An introduction to the classical theory of the Navier–Stokes equations

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Introduction

These notes are based on lectures give at a Summer School held in Campinas, Brazil, in January 2010. I am very grateful to Helena Nussenzveig Lopez and Milton Lopez Filho for the opportunity to visit them in Campinas, and to deliver these lectures there.

The intention in these notes is to give a relatively accessible overview of three major results: the existence for all time of weak solutions (due to Leray, 1934, and Hopf, 1951); the local regularity result of Serrin (1962); and the partial regularity theory of Caffarelli, Kohn, & Nirenberg (1982). The full details of the proofs are not given, but I hope that sufficient indication is given of how the proofs proceed to at least make clear the way their arguments work.

The notes give a fairly complete proof of the existence of weak solutions, and sketch a proof of the existence of strong solutions and some of their properties (Chapter 1). For much of this material I relied closely on the very nice lecture notes of Galdi (2000), with support from the books by Temam (2001) and Constantin & Foias (1988).

Chapter 2 covers Serrin's local regularity theory – his paper relies on much external material, in particular regularity theory for parabolic equations. I give an elliptic version of the parabolic results, closely following the parabolic argument given in Lieberman (1996), which does not rely on making fine estimates on the heat kernel. The proof of Campanato's Lemma that occurs in this chapter is based on online notes by Tobias Colding; and the simple proof of Young's inequality using Hölder's inequality comes from the book by Grafakos (2008; his Theorem 1.2.12).

The final chapter discusses the partial regularity theory of Caffarelli, Kohn, & Nirenberg, using a very recent argument due to Kukavica (2009a); I show in detail the proof of about half of one of the two theorems proved there which lead to the result that the dimension of the singular set is no larger than one; but I hope the details are sufficient to give a proper flavour of the proof as a whole.

It should be possible to fill in all the gaps here with the help of Galdi's notes, Constantin & Foias (1988), Serrin (1962), Lieberman (1996), Majda & Bertozzi (2002), Caffarelli, Kohn, & Nirenberg (1982), and Kukavica (2009a). Of these, the key references for me have been Galdi's notes, Serrin's paper, and Kukavica's recent work.

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Some minimal functional analysis

Hölder's inequality: if $f \in L^p$ and $g \in L^q$ with

$$\frac{1}{p} + \frac{1}{q} = 1$$

then $fg \in L^1$ with

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

We will often need the generalised form of Hölder's inequality with three exponents: if $f \in L^p$, $g \in L^q$, and $h \in L^r$ with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

then $fgh \in L^1$ with

$$\|fgh\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.$$

One nice application of Hölder's inequality is to interpolation between L^p spaces. A particular case that we will use frequently is that if $u \in L^2 \cap L^6$ then $u \in L^3$ with

$$\|u\|_{L^3} \leq \|u\|_{L^2}\|u\|_{L^6}. \tag{0.1}$$ L3L2L6

(To prove this, write $\|u\|_{L^3}^3 = \int |u|^3 = \int |u|^{3/2}|u|^{3/2}$, and use Hölder's inequality wityh exponents 4/3 and 4).

This will be particularly useful, since in 3D we have the Sobolev embedding result $H^1 \subset L^6$, i.e. if $u \in H^1$ then

$$\|u\|_{L^6} \leq c\|u\|_{H^1}. \tag{0.2}$$ L6H1

For functions in $H_0^1(\Omega)$, where Ω is a domain contained in a strip of width $2R$, we have the Poincaré inequality

$$\int_\Omega |u|^2 \,\mathrm{d}x \leq cR^2 \int_\Omega |Du|^2, \mathrm{d}x. \tag{0.3}$$ P1

We also have a Poincaré-like inequality on a ball,

$$\int_{B_R(x)} |u - \langle u \rangle_{B_R(x)}|^2 \leq cR^2 \int_{B_R(x)} |Du|^2, \tag{0.4}$$ P2

where here $\langle u \rangle_{B_R(x)}$ is the average of u over $B_R(x)$,

$$\langle u \rangle_{B_R(x)} = \frac{1}{\omega_n R^n} \int_{B_R(x)} u(y)\,\mathrm{d}y.$$

If X is a Banach space, any bounded sequence in its dual with have a weakly-* convergent subsequence. Any bounded sequence in a reflexive Banach space (in particular in a Hilbert space) will have a weakly convergent subsequence.

If $x_n \rightharpoonup x$ then $\|x\| \le \liminf_{n\to\infty} \|x_n\|$ (and similarly if f_n converges weakly-* to f).

In a Hilbert space, if we have weak convergence and 'norm convergence', $\|x_n\| \to \|x\|$, then in fact $x_n \to x$: this follows from the identity

$$(x_n - x, x_n - x) = \|x_n\|^2 - 2(x, x_n) + \|x\|^2.$$

1

Existence of weak solutions

We will study the incompressible Navier–Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \qquad \operatorname{div} u = 0 \tag{1.1}$$ NSE

in a smooth, bounded, domain $\Omega \subset \mathbb{R}^3$, with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$ and a given initial condition $u(x,0) = u_0(x)$.

Generally speaking the equations include a viscosity coefficient in front of the Laplacian,

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = 0, \tag{1.2}$$ NSEnu

but if we are interested in questions of existence and uniqueness, we lose nothing by restricting to $\nu = 1$. Indeed, if we have a solution $u(x,t)$ of (1.2) and we consider

$$u_\nu(x,t) = \nu u(x, \nu t),$$

it is easy to see that $u_\nu(x,t)$ satisfies (1.1). So, for example, if we could prove the existence and uniqueness of solutions of (1.1) for all positive times, we would have the same result for (1.2) for any $\nu > 0$.

We begin with some simple estimates, which lie at the heart of the existence of weak solutions. Take an initial condition $u_0(x)$ with

$$\int_\Omega |u_0|^2 \, \mathrm{d}x < \infty$$

(this corresponds to finite kinetic energy) and suppose that the equations have a classical solution on $[0,T]$. Then we can take the dot product of the equations with u and integrate over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = 0. \tag{1.3}$$ energyDequality

We obtain the second term after integrating by parts and using the boundary condition $u = 0$ on $\partial\Omega$. The nonlinear term vanishes – this is a particular case of the more general identity

$$\int_\Omega [(u \cdot \nabla)v] \cdot v = 0, \tag{1.4}$$ ortho

which holds provided that u has divergence zero and one of u and v vanishes on $\partial\Omega$; we will use this many times. In fact this follows from the useful antisymmetry identity

$$\int_\Omega [(u \cdot \nabla)v] \cdot w = -\int_\Omega [(u \cdot \nabla)w] \cdot v. \tag{1.5}$$ antisymm

To obtain this identity, write the integral more explicitly in components and integrate by parts

$$\int_\Omega u_i(\partial_i v_j)w_j = -\int_\Omega (\partial_i u_i)v_j w_j - \int_\Omega u_i v_j(\partial_i w_j) = -\int_\Omega u_i(\partial_i w_j)v_j.$$

The pressure term has also vanished, since

$$\int_\Omega \nabla p \cdot u = -\int_\Omega p(\nabla \cdot u) = 0,$$

using the fact that u is divergence free (and the $u = 0$ boundary condition).

