in particular if $\mathbf{f} \in \dot{\mathbb{L}}^2(Q)$ then $\mathbf{u} \in \dot{\mathbb{H}}^2_{\rm p}(Q)$. Thus the domain of A, i.e. all $\mathbf{u}$ such that $A\mathbf{u} \in \mathbb{L}^2(Q)$, is given by

$$D(A) = \{\mathbf{u} \in \dot{\mathbb{H}}^2(Q) : \ \nabla\cdot\mathbf{u} = 0\} = \mathbb{H}^2(Q) \cap V.$$

Finally, if we define

$$H = \{\mathbf{u} \in \dot{\mathbb{L}}^2(Q) : \ \nabla\cdot\mathbf{u} = 0\}, \tag{7.16}$$

i.e. the space of all $\mathbf{u} \in \dot{\mathbb{L}}^2(Q)$ whose Fourier coefficients satisfy (7.13), then, if $\mathbf{f} \in H$ the solution of the Stokes problem is given by the function $\mathbf{u}$ with Fourier coefficients

$$\hat{\mathbf{u}}_{\mathbf{k}} = \frac{\hat{\mathbf{f}}_{\mathbf{k}}}{|\mathbf{k}|^2}. \tag{7.17}$$

This expression is exactly the same as (7.19) from Exercise 7.6 (below), which gives the solution of Poisson's equation, except applied to each component of $\hat{\mathbf{u}}_{\mathbf{k}}$ in turn. So we can deduce that

$$A\mathbf{u} = -\Delta\mathbf{u} \qquad \text{for all} \qquad \mathbf{u} \in D(A). \tag{7.18}$$

Note that this particular result (7.18) is only true in the case of periodic boundary conditions.

Exercise 7.6. *Solve Poisson's equation using Fourier series, and show that*

$$\hat{u}_{\mathbf{k}} = \frac{f_{\mathbf{k}}}{|\mathbf{k}|^2}. \tag{7.19}$$

Deduce that if $f \in H^s(Q)$ then $u \in H^{s+2}(Q)$ and

$$\|u\|_{H^{s+2}} \leq \|f\|_{H^s}.$$

Exercise 7.7. *Show that the functions*

$$\mathbf{w}_{j,\mathbf{k}} := \left(\mathbf{e}_j - \frac{\mathbf{k}_j \mathbf{k}}{|\mathbf{k}|^2}\right) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \qquad \text{with} \qquad j = 1, \ldots, d, \ \mathbf{k} \in \dot{\mathbb{Z}}_L^d$$

are eigenfunctions of the Stokes operator, i.e. for some p *and* $\lambda_{j,\mathbf{k}} \neq 0$

$$-\Delta \mathbf{w}_{j,\mathbf{k}} + \nabla p = \lambda_{j,\mathbf{k}} \mathbf{w}_{j,\mathbf{k}} \qquad \text{and} \qquad \nabla \cdot \mathbf{w}_{j,\mathbf{k}} = 0,$$

or more concisely

$$A\mathbf{w}_{j,\mathbf{k}} = |\mathbf{k}|^2 \mathbf{w}_{j,\mathbf{k}}.$$

Assuming that $\mathbf{e}_j \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ *form a basis for* $\mathbb{L}^2(Q)$, *show that* $\{\mathbf{w}_{j,\mathbf{k}}\}$ *form a basis for the space* H *defined in (7.16).*

Chapter 8

The Navier-Stokes equations

This chapter has material from Chapters 7, 8, and 9 of [R].

We will now consider the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{and} \qquad \nabla \cdot \mathbf{u} = 0,$$

and we will assume further that $\mathbf{f}$ does not depend on time. This is important if we want to use the equations to define a dynamical system.

We will also make the same restrictions as in the first chapter, namely that $\int_Q \mathbf{u}(\mathbf{x}, 0) = \int_Q \mathbf{f} = 0$. Since this implies that $\int_Q \mathbf{u}(\mathbf{x}, t) = 0$ for all $t \geq 0$, this allows us to work in a space of periodic functions with zero integral, in which we can make use of the Poincaré inequality (see Lemma 5.40)

$$|\mathbf{u}| \leq \lambda_1^{-1/2} |D\mathbf{u}|, \qquad \text{with} \qquad \lambda_1 = \frac{4\pi^2}{L^2}. \tag{8.1}$$

We say that a function v of space of time is in $L^p(0, T; X)$ if

$$\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v(s)\|_X^p \, \mathrm{d}s \right)^{1/p} < +\infty.$$

Exercise 8.1. *Show that any element of $\phi \in L^q(0, T; X^*)$ gives rise to an element in the dual space of $L^p(0, T; X)$ via*

$$[\phi, u] = \int_0^T \langle \phi(t), u(t) \rangle \, \mathrm{d}t,$$

where $\langle \psi, x\rangle$ is the pairing between $\psi \in X^*$ and $x \in X$.

8.1 The weak form of the NSE

We now take the inner product of

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}$$

with an element $\mathbf{v}$ of $\mathbb{V}$ (see (7.2)), to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\mathbf{v}) - \nu\int_Q \Delta\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_Q [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_Q \nabla p\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} = (\mathbf{f},\mathbf{v}). \quad (8.2)$$

We dealt in the previous chapter with the pressure term (it vanishes) and the Laplacian term, which becomes the bilinear form

$$a(\mathbf{u},\mathbf{v}) := \int_Q \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}\mathbf{x} := \sum_{i,j=1}^{d}\int_Q (D_i u_j)(D_i v_j)\,\mathrm{d}\mathbf{x}.$$

If we also define a trilinear form

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_Q [(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{w}\,d\mathbf{x} = \sum_{i,j=1}^{d}\int_Q u_i(D_i v_j)w_j\,\mathrm{d}\mathbf{x}$$

we can rewrite (8.2) as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) \qquad \text{for all} \qquad \mathbf{v}\in\mathbb{V}. \qquad (8.3)$$

As we did with the bilinear form $a(\mathbf{u},\mathbf{v})$ in the previous chapter, we can extend the definition of $b(\mathbf{u},\mathbf{v},\mathbf{w})$ from $\mathbb{V}\times\mathbb{V}\times\mathbb{V}$ to $V\times V\times V$ respectively, where (recall (7.8))

$$V = \{\mathbf{u} :\ \mathbf{u}\in\mathbb{H}_0^1(Q),\ \nabla\cdot\mathbf{u} = 0\}.$$

In order to extend b we have to show that it is continuous with respect to the norm in V ($|D\mathbf{u}|$). This is the context of the next lemma.

Lemma 8.1. *If* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ *then*

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq k \begin{cases} |\mathbf{u}|^{1/2}|D\mathbf{u}|^{1/2}|D\mathbf{v}||\mathbf{w}|^{1/2}|D\mathbf{w}|^{1/2} & d = 2 \\ |\mathbf{u}|^{1/4}|D\mathbf{u}|^{3/4}|D\mathbf{v}||\mathbf{w}|^{1/4}|D\mathbf{w}|^{3/4} & d = 3. \end{cases} \tag{8.4}$$

Proof. To prove the inequality for $b(u, v, w)$ we will need the following, known as Ladyzhenskaya's inequality,

$$\|f\|_{L^4} \leq c \begin{cases} |f|^{1/2}|Df|^{1/2} & d = 2 \\ |f|^{1/4}|Df|^{3/4} & d = 3. \end{cases} \tag{8.5}$$

Now, we consider

$$\begin{aligned} |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| &\leq \sum_{i,j=1}^{d} \left| \int_Q u_i(D_i v_j) w_j \, \mathrm{d}\mathbf{x} \right| \\ &\leq \sum_{i,j=1}^{d} |u_i|_{L^4} |D_i v_j|_{L^2} |w_j|_{L^4} \end{aligned}$$

using the generalised version of Hölder's inequality proved as an exercise in Chapter 3 (or just use Cauchy-Schwarz twice). Now we use Ladyzhenskaya's inequality to get

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c \sum_{i,j=1}^{d} |D_i v_j| \begin{cases} |u_i|^{1/2}|Du_i|^{1/2}|w_j|^{1/2}|Dw_j|^{1/2} & d = 2 \\ |u_i|^{1/4}|Du_i|^{3/4}|w_j|^{1/4}|Dw_j|^{3/4} & d = 3. \end{cases}$$

Since $|u_i|_{L^2} \leq \|\mathbf{u}\|_{\mathbb{L}^2}$, for each fixed i, j we can bound the terms in the sum by the same expression without the suffix, and so we finally obtain (8.4). □

Exercise 8.2. *Prove the 3d Ladyzhenskaya inequality for* $f \in H_0^1(Q)$,

$$\|f\|_{L^4} \leq C|f|^{1/4}|Df|^{3/4}.$$

[Hint: use Hölder's inequality and the embedding $H^1 \subset L^6$*, i.e.* $\|f\|_{L^6} \leq c|Df|$*.]*

In particular, because of the Poincaré inequality, (8.4) shows that

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c|D\mathbf{u}||D\mathbf{v}||D\mathbf{w}|.$$

It follows that b is continuous from $\mathbb{V} \times \mathbb{V} \times \mathbb{V}$ into $\mathbb{R}$ wrt the $\mathbb{H}_0^1$ norm. Since $\mathbb{V}$ is dense in V, it follows that we can extend b to functions in products of V.

In this way, if (8.3) holds for all $\mathbf{v} \in \mathbb{V}$, it must also hold for all $\mathbf{v} \in V$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \text{for all} \qquad v \in V. \tag{8.6}$$

One can check that if $\mathbf{u}(t)$ is in $[C_{\mathrm{p}}^2(Q)]^d$ and $f \in C^1(0, T; C^0(Q))$ then (8.6) implies that u satisfies the original Navier-Stokes equation. However, note that (8.6) also makes sense if we take $\mathbf{f} \in V^* = \mathbb{H}^{-1}$.

Now, the linear operator A from V into V^* associated with the bilinear form $a(\mathbf{u}, \mathbf{v})$,

$$\langle A\mathbf{u}, \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}), \qquad \text{for all} \qquad \mathbf{v} \in V \tag{8.7}$$

is the Stokes operator of the previous chapter. By analogy with this way of defining A we can define a bilinear operator $B(\mathbf{u}, \mathbf{v})$ from $V \times V$ into V^* which is associated with $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$: we set

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \qquad \text{for all} \qquad \mathbf{w} \in V. \tag{8.8}$$

Now, we can rewrite equation (8.6) as

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}, \tag{8.9}$$

using (8.7) and (8.8). If we assume that $\mathbf{f} \in V^*$, then we can expect that (8.9) will hold as an equality in V^*.

In fact we will prove something a little weaker, that the equation holds in an integrated form: this implies in particular that it holds as an equality in V^* for almost every $t \in [0, T]$.

To analyse (8.9) we will need two additional properties of the nonlinear term $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$. We derive these in the next section before proceeding to the questions of existence and uniqueness of weak solutions.

8.2 Properties of the trilinear form

The next result contains two useful "orthogonality" identities.