We can integrate (1.3) in time to give an energy equality

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t \|Du(s)\|^2\,\mathrm{d}s = \frac{1}{2}\|u_0\|^2, \tag{1.6}$$ EE

where now we are using the notation $\|u\|$ for the $L^2(\Omega)$ norm of u.

This 'formal' calculation (we are not careful, and assume that everything is as regular as it needs to be for all our manipulations to be justified), leads us to expect that if we have an initial condition in L^2, we will obtain a solution that is bounded in L^2, and whose H^1 norm is square integrable. We write

$$u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1). \tag{1.7}$$ Li2L2H

More generally, the notation $u \in L^p(0,T;X)$, where X is a Banach space, means that the map $t \mapsto \|u(t)\|_X$ is in L^p,

$$\int_0^T \|u(t)\|_X^p \,\mathrm{d}t < \infty.$$

If we equip the space $L^p(0,T;X)$ with the natural L^p norm it is a Banach space.

A weak solution will be a function u with the regularity in (1.7) that satisfies the Navier–Stokes equations in a weak sense, which we now make precise. We will show that there exists at least one weak solution that also satisfies an energy inequality, i.e. (1.6) with the equality replaced by $\leq$.

1.1 Weak formulation of the equation

Take a function $\varphi(x,t) \in C_0^\infty([0,T) \times \Omega)$, i.e. a smooth function with compact support in $[0,T) \times \Omega$. Multiply (1.1) by φ and integrate over the space-time domain $[0,T) \times \Omega$. If we do this we obtain

$$\int_0^T (u, \varphi_t) - (\nabla u, \nabla \varphi) - ((u \cdot \nabla)u, \varphi) + (\nabla p, \varphi) = -(u_0, \varphi); \qquad (1.8)$$ weak1

we use $(\cdot\,\cdot)$ to denote the inner product in $L^2(\Omega)$.

In order to proceed, we use the (non-trivial) fact that any smooth function can be decomposed as the sum of a divergence-free function and the gradient of a scalar function (the 'Helmholtz decomposition', see Temam, 2001), and the fact that these functions are orthogonal in L^2,

$$(u, \nabla p) = 0 \qquad \text{if} \quad \nabla \cdot u = 0$$

(we have already used this to in our formal derivation of the energy equality (1.6). So if we write

$$\varphi = \phi + \nabla \chi$$

equation (1.8) actually decomposes into two equations, one involving u alone,

$$\int_0^T (u, \phi_t) - (\nabla u, \nabla \phi) - ((u \cdot \nabla)u, \phi) = -(u_0, \phi) \qquad (1.9)$$ weak2

and one that relates the pressure p to the velocity u,

$$\int_0^T ((u \cdot \nabla)u, \nabla \chi) + (\nabla p, \nabla \chi) = 0. \qquad (1.10)$$ u2P

From now on we concentrate on (1.9), which must hold for all $\phi \in \mathscr{D}([0,T)\times\Omega)$, the set of all functions in $C_0^\infty([0,T)\times\Omega)$ that are also divergence free.

We define H to be the completion of $\mathscr{D}([0,T)\times\Omega)$ in the $L^2(\Omega)$ norm. This is the space of finite-energy, divergence-free, initial conditions that also satisfy the boundary condition in a weak sense (in fact $u\cdot n = 0$ on $\partial\Omega$; the trace exists in this direction thanks to the divergence-free condition, see Temam (2001), for example).

Definition 1.1 *Given $u_0 \in H$, a weak solution of (1.1) on $[0,T)$ is a function $u \in L^\infty(0,T;H)\cap L^2(0,T;H^1)$ such that*

$$\int_0^T (u,\phi_t) - (\nabla u,\nabla\phi) - ((u\cdot\nabla)u,\phi) = -(u_0,\phi)$$

for every $\phi \in \mathscr{D}([0,T)\times\Omega)$.

In fact it more convenient to work with a formulation of the weak problem in which we test with functions that depend only on space, rather than the space-time varying functions in $\mathscr{D}([0,T)\times\Omega)$.

To do this, we consider a sequence of test functions in $\mathscr{D}([0,T)\times\Omega)$ of the form $\phi_h(x,t) = \psi(x)\theta_h(t)$, where $\psi \in \mathscr{D}(\Omega_= C_0^\infty(\Omega)$ functions with divergence zero, and $\theta_h(t)$ is a function in $C_0^\infty([0,T))$ that is identically 1 for $t \le t_0$, and identically 0 for $t \ge t_0 + h$. If we test with $\phi_h(x,t)$ and let $h \to 0$, we obtain

$$(u(t),\psi) - (u_0,\psi) + \int_0^t (\nabla u,\nabla\psi) + ((u\cdot\nabla)u,\psi) = 0 \qquad (1.11)$$ weak3

for every $t \in [0,T)$. (To obtain this equation for every $t \in [0,T)$ we may have to redefine on a set of measure zero, so there are some subtleties – see Galdi's notes, for example.)

This procedure can in fact be reversed: if $u \in L^\infty(0,T;H)\cap L^2(0,T;H^1)$ satisfies (1.11) for every $\psi \in \mathscr{D}(\Omega)$ and every $t \in [0,T)$ then u satisfies (1.9) for every $\phi \in \mathscr{D}([0,T)\times\Omega)$. (First show that (1.9) holds for functions of the form $\alpha(t)\psi$, with $\alpha(t) \in C_0^\infty([0,T))$ and $\psi \in \partial D_\Omega$ using an integration by parts for Lebesgue functions; then use density of linear combinations of such functions in $\mathscr{D}([0,T)\times\Omega)$.) So they are equivalent formulations of the problem, and we could define instead:

Definition 1.2 *A weak solution of (1.1) on* $[0,T)$ *with initial condition* $u_0 \in H$ *is a function*

$$u \in L^\infty(0,T;H) \cap L^2(0,T;H^1)$$

such that

$$(u(t),\psi) - (u_0,\psi) + \int_0^t (\nabla u, \nabla\psi) + ((u\cdot\nabla)u,\psi) = 0 \tag{1.12}$$ weakone

for every $\psi \in \mathscr{D}(\Omega)$ *and for every* $t \in [0,T)$.

One immediate consequence of the definition in this form is the weak continuity of $u(t)$:

Lemma 1.3 *Any weak solution is weakly continuous into* L^2, *i.e. for any* $v \in L^2$,

$$\lim_{t\to t_0} (u(t),v) = (u(t_0),v). \tag{1.13}$$ weakc

Proof First we show that (1.13) holds if $v \in \mathscr{D}(\Omega)$. In this case we can use (1.12) to write

$$(u(t),v) - (u(t_0),v) = -\int_{t_0}^t (\nabla u, \nabla v) + ((u\cdot\nabla)u,v)\,\mathrm{d}s$$

Since v is smooth, v and ∇v are bounded; since $u \in L^2(0,T;H^1)$ the first of the two terms on the right-hand side is integrable; since $u \in L^\infty(0,T;H)$, so is the second. It follows that the right-hand side converges to zero as $t \to t_0$, as required.

For $v \in L^2$, we approximate v by a sequence in $\mathscr{D}(\Omega)$ and use the fact that $u \in L^\infty(0,T;H)$ to pass to the limit. □

Note that this gives a sense in which the initial condition is satisfied by the weak solution: $u(t) \rightharpoonup u_0$ as $t \to 0$.

We are going to prove the existence of (at least) one weak solution that satisfies two additional properties.

Theorem 1.4 *For any* $u_0 \in H$ *there exists at least one weak solution of the*

Navier–Stokes equations. This solution is weakly continuous into L^2, and in addition satisfies the energy inequality

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t \|Du(s)\|^2 \,\mathrm{d}s \le \frac{1}{2}\|u_0\|^2 \tag{1.14}$$ EI

for every $t \in [0,T)$. As a consequence, $u(t) \to u_0$ as $t \to 0$.

Note that every weak solution is weakly continuous into L^2; but it is not known whether every weak solution satisfies (1.14). That the solution approaches the initial condition strongly is a consequence of the weak continuity and (1.14), so again is not known for every weak solution, only those of the form we construct here, termed 'Leray–Hopf' weak solutions (since they were first constructed by Leray (1934) and Hopf (1951)).

Proof We will use the Galerkin method: we construct a series of approximate solutions of the form

$$u_k(x,t) = \sum_{j=1}^k \hat{u}_{k,j}(t)\psi_j(x), \qquad u_k(0) = P_k u_0 = \sum_{j=1}^k (u_0, \psi_j)\psi_j, \tag{1.15}$$ ukis

where the $\{\psi_j\}_{j=1}^\infty$ are an orthonormal basis for L^2 formed from the eigenfunctions of the Stokes operator,

$$-\Delta\psi_j + \nabla p_j = \lambda_j \psi_j \qquad \psi_j|_{\partial\Omega} = 0 \qquad \nabla\cdot\psi_j = 0.$$

These functions ψ_j are C^∞ in Ω, are divergence-free, and satisfy the zero boundary condition. (In fact, however, the ψ_js are not elements of $\mathscr{D}(\Omega)$, since they do not have compact support in Ω. So we are being a little careless here. Nevertheless, we could fix this by proving that we can extend the test functions used in the definition of a weak solution to include the larger class of functions in $C^\infty(\bar\Omega)$ that are zero on $\partial\Omega$.)

If we use u_k in place of u in (1.12), and test with each of the functions $\{\psi_1, \dots, \psi_j\}$, we obtain a set of k coupled ordinary differential equations (ODEs) for the coefficients $\hat{u}_{k,j}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{k,j} + \lambda_j \hat{u}_{k,j} + \sum_{i,l=1}^k ((\psi_i \cdot \nabla)\psi_l, \psi_j)\hat{u}_{k,i}\hat{u}_{k,l} = 0. \tag{1.16}$$ hatu

Note that the third term is essentially quadratic in the $\hat{u}_{k,j}$s – so this is a set of locally Lipschitz ODEs. The theory of existence and uniqueness for such

ODEs is well developed: a unique solution exists while the solution stays bounded, i.e. unique solutions for the $\hat{u}_{k,j}(t)$ exist while

$$\sum_{j=1}^{k} |\hat{u}_{k,j}(t)|^2 < \infty \tag{1.17}$$ ODEis

(this is just the norm in $\mathbb{R}^k$ of the vector $(\hat{u}_{k,1}, \ldots, \hat{u}_{k,k})$ of coefficients). How can we show that this quantity stays bounded?

The easiest way is to remember that in fact the $\{\hat{u}_{k,j}\}_{j=1}^{k}$ are the coefficients in an expansion of a function $u_k(x,t)$, and look at the equation satisfied by this function. If we multiply (1.16) by ψ_j, $j = 1, \ldots, k$ and sum, we obtain

$$\frac{\partial u_k}{\partial t} - \Delta u_k + P_k((u_k \cdot \nabla)u_k) = 0,$$

where P_k denotes the orthogonal projection (in L^2) onto the space spanned by the $\{\psi_1, \ldots, \psi_k\}$ (this was defined in passing above in (1.15)). Note that if P_k did not appear the nonlinearity would generate terms that could not be expressed as a sum of $\{\psi_1, \ldots, \psi_k\}$.

We can now easily estimate norms of the solution u_k. Note that since all the ψ_j are smooth (in space), so is u_k, and therefore we can perform the manipulations that were 'formal' at the beginning of this chapter entirely rigorously: we take the inner product with u_k and integrate in time. Once again the nonlinear term vanishes, since

$$(P_k[(u_k \cdot \nabla)u_k], u_k) = ((u_k \cdot \nabla)u_k, P_k u_k) = ((u_k \cdot \nabla)u_k, u_k) = 0,$$

and we obtain

$$\frac{1}{2}\|u_k\|^2 + \|Du_k\|^2 = 0.$$

Integrating this in time gives an energy equality for the Galerkin solutions,

$$\frac{1}{2}\|u_k(t)\|^2 + \int_0^t \|Du_k(s)\|^2 \,\mathrm{d}s = \frac{1}{2}\|u_k(0)\|^2.$$

Since $\|u_k(t)\|^2 = \sum_{j=1}^{k} |\hat{u}_{k,j}|^2$, (1.17) is satisfied for all $t \geq 0$, the approximate solution u_k exists for all $t \geq 0$.

Now, since $u_k(0) = P_k u_0$, we have $\|u_k(0)\| \leq \|u_0\|$, and so we have a uniform bound (with respect to k) on the approximate solutions:

$$\frac{1}{2}\|u_k(t)\|^2 + \int_0^t \|Du_k(s)\|^2 \,\mathrm{d}s \leq \frac{1}{2}\|u_0\|^2.$$

In other words, for any $T > 0$, u_k is uniformly bounded in $L^\infty(0,T;H)$ and in $L^2(0,T;H^1)$.