Proposition 8.2. *If $d = 2$ or $d = 3$, then for $u, v, w \in V$,*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \tag{8.10}$$

whence

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0. \tag{8.11}$$

For the case $d = 2$ (and only with periodic boundary conditions)

$$b(\mathbf{u}, \mathbf{u}, A\mathbf{u}) = 0 \qquad \text{for all} \qquad \mathbf{u} \in D(A). \tag{8.12}$$

Proof. (8.10) follows from an integration by parts. Taking $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$,

$$\begin{aligned} b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_Q \sum_{i,j=1}^d u_i (D_i v_j) w_j \, \mathrm{d}\mathbf{x} \\ &= -\int_Q \sum_{i,j=1}^d D_i(u_i w_j) v_j \, \mathrm{d}\mathbf{x} \\ &= -\int_Q \sum_{i,j=1}^d (D_i u_i) w_j v_j + u_i (D_i w_j) v_j \, \mathrm{d}\mathbf{x} \\ &= -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \end{aligned}$$

since $\sum_i D_i u_i = \nabla \cdot u = 0$. (8.10) follows immediately using the density of $\mathbb{V}$ in V.

The proof of (8.12) is tedious; see Proposition 9.1 in [R]. □

8.3 Two "pseudo Sobolev" results

We could define Sobolev spaces in this context too: if $\mathrm{d}u/\mathrm{d}t \in L^2(0, T; X)$ and $u \in L^2(0, T; X)$ we could say that $u \in H^1(0, T; X)$. An argument similar to that of theorem 6.8 will then show that $u \in C^0([0,T];X)$. However, generally we will find only

$$u \in L^2(0,T;H^1) \qquad \text{and} \qquad \mathrm{d}u/\mathrm{d}t \in L^2(0,T;H^{-1}).$$

The next result, which we quote without proof, shows that from this we can still deduce that $u \in C^0([0,T];L^2)$, and a limited amount of information on the derivative of u.

Theorem 8.3. *Suppose that*

$$u \in L^2(0,T;H^1(Q)) \qquad \text{and} \qquad \mathrm{d}u/\mathrm{d}t \in L^2(0,T;H^{-1}(Q)).$$

Then

(i) u is continuous[1] *from $[0,T]$ into $L^2(Q)$, with*

$$\sup_{t\in[0,T]} |u(t)| \leq C(\|u\|_{L^2(0,T;H^1)} + \|\mathrm{d}u/\mathrm{d}t\|_{L^2(0,T;H^{-1})}), \tag{8.13}$$

and

(ii)

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 = 2\langle \mathrm{d}u/\mathrm{d}t, u\rangle \tag{8.14}$$

for almost every $t \in [0,T]$, i.e.

$$|u(t)|^2 = |u_0|^2 + 2\int_0^t \langle \mathrm{d}u/\mathrm{d}t(s), u(s)\rangle\,\mathrm{d}s.$$

(By applying this component by component we can replace L^2 by $\mathbb{L}^2$, etc.)

Another result along similar lines is a compactness result which would hold from the standard theory of Sobolev spaces if $\{u_n\}$ were a bounded sequence in $H^1(0,T;L^2)$, but is not so straightforward if u_n is bounded in $L^2(0,T;H_0^1)$ and $\mathrm{d}u_n/\mathrm{d}t$ is bounded in $L^2(0,T;H^{-1})$. Nonetheless, the conclusion is the same – that there is a subsequence of $\{u_n\}$ which converges strongly in $L^2(0,T;L^2)$.

Theorem 8.4. *Suppose that u_n is a sequence which is uniformly bounded in $L^2(0,T;H^1)$, and $\mathrm{d}u_n/\mathrm{d}t$ is uniformly bounded in $L^p(0,T;H^{-1})$, for some $p > 1$. Then there is a subsequence such that $u_{n_j} \to u$ strongly in $L^2(0,T;L^2)$.*

[1] With caveat that it may have to be adjusted on a set of measure zero.

8.4 The Galerkin expansion

In our existence proof we will expand everything in a basis for H formed by the eigenfunctions of the Stokes operator. These are

$$\left(\mathbf{e}_j - \frac{\mathbf{k}_j\mathbf{k}}{|\mathbf{k}|^2}\right) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},$$

with eigenvalues $\lambda_{j,\mathbf{k}} = |\mathbf{k}|^2$ (see Exercise 7.7). It will be more convenient to denote this countable basis by $\{\mathbf{w}_n\}_{n=1}^{\infty}$. With $A\mathbf{w}_n = \lambda_n\mathbf{w}_n$ we order the eigenfunctions so that that $\lambda_{n+1} \geq \lambda_n$.

If $\mathbf{u} = \sum_{j=1}^{\infty} c_j\mathbf{w}_j$ then we define the projection of $\mathbf{u}$ onto the space spanned by the first n eigenfunctions, $P_n\mathbf{u}$, by

$$P_n\mathbf{u} = \sum_{j=1}^{n} c_j\mathbf{w}_j.$$

Note that

$$(P_n\mathbf{u}, \mathbf{v}) = (\mathbf{u}, P_n\mathbf{v}) = (P_n\mathbf{u}, P_n\mathbf{v}).$$

Using the orthonormality of the eigenfunctions and the definition of the norm in the various spaces it is not hard to prove the following result (see Lemma 7.5 in [R]):

Lemma 8.5. *For $X = L^2$, H^1, or H^{-1}, we have: if $u \in X$ then*

$$\|P_n\mathbf{u}\|_X \leq \|\mathbf{u}\|_X \qquad \text{and} \qquad P_n\mathbf{u} \to \mathbf{u} \quad \text{in} \quad X.$$

8.5 Existence of weak solutions

With the above preparations behind us, we can now begin to investigate existence and uniqueness for the Navier-Stokes equations.

We will show the following result, valid for both $d = 2$ and $d = 3$. Note that the theorem says nothing about uniqueness.

In the statement of the theorem, H is our phase space, consisting of divergence free $\mathbb{L}^2$ functions with zero average,

$$H = \{\mathbf{u} \in \mathbb{L}^2(Q) : \int_Q \mathbf{u} = 0,\ \nabla \cdot \mathbf{u} = 0\}.$$

The norm on H is just the $\mathbb{L}^2$ norm.

Theorem 8.6. *(Weak solutions). Let* $\mathbf{f} \in \mathbb{H}^{-1}$. *Then if* $\mathbf{u}_0 \in H$, *there exists a weak solution* $\mathbf{u}(t)$ *of*

$$\mathrm{d}\mathbf{u}/\mathrm{d}t + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}$$

i.e. a function $\mathbf{u}$ *such that, for any* $T > 0$,

$$\mathbf{u} \in L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}^1_0),$$

and for each $\mathbf{v} \in V$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{u}, \mathbf{v}\rangle + \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle \tag{8.15}$$

for almost every $t \in [0, T]$. *Furthermore*

$$\mathrm{d}\mathbf{u}/\mathrm{d}t \in L^p(0, T; \mathbb{H}^{-1}),$$

with $p = 2$ *if* $d = 2$ *and* $p = 4/3$ *if* $d = 3$.

Proof. We look at the finite-dimensional equation obtained by keeping only the first n Fourier modes, the n-dimensional Galerkin approximation,

$$\mathbf{u}_n = \sum_{j=1}^{n} u_{nj}(t)\mathbf{w}_j.$$

The equation for $\mathbf{u}_n$ is

$$\frac{\mathrm{d}\mathbf{u}_n}{\mathrm{d}t} + \nu A\mathbf{u}_n + P_n B(\mathbf{u}_n, \mathbf{u}_n) = P_n\mathbf{f}. \tag{8.16}$$

We try to find a bound on $|\mathbf{u}_n|$ uniform in n.

To bound $|\mathbf{u}_n|$ we take the inner product of (8.16) with $\mathbf{u}_n$, and obtain

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_n|^2 + \nu(A\mathbf{u}_n, \mathbf{u}_n) + (P_nB(\mathbf{u}_n, \mathbf{u}_n), \mathbf{u}_n) = \langle P_n\mathbf{f}, \mathbf{u}_n\rangle,$$

which, noting that (since $\mathbf{u}_n \in P_nH$)

$$\begin{aligned}(P_nB(\mathbf{u}_n, \mathbf{u}_n), \mathbf{u}_n) &= (B(\mathbf{u}_n, \mathbf{u}_n), P_n\mathbf{u}_n) \\ &= (B(\mathbf{u}_n, \mathbf{u}_n), \mathbf{u}_n) = b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{u}_n),\end{aligned}$$

and using (8.7) and the orthogonality property (8.11), becomes

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_n|^2 + \nu|D\mathbf{u}_n|^2 = \langle \mathbf{f}, \mathbf{u}_n\rangle \leq \|\mathbf{f}\|_{\mathbb{H}^{-1}}|D\mathbf{u}_n|.$$

Using Young's inequality on the right-hand side, we obtain

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_n|^2 + \nu|D\mathbf{u}_n|^2 \leq \frac{\nu}{2}|D\mathbf{u}_n|^2 + \frac{\|\mathbf{f}\|^2_{\mathbb{H}^{-1}}}{2\nu},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}_n|^2 + \nu|D\mathbf{u}_n|^2 \leq \frac{\|\mathbf{f}\|^2_{\mathbb{H}^{-1}}}{\nu}.$$

Now integrate both sides between 0 and t, to obtain

$$|\mathbf{u}_n(t)|^2 + \nu\int_0^t |D\mathbf{u}_n(s)|^2\,\mathrm{d}s \leq |\mathbf{u}_n(0)|^2 + \frac{\|\mathbf{f}\|^2_{L^2(0,t;\mathbb{H}^{-1})}}{\nu}.$$

Since $|\mathbf{u}_n(0)| = |P_n\mathbf{u}_0| \leq |\mathbf{u}_0|$ (Lemma 7.5), we have the bounds

$$\sup_{t\in[0,T]} |\mathbf{u}_n(t)|^2 \leq K = |\mathbf{u}_0|^2 + \frac{\|\mathbf{f}\|^2_{L^2(0,T;\mathbb{H}^{-1})}}{\nu},$$

and

$$\int_0^T |D\mathbf{u}_n(s)|^2\,\mathrm{d}s \leq K/\nu,$$

uniformly in n. Thus $\mathbf{u}_n$ is bounded uniformly (in n) in

$$L^\infty(0,T;\mathbb{L}^2) \qquad \text{and} \qquad L^2(0,T;\mathbb{H}^1_0). \tag{8.17}$$

To apply the two "pseudo-Sobolev" results, we also need to obtain bounds on the derivatives, $\mathrm{d}\mathbf{u}_n/\mathrm{d}t$. Here we find a difference between $d = 2$ and $d = 3$. For $d = 2$ we can show that $\mathrm{d}\mathbf{u}_n/\mathrm{d}t$ in uniformly bounded in $L^2(0, T; \mathbb{H}^{-1})$, whereas for $d = 3$ we can only obtain a bound in $L^{4/3}(0, T; \mathbb{H}^{-1})$. We set $p = 2$ in the case $d = 2$ and $p = 4/3$ if $d = 3$.

Since

$$\mathrm{d}\mathbf{u}_n/\mathrm{d}t = -\nu A\mathbf{u}_n - P_n B(\mathbf{u}_n, \mathbf{u}_n) + P_n\mathbf{f},$$

we need to show that each term on the right-hand side is uniformly bounded in $L^p(0, T; \mathbb{H}^{-1})$. This follows for $A\mathbf{u}_n$ since $\mathbf{u}_n$ is uniformly bounded in $L^2(0, T; \mathbb{H}^1_0)$ and A is a continuous linear operator from $\mathbb{H}^1_0$ into $\mathbb{H}^{-1}$. Clearly $P_n\mathbf{f}$ is also bounded in this sense, since we have assumed that $f \in \mathbb{H}^{-1}$. It only remains to verify the same kind of bound for $P_n B(\mathbf{u}_n, \mathbf{u}_n)$, and this is where the difference arises between the cases $d = 2$ and $d = 3$.