We can therefore use weak and weak-* compactness to find a subsequence (which we relabel) such that u_k converges to u weakly-* in $L^\infty(0,T;H)$ and weakly in $L^2(0,T;H^1)$. It follows that the limit enjoys the same bounds as the approximations,

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t \|Du(s)\|^2 \,\mathrm{d}s \le \frac{1}{2}\|u_0\|^2;$$

but note that the energy equality for the approximations has now turned into an energy inequality for the (candidate) weak solution.

We may have shown that the approximations converge to a limit, but does this limit satisfy the right equation? As a first step, we need to show that for each ψ_j the terms in the equation

$$(u_k(t),\psi_j) - (u_0,\psi_j) + \int_0^t (\nabla u_k,\psi_j) + ((u_k\cdot\nabla)u_k,\psi_j) = 0 \tag{1.18}$$ ukpsij

converge to the same thing but with u_k replaced by u. We will need some better convergence of u_k to u before we can guarantee the convergence of the nonlinear term.

In fact we will show that $\mathrm{d}u_k/\mathrm{d}t$ is uniformly bounded in $L^{4/3}(0,T;H^{-1})$, which coupled with the bounds we already have in $L^2(0,T;H^1)$ will be enough to prove that there is a subsequence that converges that converges strongly in $L^2(0,T;L^2)$: this is the content of the Aubin Lemma (for a proof of a more general result see Temam, 2001).

Lemma 1.5 *Let u_k be a sequence that is bounded in $L^2(0,T;H^1)$, and has $\mathrm{d}u_k/\mathrm{d}t$ bounded in $L^p(0,T;H^{-1})$ for some $p > 1$. Then u_k has a subsequence that converges strongly in $L^2(0,T;L^2)$.*

You can think of this lemma as being a version of the familiar result that a bounded sequence in H^1 has a subsequence that converges strongly in L^2 (i.e. that H^1 is compact in L^2); things are more complicated here since for each t, $u(t)$ lies in a space of functions instead of being a real number.

We already have the bound on u_k in $L^2(0,T;H^1)$, so we need to look at the derivative of u_k. We have

$$\frac{\mathrm{d}u_k}{\mathrm{d}t} = -\Delta u_k - P_k((u_k\cdot\nabla)u_k).$$

Since $u_k \in L^2(0,T;H^1)$, we have $\Delta u_k \in L^2(0,T;H^{-1})$, so we have to estimate the nonlinear term. If we take the inner product with some $v \in H^1$

we obtain

$$\begin{aligned}|(P_k[(u_k\cdot\nabla)u_k],v)| = |((u_k\cdot\nabla)u_k,P_kv)| &= \left|\int (u_k\cdot\nabla)u_k\cdot P_kv\right|\\ &\le \int |u_k||Du_k||P_kv|\\ &\le \|u_k\|_{L^3}\|Du_k\|_{L^2}\|P_kv\|_{L^6}\\ &\le c\|u_k\|^{1/2}\|Du_k\|^{3/2}\|P_kv\|_{H^1}\\ &\le \left(c\|u_k\|^{1/2}\|Du_k\|^{3/2}\right)\|v\|_{H^1},\end{aligned}$$

and so

$$\|P_k(u_k\cdot\nabla)u_k\|_{H^{-1}} \le c\|u_k\|^{1/2}\|Du_k\|^{3/2}.$$

It follows that the nonlinear term is bounded in $L^{4/3}(0,T;H^{-1})$. So $\mathrm{d}u_k/\mathrm{d}t$ is uniformly bounded in $L^{4/3}(0,T;H^{-1})$, and we can use the Aubin Lemma to find a subsequence that converges strongly in $L^2(0,T;L^2)$.

We are now in a position to show the convergence of all the terms in (1.18).

For the first term, note that if $u_k\to u$ in $L^2(0,T;L^2)$, then $u_k(t)\to u(t)$ in L^2 for almost every $t\in(0,T)$. So the first term converges for almost every $t\in(0,T)$.

The second term doesn't depend on k, so converges trivially.

The third term converges since $u_k\rightharpoonup u$ in $L^2(0,T;H^1)$: in particular, $\int_0^t(\nabla u_k,v)\to\int_0^t(\nabla u,v)$ for any $v\in L^2$ and all $t\in(0,T)$.

The nonlinear term requires the strong convergence in $L^2(0,T;L^2)$ and a little work. We have

$$\begin{aligned}&\left|\int_0^t ((u_k\cdot\nabla)u_k,\psi_j)-((u\cdot\nabla)u,\psi_j)\,\mathrm{d}s\right|\\ &\qquad=\left|\int_0^t (((u_k-u)\cdot\nabla)u_k,\psi_j)+((u\cdot\nabla)(u-u_k),\psi_j)\,\mathrm{d}s\right|\\ &\qquad\le\|\psi_j\|_\infty\left(\int_0^t\|u_k-u\|^2\right)^{1/2}\left(\int_0^t\|\nabla u_k\|^2\right)^{1/2}\\ &\qquad\qquad+\sum_{i=1}^3\left|\int_0^t(\partial_i(u-u_k),u_i\psi_j)\right|.\end{aligned}$$

For the first term on the RHS we use the fact that u_k is uniformly bounded in $L^2(0,T;H^1)$ and that u_k converges strongly to u in $L^2(0,T;L^2)$; for the second term we use the fact that $u_i\psi_j\in L^2$ and the weak convergence of u_k to u in $L^2(0,T;H^1)$.

So for each j we have

$$(u(t),\psi_j)-(u_0,\psi_j)+\int_0^t(\nabla u,\nabla\psi_j)+((u\cdot\nabla)u,\psi_j)=0$$

for all $t\geq 0$ (we get all t by adjusting on a set of measure zero). We want this equation to hold for any $\psi\in\mathscr{D}(\Omega)$: this follows using the fact that finite linear combinations of the ψ_j are dense in $\mathscr{D}(\Omega)$.

We have therefore obtained the existence of a weak solution, and we have already seen that this weak solution satisfies the energy inequality (1.14). To see that $u(t)\to u_0$ as $t\to 0$, we have from the weak continuity

$$\|u_0\|\leq\liminf_{t\to 0}\|u(t)\|,$$

while from the energy inequality we obtain

$$\|u_0\|\geq\limsup_{t\to 0}\|u(t)\|.$$

It follows that $\|u(t)\|\to\|u_0\|$ as $t\to 0$, and since H is a Hilbert space it follows from this plus the weak continuity that $u(t)\to u_0$. □

1.2 Strong solutions

We have shown the existence of at least one weak solution that exists for all time, but it is not known whether such weak solutions are unique. We now look (in a less rigorous way) at the existence of strong solutions. If a strong solution exists then it is unique (in the larger class of weak solutions), as we will soon see.