The following bounds on $\|B(\mathbf{u}, \mathbf{u})\|_{\mathbb{H}^{-1}}$ are a consequence of (8.4):

$$\|B(\mathbf{u}, \mathbf{u})\|_{\mathbb{H}^{-1}} \leq \begin{cases} k|\mathbf{u}||D\mathbf{u}| & d = 2 \\ k|\mathbf{u}|^{1/2}|D\mathbf{u}|^{3/2} & d = 3. \end{cases} \tag{8.18}$$

Indeed, for any $\mathbf{v} \in V$ we have

$$|\langle B(\mathbf{u}, \mathbf{u}), \mathbf{v}\rangle| = |b(\mathbf{u}, \mathbf{u}, \mathbf{v})| = |b(\mathbf{u}, \mathbf{v}, \mathbf{u})|,$$

using (8.10), and so, using the bounds on $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$ in (8.4) we get

$$|\langle B(\mathbf{u}, \mathbf{u}), \mathbf{v}\rangle| \leq |D\mathbf{v}| \begin{cases} k|\mathbf{u}||D\mathbf{u}| & d = 2 \\ k|\mathbf{u}|^{1/2}|D\mathbf{u}|^{3/2} & d = 3, \end{cases}$$

and (8.18) follows immediately from the definition of the norm in $\mathbb{H}^{-1}$.

Since we also have (see lemma 8.5)

$$\|P_n B(\mathbf{u}, \mathbf{v})\|_{\mathbb{H}^{-1}} \leq \|B(\mathbf{u}, \mathbf{v})\|_{\mathbb{H}^{-1}}$$

we have

$$\|P_n B(\mathbf{u}_n, \mathbf{u}_n)\|^2_{L^2(0,T;\mathbb{H}^{-1})} \leq \int_0^T \|B(\mathbf{u}_n(s), \mathbf{u}_n(s))\|^2_{\mathbb{H}^{-1}}\,\mathrm{d}s,$$

so that for $d = 2$

$$\|P_n B(\mathbf{u}_n, \mathbf{u}_n)\|_{L^2(0,T;\mathbb{H}^{-1})} \leq k\int_0^T |\mathbf{u}_n(s)|^2|D\mathbf{u}_n(s)|^2\,\mathrm{d}s$$

$$\leq \; k\|\mathbf{u}_n\|^2_{L^\infty(0,T;\mathbb{L}^2)}\|\mathbf{u}_n\|^2_{L^2(0,T;\mathbb{H}^1_0)},$$

and for $d = 3$

$$\begin{aligned}\|P_nB(\mathbf{u}_n,\mathbf{u}_n)\|^{4/3}_{L^{4/3}(0,T;\mathbb{H}^{-1})} &\leq k\int_0^T |\mathbf{u}_n(s)|^{2/3}|D\mathbf{u}_n(s)|^2\,\mathrm{d}s \\ &\leq k\|\mathbf{u}_n\|^{2/3}_{L^\infty(0,T;\mathbb{L}^2)}\|\mathbf{u}_n\|^2_{L^2(0,T;\mathbb{H}^{-1})}.\end{aligned}$$

Since $\mathbf{u}_n$ is uniformly bounded in $L^\infty(0,T;\mathbb{L}^2)$ and $L^2(0,T;\mathbb{H}^1_0)$, see (8.17), $P_nB(\mathbf{u}_n,\mathbf{u}_n)$ is uniformly bounded (in n) in $L^2(0,T;\mathbb{H}^{-1})$ if $d = 2$ and $L^{4/3}(0,T;\mathbb{H}^{-1})$ if $d = 3$. This gives the same bounds on $\mathrm{d}\mathbf{u}_n/\mathrm{d}t$,

$$\mathrm{d}\mathbf{u}_n/\mathrm{d}t \qquad \text{is uniformly bounded in} \qquad \begin{cases} L^2(0,T;\mathbb{H}^{-1}) & d = 2 \\ L^{4/3}(0,T;\mathbb{H}^{-1}) & d = 3. \end{cases}$$

Due to these uniform bounds we can use the Alaoglu Compactness Theorem (Theorem 5.14) to find a subsequence (which we shall relabel $\mathbf{u}_n$) such that

$$\mathbf{u}_n \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0,T;\mathbb{L}^2)$$

and, extracting a further subsequence with Corollary 5.1 (and relabelling again)

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^2(0,T;\mathbb{H}^1_0),$$

with

$$\mathbf{u} \in L^\infty(0,T;\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}^1_0).$$

We can now use the bound on $\mathbf{u}_n$ in $L^2(0,T;\mathbb{H}^1_0)$ and on $\mathrm{d}\mathbf{u}_n/\mathrm{d}t$ in $L^p(0,T;\mathbb{H}^{-1})$ and apply the pseudo Rellich-Kondrachov Compactness Theorem (Theorem 8.4) to guarantee that there is a further subsequence $\{\mathbf{u}_n\}$ (after relabelling) which converges to $\mathbf{u}$ strongly in $L^2(0,T;\mathbb{L}^2)$. Finally we use the bound on the time derivative of $\mathbf{u}_n$ to end up with a subsequence that converges in all the above senses and for which

$$\mathrm{d}\mathbf{u}_n/\mathrm{d}t \overset{*}{\rightharpoonup} \mathrm{d}\mathbf{u}/\mathrm{d}t \qquad \text{in} \qquad L^p(0,T;\mathbb{H}^{-1}).$$

Now take the scalar product of the Galerkin equation

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}_n + \nu A\mathbf{u}_n + P_nB(\mathbf{u}_n,\mathbf{u}_n) = P_n\mathbf{f}$$

with a $\mathbf{v} \in V$ and integrate with respect to t between t_0 and t:

$$\begin{aligned}(\mathbf{u}_n(t), \mathbf{v}) + \nu \int_{t_0}^t (D\mathbf{u}_n(s), D\mathbf{v})\,\mathrm{d}s + \int_{t_0}^t b(\mathbf{u}_n(s), \mathbf{u}_n(s), P_n\mathbf{v})\,\mathrm{d}s \\ = (\mathbf{u}_n(t_0), \mathbf{v}) + \int_{t_0}^t (P_n\mathbf{f}, \mathbf{v})\,\mathrm{d}s.\end{aligned}$$

Since $\mathbf{u}_n \to \mathbf{u}$ in $L^2(0, T; H)$, there is a subsequence such that $\mathbf{u}_n(t) \to \mathbf{u}$ in H for almost every $t \in [0, T]$: denote by G_T the set of such t. Take $t_0 \in G_T$.

With $t, t_0 \in G_T$ we now show convergence of all the terms in the equation as $n \to \infty$. The term coming from the Laplacian converges since $D\mathbf{u}_n \rightharpoonup D\mathbf{u}$ in $L^2(0, T; H)$, and $D\mathbf{v} \in L^2$ (since $\mathbf{v} \in V$). The term arising from the forcing converges since $P_n\mathbf{f} \to \mathbf{f}$ in H, and we are left with the nonlinear term. To show that this converges is simple but a little messy. First observe that

$$b(\mathbf{u}_n, \mathbf{u}_n, P_n\mathbf{v}) - b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}) - b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{u}_n, \mathbf{u}_n, Q_n\mathbf{v}), \quad (8.19)$$

where $Q_n\mathbf{v} = \mathbf{v} - P_n\mathbf{v}$. First we deal with the last term:

$$\begin{aligned}\left|\int_{t_0}^t b(\mathbf{u}_n, \mathbf{u}_n, Q_n\mathbf{v})\,\mathrm{d}s\right| &\leq \int_{t_0}^t |b(\mathbf{u}_n(s), \mathbf{u}_n(s), Q_n\mathbf{v})|\,\mathrm{d}s \\ &\leq \int_{t_0}^t |D\mathbf{u}_n|^2 |DQ_n\mathbf{v}|\,\mathrm{d}s \\ &\leq |DQ_n\mathbf{v}| \int_{t_0}^t |D\mathbf{u}_n(s)|^2\,\mathrm{d}s \\ &= |DQ_n\mathbf{v}| \|\mathbf{u}_n\|_{L^2(t_0,t;V)},\end{aligned}$$

and so, since $\mathbf{u}_n$ is bounded in $L^2(0, T; V)$ and $P_n\mathbf{v} \to \mathbf{v}$ in V, this term tends to zero. We rewrite the first two terms on the right-hand side of (8.19) as

$$b(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}_n - \mathbf{u}, \mathbf{v}) = b(\mathbf{u}_n - \mathbf{u}, \mathbf{u}_n, \mathbf{v}) - b(\mathbf{u}, \mathbf{v}, \mathbf{u} - \mathbf{u}_n),$$

and so (we use the weaker 3d version of the inequality (8.4) for b)

$$\left|\int_{t_0}^t b(\mathbf{u}_n(s) - \mathbf{u}(s), \mathbf{u}_n(s), \mathbf{v}) + b(\mathbf{u}(s), \mathbf{v}, \mathbf{u}_n(s) - \mathbf{u}(s))\,\mathrm{d}s\right|$$

$$
\begin{aligned}
&\le \int_{t_0}^{t} |b(\mathbf{u}_n(s)-\mathbf{u}(s),\mathbf{u}_n(s),\mathbf{v})| + |b(\mathbf{u}(s),\mathbf{v},\mathbf{u}_n(s)-\mathbf{u}(s))|\,\mathrm{d}s \\
&\le k\int_{t_0}^{t} |D(\mathbf{u}_n(s)-\mathbf{u}(s))|^{3/4}|\mathbf{u}_n(s)-\mathbf{u}(s)|^{1/4}|D\mathbf{v}|\Big[|D\mathbf{u}_n(s)|+|D\mathbf{u}(s)|\Big]\,\mathrm{d}s.
\end{aligned}
$$

Using Hölder's inequality with $(8, 8/7)$ we can bound this by

$$
\begin{aligned}
&k\left(\int_{t_0}^{t} |\mathbf{u}_n(s)-\mathbf{u}(s)|^2\,\mathrm{d}s\right)^{1/8} \\
&\qquad\times\left(\int_{t_0}^{t}\left(|D(\mathbf{u}_n(s)-\mathbf{u}(s))|^{3/4}\Big[|D\mathbf{u}_n(s)|+|D\mathbf{u}(s)|\Big]\right)^{7/8}\mathrm{d}s\right)^{8/7}.
\end{aligned}
$$

Although this looks hideous, the second term is bounded, since $\mathbf{u}_n$ is uniformly bounded in $L^2(0,T;V)$, and the first term converges to zero since $\mathbf{u}_n \to \mathbf{u}$ in $L^2(0,T;H)$.

We have therefore shown that, for $t_0, t \in G_T$,

$$
\begin{aligned}
(\mathbf{u}(t),\mathbf{v}) + \nu\int_{t_0}^{t}(D\mathbf{u}(s),D\mathbf{v})\,\mathrm{d}s + \int_{t_0}^{t} b(\mathbf{u}(s),\mathbf{u}(s),\mathbf{v})\,\mathrm{d}s \\
= (\mathbf{u}(t_0),\mathbf{v}) + \int_{t_0}^{t}(\mathbf{f},\mathbf{v})\,\mathrm{d}s,
\end{aligned}
$$

which implies that (8.15) holds for almost every $t \in [0,T]$. □

(It follows from Lemma 7.5 in [R] that in fact the equation holds as an equality in $L^2(0,T;V^*)$.)

8.6 Unique weak solutions in 2d

In the two-dimensional case we can obtain much better results: continuity of the solution into H, and uniqueness of the weak solution.

Theorem 8.7. *(Unique weak solutions in 2d). If $d = 2$ then the weak solution $u(t)$ whose existence is guaranteed by Theorem 8.6 depends continuously on the initial condition u_0. In particular the solution is unique. Furthermore $u \in C^0([0,T];H)$.*

Proof. The inequalities for $b(\mathbf{u},\mathbf{v},\mathbf{w})$ play an important rôle in the proof, and this is what makes this analysis inapplicable to the 3d case.

We consider two solutions $\mathbf{u}$ and $\mathbf{v}$ of (8.9), and write the equation for their difference $\mathbf{w}=\mathbf{u}-\mathbf{v}$. Then $\mathbf{w}$ satisfies

$$\mathrm{d}\mathbf{w}/\mathrm{d}t+\nu A\mathbf{w}+B(\mathbf{u},\mathbf{u})-B(\mathbf{v},\mathbf{v})=0,$$

which we rewrite, using the bilinearity of B,

$$B(\mathbf{u}-\mathbf{v},\mathbf{u})+B(\mathbf{v},\mathbf{u}-\mathbf{v})=B(\mathbf{w},\mathbf{u})+B(\mathbf{v},\mathbf{w}),$$

as

$$\mathrm{d}\mathbf{w}/\mathrm{d}t+\nu A\mathbf{w}+B(\mathbf{w},\mathbf{u})+B(\mathbf{v},\mathbf{w})=0. \tag{8.20}$$

Now if we take the inner product of this equation with w using theorem 8.3 and use the orthogonality property of b (8.11), we obtain

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{w}|^2+\nu|D\mathbf{w}|^2=-b(\mathbf{w},\mathbf{u},\mathbf{w}). \tag{8.21}$$

Thus, using (8.4),

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{w}|^2+\nu|D\mathbf{w}|^2 &\leq |b(\mathbf{w},\mathbf{u},\mathbf{w})|\\ \frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{w}|^2+\nu|D\mathbf{w}|^2 \leq \frac{k^2}{\nu}|D\mathbf{u}|^2|\mathbf{w}|^2.\end{aligned}$$

Neglecting the term $\nu|D\mathbf{w}|^2$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\exp\Big(-\int_0^t\frac{k^2}{\nu}|D\mathbf{u}(s)|^2\,ds\Big)|\mathbf{w}(t)|^2\Big\}\leq 0.$$

We can rewrite this as

$$|w(t)|^2\leq\exp\Big(\int_0^t\frac{k^2}{\nu}|D\mathbf{u}(s)|^2\,ds\Big)|\mathbf{w}(0)|^2, \tag{8.22}$$

and since Theorem 8.6 guarantees that $\mathbf{u}\in L^2(0,T;\mathbb{H}^1_0)$, the integral in the exponential is finite. Thus if we have $\mathbf{w}(0)=0$ then $\mathbf{w}(t)=0$ for all $t\geq 0$, which gives uniqueness of the solution.

The continuity of the solution into H follows from $\mathbf{u}\in L^2(0,T;\mathbb{H}^1_0)$ and $\mathrm{d}\mathbf{u}/\mathrm{d}t\in L^2(0,T;\mathbb{H}^{-1})$, using Theorem 8.3. $\square$

The results of this section show that, when f is independent of t, for the 2d NSE we can define a semi-dynamical system on H,

$$(H, \{S_H(t)\}_{t\geq 0}),$$

where $S_H(t)\mathbf{u}_0 = \mathbf{u}(t)$, and $S_H(t)$ is a C^0 semigroup, i.e. it satisfies

$$\begin{gathered} S_H(0) = I \\ S_H(t)S_H(s) = S_H(s)S_H(t) = S_H(s+t) \\ S_H(t)x_0 \text{ is continuous in } x_0 \text{ and } t. \end{gathered}$$

We will see in the next section that V is also a suitable phase space for this problem.

Uniqueness of weak solutions is an open problem in 3d.

8.7 Strong solutions in 2d

We will now see that while weak solutions exist for all time for both the 2d and 3d equations, we can only obtain existence of strong solutions for all time in the 2d case. We will then see that the existence of strong solutions implies the existence of classical (smooth) solutions.

First we treat the 2d case.

Theorem 8.8. *(Strong solutions in 2d). Take $d = 2$. If $\mathbf{u}_0 \in V$, and $\mathbf{f} \in H$ then for any $T > 0$ there is a unique strong solution of*

$$\mathrm{d}\mathbf{u}/\mathrm{d}t + \nu A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}$$

i.e. a function u such that

$$\mathbf{u} \in L^\infty(0, T; V) \cap L^2(0, T; \mathbb{H}^2); \tag{8.23}$$

in fact $\mathbf{u} \in C^0([0, T]; V)$. Furthermore the solutions depend continuously on the initial condition $\mathbf{u}_0$.

Proof. The idea is essentially the same as in the proof of weak solutions, except that we now take the inner product with $A\mathbf{u}_n$ rather than with $\mathbf{u}_n$:

$$(\mathrm{d}\mathbf{u}_n/\mathrm{d}t, A\mathbf{u}_n) + \nu|A\mathbf{u}_n|^2 + (P_n B(\mathbf{u}_n, \mathbf{u}_n), A\mathbf{u}_n) = (P_n\mathbf{f}, A\mathbf{u}_n).$$

Since $\mathbf{u}_n \in H$ and is smooth, $A\mathbf{u}_n = -\Delta\mathbf{u}_n$, and so

$$-\int_Q \Delta\mathbf{u}_n \frac{\partial \mathbf{u}_n}{\partial t}\,\mathrm{d}x = \sum_{i=1}^{m} \int_Q \frac{\partial \mathbf{u}_n}{\partial x_i}\frac{\partial^2 \mathbf{u}_n}{\partial t \partial x_i}\,\mathrm{d}x = \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}_n|^2. \tag{8.24}$$

Therefore we have

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}_n|^2 + \nu|A\mathbf{u}_n|^2 + b(\mathbf{u}_n, \mathbf{u}_n, A\mathbf{u}_n) \le |P_n\mathbf{f}||A\mathbf{u}_n|.$$

Since for 2d periodic boundary conditions we have $b(\mathbf{u}, \mathbf{u}, A\mathbf{u}) = 0$ this can be rearranged to give

$$\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}_n|^2 + \nu|A\mathbf{u}_n|^2 \le |P_n\mathbf{f}|^2.$$

It now follows that $\mathbf{u}_n$ is uniformly bounded in

$$L^\infty(0, T; V) \qquad \text{and} \qquad L^2(0, T; D(A)).$$

We can also estimate the time derivative and show that $\mathrm{d}\mathbf{u}_n/\mathrm{d}t$ is uniformly bounded in $L^2(0, T; H)$. The same kind of limiting argument as before now applies (extract various subsequences) to show that there is a solution $\mathbf{u}$ as in the statement of the theorem.

A version of the pseudo-Sobolev result (Theorem 8.3) gives continuity into V, and a variant of the uniqueness proof used in the proof of Theorem 8.7 gives the continuous dependence on initial conditions. □

Note that uniqueness of these solutions follows from the uniqueness of weak solutions, since a strong solution is also a weak solution.

Therefore, when $\mathbf{f}$ does not depend on time, we can also define a semi-dynamical system on V,

$$(V, \{S_V(t)\}_{t\ge 0}).$$

Because solutions are unique, this is the restriction of the semi-dynamical system $S_H(t)$ to V, as so we denote both simply by $S(t)$.

8.8 Strong solutions in 3d

We now treat the 3d case, in which we only know how to prove a much weaker result.

Theorem 8.9. *(Strong solutions in 3d). Take $d = 3$. If $u_0 \in V$ and $f \in H$, then there exists a $T > 0$ depending on $|D\mathbf{u}_0|$, ν, and $|\mathbf{f}|$, such that there is a unique strong solution $\mathbf{u} \in L^\infty(0,T;V) \cap L^2(0,T;\mathbb{H}^2)$.*

Proof. If we proceed as in the 2d case then we reach

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}_n|^2 + \nu|A\mathbf{u}_n|^2 + b(\mathbf{u}_n, \mathbf{u}_n, A\mathbf{u}_n) \leq |P_n\mathbf{f}||A\mathbf{u}_n|,$$

but now we cannot simply cancel the nonlinear term. Instead we have to estimate it, and hope that it is small enough to be dominated by the $|A\mathbf{u}_n|^2$ term. However, we only have

$$|b(\mathbf{u}, \mathbf{u}, A\mathbf{u})| \leq c|D\mathbf{u}|^{3/2}|A\mathbf{u}|^{3/2},$$

so this gives

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}_n|^2 + \nu|A\mathbf{u}_n|^2 &\leq \frac{1}{\nu}|P_n\mathbf{f}|^2 + \frac{\nu}{4}|A\mathbf{u}_n|^2 + c|D\mathbf{u}_n|^{3/2}|A\mathbf{u}_n|^{3/2} \\ &\leq \frac{1}{\nu}|P_n\mathbf{f}|^2 + \frac{\nu}{4}|A\mathbf{u}_n|^2 + \frac{\nu}{4}|A\mathbf{u}_n|^2 + \frac{\tilde{c}}{\nu^3}|D\mathbf{u}_n|^6.\end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}_n|^2 + \nu|A\mathbf{u}_n|^2 \leq \frac{2}{\nu}|P_n\mathbf{f}|^2 + \frac{\tilde{c}}{\nu^3}|D\mathbf{u}_n|^6. \qquad (8.25)$$

In order to derive bounds on $\mathbf{u}_n$ that are uniform in n we will have to restrict T. Writing $X(t) = |D\mathbf{u}_n(t)|^2$, $\phi = 2|P_n\mathbf{f}|^2/\nu$ and $k = \tilde{c}/\nu^3$ we have (dropping the $\nu|A\mathbf{u}_n|^2$ term)

$$\frac{\mathrm{d}X}{\mathrm{d}t} \leq \phi + kX^3.$$

In order to deduce bounds from this differential inequality divide both sides by $(1+X)^3$, and then

$$\frac{\mathrm{d}X}{(1+X)^3} \leq (\phi + k)\,\mathrm{d}t$$

(since $(1+X)^3 \geq 1$, and $(1+X)^3 > X^3$). Integrating both sides of this new inequality we obtain

$$\frac{1}{(1+X(0))^2} - \frac{1}{(1+X(t))^2} \leq Kt$$

where $K = 2(\phi + k)$. Therefore

$$|D\mathbf{u}_n(t)|^2 \leq \frac{1+|D\mathbf{u}_n(0)|^2}{\sqrt{1-Kt(1+|D\mathbf{u}_n(0)|)^2}}.$$

This bound is only finite while $Kt(1+|D\mathbf{u}_n(0)|^2) < 1$; if we choose T satisfying

$$T < \frac{1}{K(1+|D\mathbf{u}(0)|^2)}$$

then it follows that $\mathbf{u}_n$ is uniformly bounded in $L^\infty(0,T;V)$ and $L^2(0,T;D(A))$, and the proof proceeds as in the 2d case.