First suppose that $u_0\in H\cap H^1$; take the inner product of (1.1) with $-P\Delta u$ and integrate over Ω. Here P is the orthogonal projection (in L^2) onto the space of divergence-free functions ($-P\Delta$ is the 'Stokes operator'). Equivalently, take the Galerkin approximations, multiply each by λ_j, and then sum; estimate each equation and take a limit as $k\to\infty$ – this is the rigorous way of performing these calculations. We obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2+\|P\Delta u\|^2&=-((u\cdot\nabla)u,P\Delta u)\\&\leq\|u\|_{L^6}\|Du\|_{L^3}\|P\Delta u\|_{L^2}\\&\leq\|Du\|^{3/2}\|P\Delta u\|^{3/2}\\&\leq\frac{1}{2}\|P\Delta u\|^{3/2}+c\|Du\|^6,\end{aligned}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 + \|P\Delta u\|^2 \leq c\|Du\|^6. \tag{1.19}$$ DI

The differential inequality (1.19) has a number of important consequences.

Consequence (i). Local existence of strong solutions.

If we drop the $\|P\Delta u\|^2$ term then we have

$$\frac{\mathrm{d}X}{\mathrm{d}t} \leq cX^3,$$

where $X(t) = \|Du(t)\|^2$. It follows that

$$\frac{\mathrm{d}X}{X^3} \leq c\,\mathrm{d}t,$$

and so

$$\left[\frac{1}{-2X^2}\right]_{X(0)}^{X(t)} \leq ct,$$

i.e.

$$\|Du(t)\|^4 \leq \frac{\|Du_0\|^4}{1 - 2ct\|Du_0\|^4}. \tag{1.20}$$ Duuppert

It follows that if $\|Du_0\| < \infty$, then $\|Du(t)\| < \infty$, at least for $t < 1/2c\|Du_0\|^4$.

In other words, we have small time existence of what we term a *strong solution* ($u \in L^\infty(0,T;H^1)$) if $u_0 \in H^1$, with the time interval on which we can guarantee existence of such a solution tending to zero as $\|u_0\|_{H^1} \to \infty$.

For such a strong solution, we can return to (1.19) and integrate in time, keeping the Laplacian term, to deduce that

$$\|Du(t)\|^2 + \int_0^t \|P\Delta u(s)\|^2\,\mathrm{d}s \leq \|Du_0\|^2 + \int_0^t \|Du(s)\|^6\,\mathrm{d}s.$$

Since we have a strong solution with $u \in L^\infty(0,T;H^1)$, it follows that we must also have $u \in L^2(0,T;H^2)$. (There is some elliptic regularity theory for the Stokes operator hidden here, of course, since what we actually show is that $\|P\Delta u(t)\|$ is square integrable; we need an inequality that says $\|u\|_{H^2} \leq c\|P\Delta u\|$, which is what comes from elliptic regularity.)

So in fact a strong solution is a weak solution with the additional regularity $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$.

Strong solutions are unique in the larger class of weak solutions. This is indicated by the following argument (the argument has a flaw in, but the result is true).

Lemma 1.6 *Suppose that u is a strong solution of the Navier–Stokes equations with initial condition $u_0 \in H \cap H^1$. Then u is unique in the class of weak solutions.*

Proof We give a formal proof (with a mistake). Take u to be a strong solution satisfying

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \qquad u(0) = u_0$$

and v to be a weak solution satisfying

$$v_t - \Delta v + (v \cdot \nabla)v + \nabla q = 0 \qquad v(0) = u_0.$$

Consider $w = u - v$: this satisfies the equation

$$w_t - \Delta w + (w \cdot \nabla)u + (v \cdot \nabla)w + \nabla r = 0 \qquad w(0) = 0.$$

Take the inner product with w to give

$$\frac{1}{2}\|w\|^2 + \|Dw\|^2 = -((w \cdot \nabla)u, w). \tag{1.21}$$ wrong

Note that one of the two nonlinear terms vanishes using the orthogonality identity $((u \cdot \nabla)v, v) = 0$ (see (1.4)).

In fact the equation (1.21 is not valid - there is not enough regularity of w (essentially limited by v) to deduce that $(w, w_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2$. While we can pair w with w_t for a.e. $t \in [0, T)$, we cannot integrate in time (which we are about to do) since we only have $w_t \in L^{4/3}(0, T; H^{-1})$ and $w \in L^2(0, T; H^1)$. But assuming that all is well...

Bound the right-hand side according to

$$\begin{aligned} |((w \cdot \nabla u), w)| &\leq \int |w||Du||w| \leq \|Dw\|\|Du\|^{1/2}\|P\Delta u\|^{1/2}\|w\| \\ &\leq \frac{1}{2}\|Dw\|^2 + \frac{1}{2}\|Du\|\|P\Delta u\|\|w\|^2, \end{aligned}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \|Dw\|^2 \leq c\|Du\|\|P\Delta u\|\|w\|^2.$$

Dropping the $\|Dw\|^2$ term and integrating gives

$$\|w(t)\|^2 \leq \exp\left(\int_0^t \|Du(s)\|\|P\Delta u(s)\| \,\mathrm{d}s\right) \|w(0)\|^2.$$

Since u is a strong solution, $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$, so the integrand on the RHS is integrable. It follows that $w(t) = 0$ for all $t \geq 0$, and hence $u(t) = v(t)$, i.e. the solution is unique. □

Consequence (ii). Global existence of strong solutions for small initial data.

If we return to (1.19) and use Poincaré's inequality (0.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 + \alpha\|Du\|^2 \leq c\|Du\|^6,$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 \leq \|Du\|^2(c\|Du\|^4 - \alpha).$$

It follows that if $\|Du_0\|^4 < \alpha/c$, then $\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 < 0$. Thus $\|Du(t)\|^2$ can never increase, and hence remains bounded for all $t \geq 0$.

This result is correct, but in some sense H^1 is the 'wrong' space in which to require that the initial data is small. This follows from an observation which will be extremely useful later when we come to consider the partial regularity results of Caffarelli, Kohn, & Nirenberg.

If $u(x,t)$ is a solution of the Navier–Stokes equations (with pressure $p(x,t)$), then so is the rescaled solution $\lambda u(\lambda x, \lambda^2 t)$ (and pressure $\lambda^2 p(\lambda x, \lambda^2 t)$). If an initial condition $u_0(x)$ is 'small' and this implies existence for all time, we have the option of considering all the rescaled versions of $u_0(x)$, $\lambda u_0(\lambda x)$ – if one of these rescaled initial conditions is 'small' then the solution with initial condition $\lambda u_0(\lambda x)$ exists for all time, and hence so does the solution with initial condition $u_0(x)$. Ideally we choose to measure the 'smallness' of u_0 is a sense that is invariant under this scaling transformation.

If we consider the H^1 norm on $\mathbb{R}^3$, this is not invariant under such a scaling, since

$$\int_{\mathbb{R}^3} |u_\lambda(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} |\lambda u(\lambda x)|^2 = \lambda^2 \int |u(y)|^2 \lambda^{-3} \mathrm{d}y = \frac{1}{\lambda}\int_{\mathbb{R}^3} |u(y)|^2 \,\mathrm{d}y$$

and by similar reasoning

$$\int_{\mathbb{R}^3} |Du_\lambda(x)|^2 \,\mathrm{d}x = \lambda \int_{\mathbb{R}^3} |Du(y)|^2 \,\mathrm{d}y.$$

So the smallness condition

$$\|u_\lambda\|^2 + \|Du_\lambda\|^2 = \frac{1}{\lambda}\|u\|^2 + \lambda\|Du\|^2 < \epsilon$$

can in some sense be 'optimised' by choosing $\lambda = \|u\|/\|Du\|$, in which case it becomes

$$\|u\|\|Du\| < \epsilon. \tag{1.22}$$ product

This product of norms gives a measure of the 'size' of u that is invariant under the scaling transformation.

Spaces in which the norm is invariant under this transformation are termed 'critical' for the Navier–Stokes equations; examples are L^3 and $H^{1/2}$ (if $u \in H^1$, then one can bound the $H^{1/2}$ by interpolation between L^2 and H^1, $\|u\|^2_{H^{1/2}} \le \|u\|\|Du\|$, c.f. (1.22).

Consequence (iii). Eventual regularity of weak solutions.

If u is a weak solution then we know that

$$\int_0^t \|Du(s)\|^2 \, \mathrm{d}s \le \frac{1}{2}\|u_0\|^2.$$

It follows that if one chooses T large enough, there must be a $t^* < T$ such that $\|Du(t^*)\|^2 < \sqrt{\alpha/c}$, i.e. such that the existence of strong solution for all $t \ge t^*$ is guaranteed. So every weak solution is eventually strong.

Consequence (iv). A lower bound on solutions that blow up.

If a solution that starts with $u_0 \in H^1$ ceases to be a strong solution, this will be because $\|Du(t)\|^2 \to \infty$ as $t \to T$ (T is the 'blow up time'). One can deduce that the solution must be bounded below according to

$$\|Du(T-t)\|^2 \ge \frac{\sqrt{2c}}{t^{1/2}}, \tag{1.23}$$ lb

(where c is the constant occurring in (1.20)) for otherwise we would have $\|Du(T-t)\|^2 < \sqrt{2c}\,t^{-1/2}$ for some $t > 0$, and then one can use (1.20) to show that in fact $\|Du(T)\|^2 < \infty$:

$$\|Du(T)\|^4 \le \frac{\|Du(T-t)\|^4}{1 - 2ct\|Du(T-t)\|^4},$$

which is finite since $2ct\|Du(T-t)\|^4 < 1$.

Consequence (v). Limits on time singularities of weak solutions.

One can use the lower bound (1.23) along with the fact that Du is square integrable in time to show that for any Leray–Hopf weak solution the set of singular times

$$\Sigma = \{t \in \mathbb{R} : \ \|Du(t)\| = \infty\}$$

can have box-counting dimension no larger than 1/2. The simple argument here is due to Robinson & Sadowski (2007). (An older argument due to Scheffer (1976) shows that $H^{1/2}(\Sigma) = 0$, where $H^{1/2}$ is the 1/2-dimensional Hausdorff measure; we will define something similar in Chapter 3 to deal with space-time singularities.)

The (upper) box-counting dimension is defined as follows. Let $N(X, \epsilon)$ denote the number of balls of radius ϵ needed to cover X, and set

$$d_{\rm box}(X) = \limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{-\log \epsilon}.$$

Essentially (but not exactly) $d_{\rm box}(X)$ captures the exponent d in a relationship of the form $N(X, \epsilon) \sim \epsilon^{-d}$ as $\epsilon \to 0$. What certainly is true is that if $\delta < d_{\rm box}(X)$, there exists a sequence $\epsilon_j \to 0$ such that

$$N(X, \epsilon_j) > \epsilon_j^{-\delta}.$$

For our purposes, and equivalent definition is useful, where we replace $N(X, \epsilon)$ by $N_d(X, \epsilon)$, the maximum number of disjoint balls that one can find whose centres lie in X (a relatively simple argument shows that $N_d(X, \epsilon)$ is comparable to $N(X, \epsilon)$, and hence that the two definitions yields the same quantity).

Suppose then that $d_{\rm box}(X) > 1/2$, and choose a δ with $1/2 < \delta < d_{\rm box}(X)$. For such a δ, there exists a sequence $\epsilon_j \to 0$ such that $N_d(X, \epsilon_j) > \epsilon_j^{-\delta}$; let the intervals in this disjoint collection be $[t_k - \epsilon_j, t_k + \epsilon_j]$.

Now, consider

$$\begin{aligned}
\int_0^T \|Du(s)\|^2 \, {\rm d}s &\geq \sum_{j=1}^{N_d(X,\epsilon_j)} \int_{t_k - \epsilon_j}^{t_k + \epsilon_j} \|Du(s)\|^2 \, {\rm d}s \\
&\geq \sum_{j=1}^{N_d(X,\epsilon_j)} \int_{t_k - \epsilon_j}^{t_k} \|Du(s)\|^2 \, {\rm d}s \\
&\geq \sum_{j=1}^{N_d(X,\epsilon_j)} \int_{t_k - \epsilon_j}^{t_k} \frac{\sqrt{2c}}{(t_k - s)^{1/2}} \, {\rm d}s \qquad \geq \epsilon_j^{-d} \ \sqrt{c/2} \, \epsilon_j^{1/2}.
\end{aligned}$$

Since $d > 1/2$, the right-hand side tends to infinity as $j \to \infty$; but we know that the left-hand side is bounded. It follows that $d_{\mathrm{box}}(\Sigma) \leq 1/2$ as claimed.

2

Serrin's local regularity result

Suppose that we return to the question of when a weak solution is unique, and rather than looking at strong solutions, assume rather than

$$u \in L^{s'}(0,T;L^s(\Omega)),$$

i.e. that

$$\int_0^T \left(\int_\Omega |u|^s\right)^{s'/s} < \infty.$$

Can we find conditions on s and s' that guarantee uniqueness?