□

Exercise 8.3. *Suppose that $\mathbf{f} = 0$. Show that if the initial condition u_0 satisfies*

$$|D\mathbf{u}_0|^2 \leq \tilde{c}^{-1/2}\nu^2\lambda_1^{1/2}$$

then

$$|D\mathbf{u}_n(t)|^2 \leq \tilde{c}^{-1/2}\nu^2\lambda - 1^{1/2} \qquad \text{for all} \qquad t \geq 0$$

and hence that in this case the equation has a strong solution that exists for all positive times. [Hint: use (8.25) to show that $\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}_n|^2 \leq 0$.]

8.9 Uniqueness of 3d strong solutions

To end this chapter, we show that strong solutions of the 3d equations are unique in the class of weak solutions:

Theorem 8.10. *Let $\mathbf{u}$ be a strong solution of the 3D Navier-Stokes equations, i.e.*

$$\mathbf{u} \in L^\infty(0,T;V) \cap L^2(0,T;\mathbb{H}^2),$$

as in (8.23). Then u is unique in the class of all weak solutions.

We need to use another inequality for $b(\mathbf{u},\mathbf{v},\mathbf{w})$,

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leq k|D\mathbf{u}||D\mathbf{v}|^{1/2}|A\mathbf{v}|^{1/2}|\mathbf{w}| \tag{8.26}$$

Exercise 8.4. *Prove (8.26) for d = 3. You will need to use the generalised Hölder inequality, the embedding $H^1 \subset L^6$, and*

$$\|u\|_{L^3} \leq c|u|^{1/2}|Du|^{1/2}$$

(which you may assume).

Proof. Once again we consider the equation for the difference of two solutions, $\mathbf{w} = \mathbf{u} - \mathbf{v}$, where $\mathbf{u}$ is a strong solution and $\mathbf{v}$ is a weak solution. We take the inner product with $\mathbf{w}$, and obtain as before (8.21)

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{w}|^2 + \nu|D\mathbf{w}|^2 = -b(\mathbf{w},\mathbf{u},\mathbf{w}). \tag{8.27}$$

[This proof is only formal, since we do not have $\mathrm{d}\mathbf{w}/\mathrm{d}t \in L^2(0,T;V^*)$ which we need to justify this step rigorously; however, it can be made rigorous using the Galerkin procedure.] We now apply the inequality (8.26) for $b(\mathbf{w},\mathbf{u},\mathbf{w})$, to obtain

$$\begin{aligned} \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{w}|^2 + \nu|D\mathbf{w}|^2 &\leq k|D\mathbf{w}||D\mathbf{u}|^{1/2}|A\mathbf{u}|^{1/2}|\mathbf{w}| \\ &\leq \frac{\nu}{2}|D\mathbf{w}|^2 + \frac{k^2}{2\nu}|D\mathbf{u}||A\mathbf{u}||\mathbf{w}|^2. \end{aligned}$$

Rewrite this as

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{w}|^2 + \nu|D\mathbf{w}|^2 \leq \frac{k^2}{\nu}|D\mathbf{u}||A\mathbf{u}||\mathbf{w}|^2,$$

and ignoring the $\nu|D\mathbf{w}|^2$ term (as before) gives

$$|\mathbf{w}(t)|^2 \leq \exp\Big(\frac{k^2}{\nu}\int_0^t |D\mathbf{u}(s)||A\mathbf{u}(s)|\,\mathrm{d}s\Big)|\mathbf{w}(0)|^2. \tag{8.28}$$

Since

$$\int_0^t |D\mathbf{u}(s)||A\mathbf{u}(s)|\,\mathrm{d}s \leq \|\mathbf{u}\|_{L^2(0,T;V)}\|\mathbf{u}\|_{L^2(0,T;D(A))},$$

and both these quantities are finite, we have uniqueness. □

Exercise 8.5. *Show that if $\mathbf{u} \in L^4(0,T;V)$ is a solution of the 3d Navier-Stokes equations then it is unique in the class of weak solutions. (Use inequality (8.4).)*

8.10 Strong solutions are regular

We now give an indication of why we have concentrated on the problem of strong solutions: essentially, if a solution is strong then it will be as smooth as the forcing function f and the initial condition allow.

The easiest way to prove the result relies on the introduction of the notion of fractional powers of the operator A. If $\mathbf{u} \in D(A)$ is given by

$$\mathbf{u} = \sum_{\mathbf{k}} \hat{\mathbf{u}}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

(recall that all such functions are divergence free, so we must have $\mathbf{k}\cdot\hat{\mathbf{u}}_{\mathbf{k}} = 0$) then

$$A\mathbf{u} = \sum_{\mathbf{k}} |\mathbf{k}|^2 \hat{\mathbf{u}}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

We therefore define

$$A^{s/2}u = \sum_{k} |k|^s \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}.$$

Note that

$$|A^{s/2}\mathbf{u}|^2 = \sum_{k} |\mathbf{k}|^{2s} |\hat{\mathbf{u}}_{\mathbf{k}}|^2 = \|\mathbf{u}\|_{\mathbb{H}^s}^2$$

and that

$$(A^s\mathbf{u}, \mathbf{v}) = (A^{s/2}\mathbf{u}, A^{s/2}\mathbf{v}).$$

We denote by $D(A^{s/2})$ all those $\mathbf{u}$ for which $|A^{s/2}\mathbf{u}| < \infty$. In the case of periodic boundary conditions, $D(A^{s/2}) = \mathbb{H}^s(Q) \cap V$.

The following inequality, valid only for periodic boundary conditions, is proved in Constantin & Foias (Lemma 10.4):

$$|A^{s/2}B(\mathbf{u},\mathbf{v})| \le c|A^{s/2}\mathbf{u}||A^{(s+1)/2}\mathbf{v}| \qquad \text{for} \qquad s > d/2.$$

Theorem 8.1. *Suppose that $\mathbf{u}_0, \mathbf{f} \in H^s(Q) \cap V$ with $s > d/2$. Then any strong solution is in fact in $L^\infty(0,T;H^s(Q)) \cap L^2(0,T;H^{s+1}(Q))$.*

Proof. We first show that if

$$\mathbf{u} \in L^\infty(0,T;D(A^{(k-1)/2})) \cap L^2(0,T;D(A^{k/2}))$$

and $d/2 < k \leq s$ then

$$\mathbf{u} \in L^\infty(0,T;D(A^{k/2})) \cap L^2(0,T;D(A^{(k+1)/2})).$$

Take the inner product of

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{f}$$

with $A^k\mathbf{u}$. Then

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A^{k/2}\mathbf{u}|^2 + \nu|A^{(k+1)/2}\mathbf{u}|^2 \leq |A^{k/2}\mathbf{u}||A^{k/2}\mathbf{f}| + c|A^{k/2}\mathbf{u}|^2|A^{(k+1)/2}\mathbf{u}| \quad (8.29)$$

and so, using Young's inequality and the Poincaré inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}|A^{k/2}\mathbf{u}|^2 \leq c|A^{k/2}\mathbf{f}|^2 + c|A^{k/2}\mathbf{u}|^4.$$

The more general version of Gronwall's Lemma (see Exercise 2.1) now gives

$$|A^{k/2}\mathbf{u}(t)|^2 \leq |A^{k/2}\mathbf{u}_0|^2\mathrm{e}^{G(t)} + c|A^{k/2}f|^2\int_0^t \mathrm{e}^{G(t)-G(s)}\,\mathrm{d}s,$$

where

$$G(t) = \int_0^t |A^{k/2}\mathbf{u}(s)|^2\,\mathrm{d}s.$$

Since $\mathbf{u} \in L^2(0,T;D(A^{k/2}))$ it follows that $G(t) < \infty$ and so in fact $\mathbf{u} \in L^\infty(0,T;D(A^{k/2}))$; (8.29) then shows that $\mathbf{u} \in L^2(0,T;D(A^{(k+1)/2})$.

Now, observe that a strong solution has $\mathbf{u} \in L^2(0,T;D(A))$; in particular this gives $\mathbf{u} \in L^2(0,T;D(A^{2/2}))$, and since $2 > 3/2$ (and $2 > 2/2$) we can use induction to prove the result. □

Chapter 9

Global attractors for the NSE

We start this chapter by summarising the existence and uniqueness results of the previous chapter. When f is independent of time, the 2d Navier-Stokes equations can be used to generate a semi-dynamical system either on H (if $f \in V^*$) or on V (if $f \in H$). In the 3d case we have no way to define a dynamical system, since we cannot prove uniqueness of weak solutions, nor existence (for all time) of strong solutions.

The general result on the existence of global attractors that we proved in Chapter 3 only requires the existence of a compact absorbing set. We now show that such a set exists for the 2d Navier-Stokes equations.

9.1 A global attractor for the 2d NSE

We show the existence of an absorbing set in $\mathbb{L}^2$ and in $\mathbb{H}_0^1$. The first calculation is rigorous really should be justified using the Galerkin method (an "exercise").

9.2 Absorbing set bounded in $\mathbb{L}^2$

The existence of an absorbing set in $\mathbb{L}^2$ is straightforward.

Proposition 9.1. *For the 2d Navier-Stokes evolution equation there exists an absorbing set that is bounded in* $\mathbb{L}^2(\Omega)$.

[This result is also valid for the weak solutions in 3d that result as limits of the Galerkin procedure.]

Proof. We take the inner product of

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \nu A\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{f} \tag{9.1}$$

with $\mathbf{u}$ to obtain

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 + (B(\mathbf{u},\mathbf{u}),\mathbf{u}) = (\mathbf{f},\mathbf{u}).$$

Since $(B(\mathbf{u},\mathbf{u}),\mathbf{u}) = 0$ this gives

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 \le \|\mathbf{f}\|_{-1}|D\mathbf{u}|. \tag{9.2}$$

We now use Young's inequality on the right-hand side to write

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 \le \frac{\nu}{2}|D\mathbf{u}|^2 + \frac{1}{2\nu}\|\mathbf{f}\|_{-1}^2.$$

Tidying this up gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 \le \frac{1}{\nu}\|\mathbf{f}\|_{-1}^2, \tag{9.3}$$

and now if we use the Poincaré inequality on the $|D\mathbf{u}|$ term,

$$|D\mathbf{u}| \ge \lambda_1^{1/2}|\mathbf{u}|$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu\lambda_1|\mathbf{u}|^2 \le \frac{1}{\nu}\|\mathbf{f}\|_{-1}^2.$$

Gronwall's inequality gives

$$|\mathbf{u}(t)|^2 \le |\mathbf{u}_0|^2 \mathrm{e}^{-\nu\lambda_1 t} + \frac{\|\mathbf{f}\|_{-1}^2}{\nu\lambda_1}(1 - \mathrm{e}^{-\nu\lambda_1 t}).$$

So for t large enough (depending only on ϵ and $|\mathbf{u}_0|$)

$$|\mathbf{u}(t)|^2 \le \rho_0^2 := \frac{2\|\mathbf{f}\|_{-1}^2}{\nu\lambda_1}. \tag{9.4}$$

The number 2 could be replaced by $1+\epsilon$ for any $\epsilon > 0$. □

9.3 Absorbing set bounded in $\mathbb{H}^1$

The existence of an absorbing set in $\mathbb{H}^1$ is the crucial ingredient for proving the existence of a global attractor, and is (essentially) the missing estimate that would allow us to prove regularity for the 3d equations.

Proposition 9.2. *The 2d Navier-Stokes equations have an absorbing set that is bounded in* $\mathbb{H}^1(\Omega)$*.*

Before proving the proposition we make an additional estimate from

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 \le |\mathbf{f}||\mathbf{u}|.$$

(this was (9.2)). Integrating both sides between t and $t+1$ we obtain

$$|\mathbf{u}(t+1)|^2 + \int_t^{t+1} |D\mathbf{u}(s)|^2 \,\mathrm{d}s \le |\mathbf{u}(t)|^2 + |f| \int_t^{t+1} |\mathbf{u}(s)| \,\mathrm{d}s.$$

Since $|\mathbf{u}(t)| \le \rho_0$ for t large enough, for all such t we also have

$$\int_t^{t+1} |D\mathbf{u}(s)|^2 \,\mathrm{d}s \le I_1 := \rho_0^2 + |\mathbf{f}|\rho_0. \tag{9.5}$$

Although in the proof we use the orthogonality condition $b(\mathbf{u}, \mathbf{u}, A\mathbf{u}) = 0$, which is only valid for periodic boundary conditions in 2d, the same result (with a slightly more involved argument and weaker estimates) holds for Dirichlet boundary conditions.

Proof. To prove the existence of this absorbing set we use a 'trick', which can be formalised as the 'uniform Gronwall lemma' (see Temam (1988), for example, although the statement of this as a formal lemma tends to hide the underlying idea). We take the inner product of (9.1) with Au to give

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}|^2 + \nu|A\mathbf{u}|^2 + b(\mathbf{u},\mathbf{u},A\mathbf{u}) = (\mathbf{f}, A\mathbf{u}).$$

We now use an orthogonality condition (8.12), $b(\mathbf{u},\mathbf{u},A\mathbf{u}) = 0$ and the Cauchy-Schwarz inequality to rewrite this as

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}|^2 + \nu|A\mathbf{u}|^2 \leq \frac{|\mathbf{f}|^2}{2\nu} + \frac{\nu}{2}|A\mathbf{u}|^2. \tag{9.6}$$

Dropping the $|A\mathbf{u}|^2$ terms we have

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D\mathbf{u}|^2 \leq \frac{|\mathbf{f}|^2}{2\nu}.$$

We integrate this equation between s and $t+1$, with $t \leq s < t+1$, which gives

$$|D\mathbf{u}(t+1)|^2 \leq \frac{|\mathbf{f}|^2}{2\nu} + |D\mathbf{u}(s)|^2$$

(since $0 < t+1-s \leq 1$). We now integrate both sides with respect to s between $t-1$ and t, and obtain

$$|D\mathbf{u}(t+1)|^2 \leq \frac{|f|^2}{2\nu} + \int_t^{t+1} |D\mathbf{u}(s)|^2 \,\mathrm{d}s.$$

We can now use (9.5): if t is large enough then we have

$$|D\mathbf{u}(t+1)|^2 \leq \rho_1^2 := \frac{|\mathbf{f}|^2}{2\nu} + I_1.$$

□

Theorem 9.1. *If $\mathbf{f} \in H$ then the dynamical system on H generated by the 2D Navier-Stokes equations has a global attractor $\mathcal{A}_H$, and the solutions on $\mathcal{A}_H$ are strong solutions of the original equation.*

9.4 The 3d Navier-Stokes equations

At present we cannot show that the 3D Navier-Stokes equations generated unique weak solutions, nor could we show that the strong solutions, which are unique, exist for all time. Trying to investigate the existence of attractors without the guarantee of a sensible semigroup seems futile.

However, the result given here shows that if we are prepared to assume that the equations generate a semigroup on V, i.e. if we assume the existence of strong solutions, then we can show that the equations must have a global attractor. In fact the result here just shows the existence of an absorbing set bounded in V, and to show that there is a global attractor we would need an absorbing set that is compact in V. A relatively straightforward argument can be used to prove the existence of an absorbing set that is bounded in $D(A)$ once we have the absorbing set in V, and hence of a global attractor.

What we are doing here is making a physically reasonable assumption in a mathematically precise way, and then deducing an entirely mathematical consequence. It allows us to consider the asymptotic regimes of the "true" Navier-Stokes equations, and so fully-developed turbulence, within a mathematical framework.

Another way to view this theorem, which does not require us to make any "unjustified" assumptions, is as a description of the way in which the 3D Navier-Stokes equations must break down if they are not well-posed. The theorem shows that existence and uniqueness fail only if there is some solution $u(t)$ such that $|Du(t)|$ becomes infinite in some finite time.

Theorem 9.2. *Suppose that the 3d Navier-Stokes equations are well-posed on V, so that for any $\mathbf{f} \in H$ and $\mathbf{u}_0 \in V$,*

$$\mathrm{d}\mathbf{u}/\mathrm{d}t + A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f}$$

has a strong solution $\mathbf{u}(t)$, i.e. a solution $\mathbf{u}$ with

$$\mathbf{u} \in L^\infty(0, T; V) \cap L^2(0, T; D(A))$$

for all $T > 0$. Then there exists an absorbing set in V.

For a proof see Theorem 12.10 in [R].

Chapter 10

Finite-dimensional attractors

10.1 Fractal (box-counting) dimension

The "fractal" dimension, which we will write as $d_f(X)$, is based on counting the number of closed balls of a fixed radius ϵ needed to cover X.

We denote the minimum number of balls in such a cover by $N(X, \epsilon)$. If X were a line, we would expect $N(X, \epsilon) \sim \epsilon^{-1}$, if X a surface we would have $N(X, \epsilon) \sim \epsilon^{-2}$, and for a (3-) volume we would have $N(X, \epsilon) \sim \epsilon^{-3}$. So one possible method for obtaining a general measure of dimensions would be to say that X has dimension d if $N(X, \epsilon) \sim \epsilon^{-d}$. Accordingly, we make the following definition. (The fractal dimension is also known as the (upper) box-counting dimension[1] and the entropy dimension.)

Definition 10.1. *The* fractal dimension *of X, $d_f(X)$ is given by*

$$d_f(X) = \limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{\log(1/\epsilon)}, \tag{10.1}$$

[1] At least in the finite-dimensional spaces $\mathbb{R}^d$ definition 10.1 is equivalent to the "upper box-counting dimension". The idea here is take a grid of cubes of side ϵ which cover $\mathbb{R}^d$, and let $N(X, \epsilon)$ be the number of cubes which intersect X. The dimension is then defined exactly as in 10.1. Although these definitions coincide in $\mathbb{R}^d$, the "unit cube" $[0, 1]^\infty$ in an infinite-dimensional Hilbert space has elements with arbitrarily large norm, so one cannot "count boxes" in this context.

where we allow the limit in (10.1) to take the value $+\infty$.

Note that it follows from the definition that if $d > d_f(X)$, then for sufficiently small ϵ,

$$N(X, \epsilon) \leq \epsilon^{-d}. \tag{10.2}$$

The fractal dimension has the following properties:

Proposition 10.1. *(Properties of fractal dimension).*

(i) Stability under finite unions:

$$d_f\Big(\bigcup_{k=1}^{N} X_k\Big) \leq \max_k \; d_f(X_k), \tag{10.3}$$

(ii) if $f : H \to H$ *is Hölder continuous with exponent* θ*, i.e.*

$$|f(x) - f(y)| \leq L|x - y|^{\theta}, \tag{10.4}$$

then $d_f(f(X)) \leq d_f(X)/\theta$*,*

(iii) $d_f(X \times Y) \leq d_f(X) + d_f(Y)$*, and*

(iv) if $\overline{X}$ *is the closure of* X *in* H*, then* $d_f(\overline{X}) = d_f(X)$*,*

See Proposition 13.2 in [R] for a proof.

10.2 Dimension estimates

There is an analytical method by which we can estimate the fractal dimension of the global attractor. The idea is to study the evolution of infinitesimal n-dimensional volumes as they evolve under the flow, and try to find the smallest dimension n at which we can guarantee that all such n-volumes contract asymptotically. We will not give the analysis in detail, but merely in outline.

We will consider an abstract problem, written as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u(t)) \qquad u(0) = u_0,$$

with u_0 contained in a Hilbert space H, whose norm we denote by $|\cdot|$. We assume that the equation has unique solutions given by $u(t;u_0) = S(t)u_0$, and a compact global attractor $\mathcal{A}$.

We want to start off with an orthogonal set of infinitesimal displacements near an initial point $u_0 \in \mathcal{A}$, and then watch how the volume they form evolves under the flow.

To study the evolution of this volume we have to study the evolution of a set of infinitesimal displacements $\delta x^{(i)}(t)$ about the trajectory $u(t)$. We suppose that the evolution of these displacements is given by the linearised equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = F'(u(t))U(t) \qquad U(0) = \xi,$$

which we write as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = L(t;u_0)U(t) \qquad U(0) = \xi. \tag{10.5}$$

The validity of such a linearisation is one of the main points to check when applying this theory rigorously. To this end we make the following definition:

Definition 10.2. *We say that $S(t)$ is* uniformly differentiable on $\mathcal{A}$ *if for every $u \in \mathcal{A}$ there exists a linear operator $\Lambda(t,u)$, such that, for all $t \geq 0$,*

$$\sup_{u,v\in\mathcal{A};\,0<|u-v|\leq\epsilon} \frac{|S(t)v - S(t)u - \Lambda(t,u)(v-u)|}{|v-u|} \to 0 \text{ as } \epsilon \to 0 \tag{10.6}$$

$$\text{and} \qquad \sup_{u\in\mathcal{A}} \|\Lambda(t,u)\|_{\mathrm{op}} < \infty \qquad \textit{for each } t \geq 0.$$

Although this is straightforward to check for ordinary differential equations, its proof in the PDE will often involve technical difficulties.

Heuristically speaking the growth rate of each infinitesimal displacement $\delta x^{(j)}$ will be related to the eigenvalues of L. In particular, the length of an infinitesimal displacement in the λ-eigendirection is $e^{\lambda t}\delta(0)$ at time t: the growth rate is λ, attached to the eigen-direction of L with eigenvalue λ. The size of a small 2-volume with sides in two different eigendirections would be

$$e^{(\lambda_1+\lambda_2)t},$$

and so the growth "rate" is $\lambda_1 + \lambda_2$. The growth rate of an n-volume made of infinitesimal directions in n eigendirections would be

$$\lambda_1 + \ldots + \lambda_n.$$

If we can makes sure that this growth rate must be negative then we know that n-volumes contract. It is true, although by no means immediate, that if all n-volumes contract then the dimension of the attractor must be smaller than n. In the finite-dimensional setting this result is due to Douady & Oesterlé (1980), while in the infinite-dimensional case it was proved by Constantin & Foias (1985) and Constantin et al. (1985).