We'll use the same 'wrong' argument that gave us uniqueness of strong solutions; again, we'll end up with the right answer (and essentially for the right reasons) even if the proof is less than watertight.

Recall that we obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \|Dw\|^2 = ((w\cdot\nabla)u, w). \tag{2.1}$$ unique2

We now estimate the right-hand side differently:

$$\begin{aligned}\left|\int((w\cdot\nabla)u,w)\right| &= \left|\int((w\cdot\nabla)w,u)\right| \\ &\le \int |w||Dw||u| \\ &\le \|w\|^{L^{(2+s)/(s-2)}}\|Dw\|\|u\|_{L^s},\end{aligned}$$

using Hölder inequality with exponents $(2+s)/(s-2)$, 2, and s. Now use Lebesgue space interpolation (just Hölder's inequality again) on the first term to give

$$\|w\|_{L^{(2+s)/(s-2)}} \leq \|w\|^{1-(3/s)} \|w\|_{L^6}^{3/s} \leq c\|w\|^{1-(3/s)} \|Dw\|^{3/s}$$

after also using $H^1 \subset L^6$ and Poincaré's inequality (0.3). Feeding this back in to the inequality (2.1) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \|Dw\|^2 \leq c\|Dw\|\|u\|_{L^s}\|w\|^{1-(3/s)}\|Dw\|^{3/s}.$$

Integrating in time and using Hölder's inequality with exponents 2, r, and $2s/3$ – so that

$$\frac{1}{2} + \frac{1}{r} + \frac{3}{2s} = 1 \tag{2.2}$$ rsfrom

we obtain

$$\|w(t)\|^2 + \int_0^t \|Dw(s)\|^2$$
$$\leq \left(\int_0^t \|Dw\|^2\right)^{1/2} \left(\int_0^t \|u\|_{L^s}^r \|w\|^2\right)^{1/r} \left(\int_0^t \|Dw\|^2\right)^{3/2s},$$

which after using Young's inequality (again with exponents 2, r, and $2s/3$) yields

$$\|w(t)\|^2 \leq c\int_0^t \|u(\tau)\|_{L^s}^r \|w(\tau)\|^2 \,\mathrm{d}\tau.$$

This implies that

$$\|w(t)\|^2 \leq \exp\left(\int_0^t \|u(\tau)\|_{L^s}^r \,\mathrm{d}\tau\right) \|w(0)\|^2,$$

so that $w(t) = 0$ for all $t \geq 0$ provided that $u \in L^r(0,T;L^s)$, where from (2.2) we must have

$$\frac{3}{s} + \frac{2}{r} = 1.$$

[Note that on $\mathbb{R}^3 \times \mathbb{R}$, the norm in $L^r(0,T;L^s)$ is scale invariant.]

In this chapter we show that if u is a weak solution that satisfies the assumption

$$u \in L^{s'}(T_1, T_2(L^s(U))) \qquad \frac{3}{s} + \frac{2}{s'} < 1$$

for some region $U \times (T_1, T_2)$ of space-time, then $u(t) \in C^\infty(U)$ for $t \in (T_1, T_2)$, a result due to Serrin (1962).

2.1 Serrin I: vorticity formulation and a representation formula

The key idea is to work with the vorticity form of the Navier–Stokes equations. If we take the curl of the equations and define $\omega = \nabla \wedge u$ (the vorticity), we obtain

$$\frac{\partial \omega}{\partial t} - \Delta\omega + \nabla \wedge ((u \cdot \nabla)u) = 0$$

(the pressure term vanishes since curl grad= 0). To simplify the nonlinear term we require two vector identities. The first is

$$\frac{1}{2}\nabla|u|^2 = (u \cdot \nabla)u + u \wedge \omega,$$

which means that the nonlinear term becomes

$$\nabla \wedge [\frac{1}{2}\nabla|u|^2 - u \wedge \omega] = -\nabla \wedge (u \wedge \omega);$$

the second identity is

$$\nabla \wedge (a \wedge b) = a(\text{div } b) - b(\text{div } a) + (b \cdot \nabla)a - (a \cdot \nabla)b,$$

from which (since both u and ω are divergence free)

$$-\nabla \wedge (u \wedge \omega) = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u,$$

and so the vorticity form of the Navier-Stokes equations is

$$\omega_t - \Delta\omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0.$$

If we rewrite this as

$$\omega_t - \Delta\omega = (\omega \cdot \nabla)u - (u \cdot \nabla\omega) = \text{div}(\omega u - u\omega) \qquad (2.3)$$ omegaheat

then we have recast the equation as a heat equation for ω, although admittedly the right-hand side depends on ω and on u (which also depends on ω – we will discuss recovering u from ω later).

However, we know a lot about solutions of the heat equation. In particular if we set

$$K(x,t) = \begin{cases} ct^{-3/2}\mathrm{e}^{-|x|^2/4t} & t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(K is the 'heat kernel') then the convolution of K with f, $K \star f$, defined as

$$K \star f = \int_{T_1}^{T_2} \int_U K(x - \xi, t - \tau) f(\xi, \tau) \, \mathrm{d}\xi \, \mathrm{d}\tau,$$

is the solution in $t \geq 0$ of

$$u_t - \Delta u = f(x,t) \qquad \text{with} \quad u(x,t) = 0.$$

We can use this to write down the following representation formula for ω in terms of $g = \omega u - u\omega$:

$$\omega = K \star (\operatorname{div} g) + H(x,t) = \nabla K \star g + H(x,t),$$

where $H(x,t)$ is a solution of the heat equation (and hence smooth). We therefore neglect H in what follows (we should continue to include it, but it would cause no difficulty), and consider what information we can obtain from the (not quite true) representation

$$\omega = \nabla K \star g.$$

(Note that there are also some subtleties here that we have ignored. For example, are u and ω (and so g) actually smooth enough to use this representation? To derive it rigorously Serrin mollified the equation and took limits.)

In order to exploit this representation formula, we need some results about the smoothing properties of convolutions.

2.2 Convolutions and Young's inequality

If we know something about f and something about g, we would like to be able to deduce properties of the convolution

$$f * g(x) = \int f(y)g(x-y)\,\mathrm{d}y.$$

We start with two general such results, the first of which is a fundamental inequality due to Young, which we prove here by a carefully chosen application of Hölder's inequality. Note that $r = \infty$ is included, and we obtain $f * g \in L^\infty$ if $p^{-1} + q^{-1} \leq 1$.