To extract the "growth rate", we consider

$$\sum_{j=1}^{n}(\phi_j, L(t;u_0)\phi_j), \tag{10.7}$$

over all possible orthonormal collections of n elements $\{\phi_j\}_{j=1}^n$ of H. The idea is, essentially, that the maximum over all choices of ϕ_j gives the largest possible growth rate, i.e. the sum of the n largest eigenvalues of L. A more compact notation for (10.7) is

$$\mathrm{Tr}(L(t;u_0)P),$$

where Tr denotes the trace in H and P is the orthogonal projection onto the space spanned by the $\{\phi_j\}_{j=1}^n$.

The following theorem is given in a form suitable for calculations. We denote by $\langle f\rangle$ the time average of $f(t)$, namely

$$\langle f\rangle = \limsup_{t\to\infty}\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s.$$

Theorem 10.2. *Suppose that $S(t)$ is uniformly differentiable on $\mathcal{A}$, and there exists a t_0 such that $\Lambda(t, u_0)$ is compact for all $t \geq t_0$. If for some n we have*

$$\sup_{u_0 \in \mathcal{A}} \left\langle \sup_{\{\phi_j\}_{j=1}^n} \sum_{j=1}^n (\phi_j, L(u(t))\phi_j) \right\rangle < 0$$

for any choice of n orthogonal elements $\{\phi_j\}$ of H, then $d_f(\mathcal{A}) < n$. [Here $u(t)$ is the solution through u_0.]

A proof, due to Hunt, is given in Appendix B in [R].

Before we apply this result to give dimension bounds for the Navier-Stokes equations, we will need an auxiliary lemma which gives a lower bound on

$$\sum_{j=1}^n (\phi_j, -\Delta\phi_j),$$

valid for any choice of n orthogonal elements of H.

Indeed, we use exactly this idea in the next lemma, which gives a bound on this quantity over all choices of ϕ_j when $L(t; u_0) = -\Delta$.

Lemma 10.3. *For any choice of n orthogonal elements $\{\phi_j\}_{j=1}^n$ of $\mathbb{L}^2(Q)$,*

$$\sum_{j=1}^n |D\phi_j|^2 = \sum_{j=1}^n (\phi_j - \Delta\phi_j) \geq CL^{-d} n^{(d+2)/d}. \tag{10.8}$$

The same result is also valid for any bounded C^2 domain $\Omega \subset \mathbb{R}^d$.

Proof. Write $A = -\Delta$, and denote its orthonormal eigenfunctions as w_j, with corresponding eigenvalues λ_j ordered so that $\lambda_{j+1} \geq \lambda_j$. We show first that

$$\sum_{j=1}^n (\phi_j, A\phi_j) \geq \sum_{j=1}^n \lambda_j. \tag{10.9}$$

We rewrite the left-hand side of (10.9), expanding the ϕ_j in terms of the eigenbasis $\{w_j\}$, and obtain

$$\mathrm{Tr}(AP_n) \quad = \quad \sum_{j=1}^n \sum_{k=1}^\infty \lambda_k |(\phi_j, w_k)|^2$$

$$= \sum_{k=1}^{\infty} \lambda_k \Big(\sum_{j=1}^{n} |(\phi_j, w_k)|^2 \Big).$$

Now, since $|\phi_j| = 1$ we have

$$\sum_{j=1}^{n} \sum_{k=1}^{\infty} |(w_k, \phi_j)|^2 = n,$$

and since $\{\phi_j\}$ are orthonormal but do not span H we have

$$\sum_{j=1}^{n} |(w_k, \phi_j)|^2 \leq 1.$$

(10.9) now follows.

The explicit bound in (10.8) follows easily, using the property of the eigenvalues of the Laplacian

$$cj^{2/d} \leq \lambda_j \leq Cj^{2/d},$$

since then

$$\sum_{j=1}^{n} \lambda_j \geq c \sum_{j=1}^{n} j^{2/d} \geq cn^{(2/d)+1} = cn^{(d+2)/d}.$$

□

10.3 Dimension estimate for the 2d NSE

We first state the differentiability result.

Theorem 10.3. *The solutions of the Navier-Stokes equations in 2d satisfy (10.6) with* $\Lambda(t; \mathbf{u}_0)\xi$ *the solution of the equation*

$$\frac{\mathrm{d}U}{\mathrm{d}t} + \nu AU + B(\mathbf{u}, U) + B(U, \mathbf{u}) = 0 \qquad U(0) = \xi. \tag{10.10}$$

Furthermore, $\Lambda(t; \mathbf{u}_0)$ *is compact [maps bounded sets into compact sets] for all* $t > 0$*.*

See Theorem 13.20 in [R] for a proof.

With the differentiability ensured we can apply the trace formula to find a bound on the dimension.

Theorem 10.4. *The attractor for the 2d periodic Navier-Stokes equations is finite dimensional, with*

$$d_f(\mathcal{A}) \leq \frac{c}{\lambda_1^{1/2}\nu}\langle |D\mathbf{u}|_{L^2}^2\rangle^{1/2}. \tag{10.11}$$

The result, due to Constantin et al. (1985), is also valid as stated for Dirichlet boundary conditions.

Proof. The correct form of the linearised equation is given in (10.10), and so

$$L(\mathbf{u})\mathbf{w} = \nu A\mathbf{w} - B(\mathbf{w},\mathbf{u}) - B(\mathbf{u},\mathbf{w}).$$

Thus the time-averaged trace $\langle P_n L(\mathbf{u}(t))\rangle$ is bounded by

$$\begin{aligned}\langle P_n L(\mathbf{u})\rangle &= \Big\langle \sum_{j=1}^n (L(\mathbf{u})\phi_j,\phi_j)\Big\rangle \\ &= -\Big\langle \sum_{j=1}^n (-\nu\Delta\phi_j,\phi_j)\Big\rangle - \Big\langle \sum_{j=1}^n b(\phi_j,u,\phi_j)\Big\rangle.\end{aligned}$$

In order to bound the contribution from the nonlinear term we could proceed using the standard bound on $b(\mathbf{u},\mathbf{v},\mathbf{w})$ in (8.4), which leads to an estimate similar to (10.11) but with a factor of ν^2 rather than ν.

However, a better estimate can be obtained as follows. Note that we have

$$\sum_{j=1}^n b(\phi_j,\mathbf{u},\phi_j) = \int_\Omega \sum_{j=1}^n \sum_{i,k=1}^2 \phi_{ji}(\mathbf{x})\frac{\partial u_k}{\partial x_i}(\mathbf{x})\phi_{jk}(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

and we have for each $\mathbf{x} \in Q$

$$\left|\sum_{i,k=1}^2 \phi_{ji}(\mathbf{x})\frac{\partial u_k}{\partial x_i}(\mathbf{x})\phi_{jk}(\mathbf{x})\right| \leq \left(\sum_{i=1}^2\sum_{j=1}^n \phi_{ji}(\mathbf{x})^2\right)\left(\sum_{i,k=1}^2 \left|\frac{\partial u_k}{\partial x_i}(\mathbf{x})\right|^2\right)^{1/2}.$$

It follows use the Cauchy-Schwarz inequality that

$$|\sum_{j=1}^{n} b(\phi_j, \mathbf{u}, \phi_j)| \leq |D\mathbf{u}||\rho|_{L^2},$$

where

$$\rho(\mathbf{x}) = \sum_{i=1}^{2}\sum_{j=1}^{n} \phi_{ji}(\mathbf{x})^2.$$

An inequality due to Lieb & Thirring (1976), adapted appropriately to this case (details are given in Temam, 1988) allows us to bound $|\rho|_{L^2}$ by

$$|\rho|_{L^2}^2 \leq c \sum_{j=1}^{n} |D\phi_j|^2.$$

It follows that

$$P_n L(\mathbf{u}) \leq -\nu \sum_{j=1}^{n} |D\phi_j|^2 + c|D\mathbf{u}| \left(\sum_{j=1}^{n} |D\phi_j|^2 \right)^{1/2},$$

and so, using the Cauchy-Schwarz inequality we obtain

$$P_n L(\mathbf{u}) \leq -\frac{\nu}{2} \sum_{j=1}^{n} |D\phi_j|^2 + \frac{c}{\nu}|D\mathbf{u}|^2.$$

Now taking the time average and using lemma 10.3 we obtain

$$\langle P_n L(\mathbf{u})\rangle \leq -\frac{\nu}{2} n^2 + \frac{c}{\nu}\langle |D\mathbf{u}|\rangle^2.$$

We therefore have $\langle P_n L(\mathbf{u})\rangle < 0$ provided that $n > \lambda_1^{-1/2}\nu^{-1}\langle |D\mathbf{u}|^2\rangle^{1/2}$ as claimed. □

In order to make the dimension estimate more explicit, we define the dimensionless Grashof number G, which measures the relative strength of the forcing and viscosity by

$$G = \frac{|f|_{L^2}}{\nu^2 \lambda_1},$$

and estimate $\langle |D\mathbf{u}|^2 \rangle$ in terms of G.

Returning to the equation (9.2) that we obtained on the way to finding an absorbing set in $\mathbb{L}^2$,

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 \leq |\mathbf{f}||\mathbf{u}| \tag{10.12}$$

we can use the Poincaré inequality and Cauchy-Schwarz inequalities on the right-hand side to obtain

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 \leq \lambda_1^{-1/2}|\mathbf{f}||D\mathbf{u}| \leq \frac{|\mathbf{f}|^2}{2\nu\lambda_1} + \frac{\nu}{2}|D\mathbf{u}|^2,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 \leq \frac{|\mathbf{f}|^2}{\nu\lambda_1}.$$

Integrating between 0 and t we obtain

$$\nu\int_0^t |D\mathbf{u}(s)|^2\,\mathrm{d}s \leq \big(|\mathbf{u}(0)|^2 - |\mathbf{u}(t)|^2\big) + \frac{t|\mathbf{f}|^2}{\nu\lambda_1},$$

which gives

$$\limsup_{t\to\infty} \langle |D\mathbf{u}|^2 \rangle \leq \frac{|\mathbf{f}|^2}{\nu^2\lambda_1}.$$

Therefore

$$d_f(\mathcal{A}) \leq \frac{c}{\lambda_1^{1/2}\nu}\frac{|\mathbf{f}|}{\nu\lambda_1^{1/2}} = c\frac{|\mathbf{f}|}{\nu^2\lambda_1} = cG.$$

We have shown:

Theorem 10.5. *The global attractor for the 2d Navier-Stokes equations is finite-dimensional, and*

$$d_f(\mathcal{A}) \leq cG.$$

This is the best bound in the case of Dirichlet boundary conditions. By working with the equation for $\omega = \nabla\wedge\mathbf{u}$ and using the identity $(B(\mathbf{u},\mathbf{u}), A\mathbf{u}) = 0$, Constantin et al. (1988) were able to improve this for periodic boundary conditions to

$$d_f(\mathcal{A}) \leq cG^{2/3}(1 + \log G)^{1/3}. \tag{10.13}$$

This bound is known to be sharp (see Babin & Vishik (1988), Liu (1993), Ziane (1997)).

10.4 Reflecting the scales in the forcing

Note that G reflects only the amount of energy being put into the flow, and says nothing of the scales at which the energy is supplied. The following simple calculation, inspired by the paper of Olson & Titi (2003), shows that it is in fact possible to improve on the estimate in the Dirichlet boundary condition case, and on the estimate (10.13) in the periodic case when the forcing is at very small scales, by making a small modification to the above argument (see Robinson 2003). If we estimate $\langle |D\mathbf{u}|^2 \rangle$ from

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}|^2 + \nu|D\mathbf{u}|^2 \leq \frac{\|\mathbf{f}\|_{-1}^2}{\nu}.$$

(this was (9.3)) rather than from (10.12) as above, then we obtain (the argument is similar)

$$\langle |D\mathbf{u}|^2 \rangle \leq \frac{\|\mathbf{f}\|_{-1}^2}{\nu^2} = \lambda_1 \nu^2 G_*^2,$$

where now we have defined an alternative Grashof number G_* based on $\|\mathbf{f}\|_{-1}$,

$$G_* := \frac{\|\mathbf{f}\|_{-1}}{\nu^2 \lambda_1^{1/2}}.$$

It follows that

$$d_f(\mathcal{A}) \leq cG_*.$$

Exercise 10.1. *Use the Fourier expansion of* $\mathbf{f}$ *to show that* $\|\mathbf{f}\|_{-1} \leq \lambda_1^{-1/2}|\mathbf{f}|_{L^2}$, *and hence that* $G_* \leq G$.

10.4.1 The 'effective lengthscale' of the forcing

If we expand $\mathbf{f}$ as a Fourier series,

$$\mathbf{f} = \sum_{\mathbf{k}} \hat{\mathbf{f}}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

then

$$|\mathbf{f}|_{L^2}^2 = \sum_{\mathbf{k}} |\hat{\mathbf{f}}_{\mathbf{k}}|^2 \qquad \text{and} \qquad \|\mathbf{f}\|_{-1}^2 = \sum_{\mathbf{k}} \frac{|\hat{\mathbf{f}}_{\mathbf{k}}|^2}{|\mathbf{k}|^2}.$$

It follows that

$$\frac{\|\mathbf{f}\|_{-1}^2}{|\mathbf{f}|^2} = \sum_{\mathbf{k}} |\mathbf{k}|^{-2} \frac{|\hat{\mathbf{f}}_{\mathbf{k}}|^2}{\sum_{\mathbf{j}} |\hat{\mathbf{f}}_{\mathbf{j}}|^2}$$

The ratio $\|\mathbf{f}\|_{-1}^2/|\mathbf{f}|^2$ appears as an average of squared lengthscales, weighted according to the amount of energy injected at each scale. Accordingly, it is natural to define an effective lengthscale of the forcing by

$$l_{\text{eff}} = \|\mathbf{f}\|_{-1}/|\mathbf{f}|,$$

in which case the relationship

$$G_* = \frac{l_{\text{eff}}}{l_{\text{ref}}} G \tag{10.14}$$

is essentially a tautology.

This implies that the dimension of the attractor is less than one if the forcing is applied at sufficiently small scales. Since the attractor is a compact connected set, this in fact implies that the attractor is a point (Falconer, 1985), so that the dynamics is trivial (no matter how 'complicated' the forcing is at these scales).

10.5 Physical interpretation of the attractor dimension

One way of interpreting the physical significance of an attractor is as a means of giving a rigorous notion of the number of independent "degrees of freedom" of the asymptotic dynamics of the system.

One way of making this precise is the following result. Essentially it says that if k is large enough (roughly twice the dimension of the attractor), then the attractor can be embedded into $\mathbb{R}^k$. This result can be considered in two ways. Firstly, the idea is that we can take the attractor "out" of the infinite-dimensional space and map it, using some linear map L, homeomorphically onto a subset of $\mathbb{R}^k$. This makes sense of the idea that $\mathcal{A}$ is a finite-dimensional set. Secondly, and perhaps more importantly, it follows that L^{-1} provides a way of parametrising the attractor using a finite set of coordinates.

10.5.1 Embedding subsets of $\mathbb{R}^N$

First we give a proof of the corresponding result for finite-dimensional subsets of finite-dimensional spaces.

Theorem 10.6. *If X is a compact subset of $\mathbb{R}^N$ with $d_f(X) \leq d$ ($d \in \mathbb{N}$) then there exists an orthogonal projection P, of rank $2d+2$, which is injective on X.*

The theorem is a corollary of the following result.

Proposition 10.7. *If Y is a compact subset of $\mathbb{R}^N$ with $d_f(Y) \leq k \leq N-3$ then there exists an orthogonal projection P, of rank $k+2$, with*

$$\operatorname{Ker} P \cap Y = \{0\}.$$

Given this proposition, the proof of Theorem 10.6 is straightforward:

Proof (Theorem 10.6). Let

$$Y = X - X = \{x_1 - x_2 : \ x_1 \in X, \ x_2 \in X\}.$$

Then since $d_f(X) \leq d$, we have $d_f(Y) \leq 2d$ (using Proposition 10.1). By Proposition 10.7 there is an orthogonal projection P, of rank $2d+2$, with

$$\ker P \cap Y = \{0\}.$$

Thus, if $x_1, x_2 \in X$, $P(x_1 - x_2) = 0$ implies that $x_1 - x_2 = 0$, i.e.

$$Px_1 = Px_2 \qquad \text{implies that} \qquad x_1 = x_2.$$

so P is injective on X. □

To prove Proposition 10.7 we need the following simple lemma.

Lemma 10.8. *Let Y be a compact subset of $\mathbb{R}^N$ with $d_f(Y) \leq N-3$. Then there exists a unit vector b such that $b \notin \mathbb{R}Y$ (i.e. $b \neq \alpha y$ for any $\alpha > 0$, $y \in Y$).*

Proof (Lemma 10.8). $d_f(\mathbb{R}Y) \leq N-2$, but $d_f(\{b : |b| = 1\}) = N-1$. □

Proof (Proposition 10.7). By Lemma 10.8, there exists a unit vector $a \notin \mathbb{R}Y$. Then if

$$Q_1 = I - aa^*, \qquad \text{where} \qquad (aa^*)x = a(a,x),$$

suppose that $y \in \ker Q_1 \cap Y$; this means that

$$(a,y)a = y.$$

But $a \notin \mathbb{R}Y$, so $y = 0$. Thus

$$\ker Q_1 \cap Y = \{0\}.$$

We now proceed by induction. Suppose that we have an orthogonal projector Q_m of rank $N-m$, with

$$\ker Q_m \cap Y = \{0\}$$

and $k \leq N-m-3$. We apply Lemma 10.8 to $Q_m\mathbb{R}^N \simeq \mathbb{R}^{N-m}$, so that there is a unit vector $a \notin \mathbb{R}[Q_mY]$, and consider

$$Q_{m+1} = Q_m - aa^*.$$

Again, if $y \in \ker Q_{m+1} \cap Y$, we have

$$Q_m y = (a,y)a,$$

and as before this implies that $y = 0$, so that

$$\ker Q_{m+1} \cap Y = \{0\}.$$

Thus we obtain, by induction, a projection as in the statement. □

10.5.2 Embedding subsets of H

Theorem 10.4. *Let X be a compact subset of H, with $d_f(X) < d$, d an integer, and let $k \geq 2d+1$. Then if L_0 is a bounded linear map into $\mathbb{R}^k$, for any $\epsilon > 0$ there exists another bounded linear map into $\mathbb{R}^k$, $L = L(\epsilon)$, such that L is injective on X and*

$$\|L - L_0\|_{\text{op}} \leq \epsilon.$$

Corollary 10.5. *Let X be a compact subset of H, with $d_f(X) < d$, d an integer, and take $k \geq 2d+1$. Then there exists a continuous parametrisation of X using k coordinates.*

Proof. To show that the inverse is continuous, suppose not. Then there exists an $\epsilon > 0$ and a sequence $\{x_n\} \in L(X)$ with $x_n \to y \in L(X)$ but $|L^{-1}(x_n) - L^{-1}(y)| \geq \epsilon$. However, $L^{-1}(x_n) \in X$, and since X is compact there exists a subsequence x_{n_j} such that $L^{-1}(x_{n_j}) \to z$. Since L is continuous (it is a bounded linear map), it follows that $x_{n_j} \to L(z)$. Since L is injective, it follows from $L(z) = y$ that $z = L^{-1}(y)$, which is a contradiction. So L^{-1} is continuous on $L(X)$. □

10.6 Physical implications

To obtain a more physical conclusion from the attractor dimension, we assume that it is a good indication of the number of "degrees of freedom", and use this to relate $d_f(\mathcal{A})$ to a possible fundamental length-scale of the original problem.

Assume that the original equation is posed on some domain $\Omega \subset \mathbb{R}^d$, with volume $|\Omega|$. Suppose that there is a smallest physically relevant length-scale l in the problem, the idea being that interactions on scales of less than l do not affect the dynamics (for example, in fluid mechanics the viscosity has a large effect on the very small scales, and we hope that this means that fluctuations on these scales have negligible effects). A heuristic indication of the number of degrees of freedom would then given by how many "boxes" of side l fit into Ω,

$$n_{\text{heuristic}} \sim |\Omega| l^{-m}.$$

If we assume that $n_{\text{heuristic}}$ is a good estimate of the true number of degrees of freedom, and in turn (!) that this is well estimated by the attractor dimension, we can isolate a length scale, given in terms of $d_f(\mathcal{A})$ by

$$l \sim \left(\frac{|\Omega|}{d_f(\mathcal{A})} \right)^{1/m}. \tag{10.15}$$

Notice that tighter bounds on $d_f(\mathcal{A})$ raise the estimate of the smallest length scale. The best current estimates of the attractor dimension for the 2d NSE are

$$d_f(\mathcal{A}) \leq cG^{2/3}(1 + \log G)^{1/3} \tag{10.16}$$

in the case of periodic boundary conditions, and

$$d_f(\mathcal{A}) \leq cG$$

in the case of Dirichlet boundary conditions, both due to Constantin, Foias & Temam.

What is remarkable about the bound in (10.16) is that, using our "very heuristic" estimate in (10.15), it corresponds to a length scale l which satisfies

$$\frac{l}{L} \sim G^{-1/3}, \tag{10.17}$$

to within logarithmic corrections (L denotes the size of one side of our 2D periodic domain). The length scale in (10.17) is precisely the "Kraichnan length", derived by other (also heuristic) methods as the natural minimum scale in two-dimensional turbulent flows (see problems). This links the rigorous analytical bound on the attractor dimension with an "intuitive" estimate from fluid dynamics.

Exercise 10.2. *The Kraichnan length scale L_χ is the only quantity with dimensions of length which can be formed from the viscous enstrophy dissipation χ and the viscosity ν. The quantity χ is defined by*

$$\chi = \frac{\nu}{L^2} \limsup_{t\to\infty} \sup_{u_0\in\mathcal{A}} \frac{1}{t}\int_0^t |A\mathbf{u}(s)|^2\,ds$$

(it is the rate of decrease of $|\nabla \mathbf{u}|^2$). Use the estimate in (9.6) to show that

$$\chi \leq \nu^3 L^{-6} G^2,$$

and hence that L_χ satisfies

$$\frac{L_\chi}{L} \geq G^{-1/3}. \tag{10.18}$$

Compare this with (10.17).