Lemma 2.1 (Young's inequality) *Let $1 \leq p, q, r \leq \infty$ satisfy*

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}.$$

Then for all $f \in L^p$, $g \in L^q$, we have $f * g \in L^r$ with

$$\|f * g\|_L^r \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Proof We use p' to denote the conjugate of p. Then we have

$$\frac{1}{q'} + \frac{1}{r} + \frac{1}{p'} = 1, \qquad \frac{p}{r} + \frac{p}{q'} = 1, \quad \text{and} \quad \frac{q}{r} + \frac{q}{p'} = 1.$$

First use Hölder's inequality with exponents q', r, and p':

$$\begin{aligned} |(f * g)(x)| &\le \int |f(y)||g(x-y)| \,\mathrm{d}y \\ &= \int |f(y)|^{p/q'} \left(|f(y)|^{p/r}|g(x-y)|^{q/r}\right) |g(x-y)|^{q/p'} \,\mathrm{d}y \\ &\le \|f\|_{L^p}^{p/q'} \left(\int |f(y)|^p |g(x-y)|^q \,\mathrm{d}y\right)^{1/r} \left(\int |g(x-y)|^q \,\mathrm{d}y\right)^{1/p'} \\ &\le \|f\|_{L^p}^{p/q'} \left(\int |f(y)|^p |g(x-y)|^q \,\mathrm{d}y\right)^{1/r} \|g\|_{L^q}^{q/p'}. \end{aligned}$$

Now take the L^r norm (wrt x):

$$\begin{aligned} \|f * g\|_{L^r} &\le \|f\|_{L^p}^{p/q'} \|g\|_{L^q}^{q/p'} \left(\int\int |f(y)|^p |g(x-y)|^q \,\mathrm{d}y \,\mathrm{d}x\right)^{1/r} \\ &= \|f\|_{L^p}^{p/q'} \|g\|_{L^q}^{q/p'} \|f\|_{L^p}^{p/r} \|g\|_{L^q}^{q/r} \\ &= \|f\|_{L^p} \|g\|_{L^q}. \end{aligned}$$

□

We will need a space-time version of this inequality, which is provided by the following lemma. Essentially it says that we can use Young in space, and Young in time independently.

Note that in the following lemma, $h \in L^\infty_x(L^\infty_t)$ provided that

$$\frac{1}{p} + \frac{1}{q} \le 1 \qquad \text{and} \qquad \frac{1}{p'} + \frac{1}{q'} \le 1.$$

rrppqq **Lemma 2.2** *Let $h = K \star g$, with $K \in L_t^{p'}(L_x^p)$ and $g \in L_t^{q'}(L_x^q)$. Then $h \in L_t^{r'}(L_x^r)$ with*

$$\|h\|_{L_t^{r'}(L_x^r)} \le \|k\|_{L_t^{p'}(L_x^p)} \|g\|_{L_t^{q'}(L_x^q)}, \tag{2.4}$$ 2young

where $1 \le q \le r$, $1 \le q' \le r$, and

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, \qquad \frac{1}{r'} = \frac{1}{p'} + \frac{1}{q'} - 1. \tag{2.5}$$ exponents1

Proof Use Minkowski's inequality (which works for integrals as well as simple sums) to write

$$\begin{aligned}\|h(\cdot,t)\|_{L^r_x} &= \left\| \int_{T_1}^{T_2} \int_G k(x-\xi,t-\tau)g(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \right\|_{L^r_x} \\ &\leq \int_{T_1}^{T_2} \left\| \int_G k(x-\xi,t-\tau)g(\xi,\tau)\,\mathrm{d}\xi \right\|_{L^r_x} \mathrm{d}\tau \\ &\leq \int \|k(t-\tau)\|_{L^p_x} \|g(\tau)\|_{L^q_x}\,\mathrm{d}\tau,\end{aligned}$$

using Young's inequality in x to obtain the last line. Now use Young's inequality in t to obtain the result. □

If we apply this result using ∇K as the kernel, then we obtain the following, which is fundamental to the first part of Serrin's argument.

convolve **Lemma 2.3** *Let* $\omega = \nabla K \star g$, *with* $g \in L^{q'}_t(L^q_x)$. *Then*

$$\|h\|_{L^r_x L^{r'}_t} \leq c\|g\|_{L^q_x L^{q'}_t}$$

provided that $1 \leq q \leq r$, $1 \leq q' \leq r'$, *and*

$$3\left(\frac{1}{q}-\frac{1}{r}\right) + 2\left(\frac{1}{q'}-\frac{1}{r'}\right) < 1. \tag{2.6}$$ exponents2

Proof Set $T = T_2 - T_1$. Clearly

$$|k(x,t)| \leq c|x|t^{-(n/2)-1}\mathrm{e}^{-|x|^2/4t},$$

and hence one can calculate

$$\begin{aligned}\|k\|^{p'}_{L^p_x L^{p'}_t} &\leq \int_0^T \left(\int_U c|x|^p t^{-p[(n/2)+1]}\mathrm{e}^{-p|x|^2/4t}\,\mathrm{d}x\right)^{p'/p} \mathrm{d}t \\ &= c\int_0^T t^{-p'[(n/2)+1]} t^{p'/2+(np'/2p)} \left(\int_U |y|^p \mathrm{e}^{-|y|^2}\,\mathrm{d}y\right)^{p'/p} \mathrm{d}t \\ &= c\int_0^T t^{-p'[(n/2)+1]+(p'/2)+(np'/2p)}\,\mathrm{d}t \\ &= cT^{-\alpha p'+1},\end{aligned}$$

where $\alpha = \frac{n}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}$, provided that $\alpha p' < 1$, and then

$$\|k\|_{L^p_x L^{p'}_t} \leq cT^{-\alpha p'+1}.$$

To apply Lemma 2.2, we must choose the exponents to satisfy (2.5). Eliminating p and p' in the condition $\alpha < 1/p'$ yields (2.6). □

2.3 Serrin II: Using convolutions to show that $\omega \in L_t^\infty(L_x^\infty)$.

We are now in a position to give the first part of Serrin's regularity argument. Note that if u is a weak solution we know that $u \in L^2(0,T;H^1)$, so in particular $\omega \in L_t^2(L_x^2)$.

Proposition 2.4 *Suppose that $\omega \in L_t^2(L_x^2)$ and $u \in L_t^{s'}(L_x^s)$ with*

$$\frac{3}{s} + \frac{2}{s'} < 1.$$

Then in fact $\omega \in L_t^\infty(L_x^\infty)$.

Proof Suppose that $\omega \in L_t^\rho(L_x^\rho)$. Then since $u \in L_t^{s'}(L_x^s)$ by assumption, and pointwise we have $|g| \le 2|u||\omega|$, it follows that

$$g \in L_x^q L_t^{q'}$$

provided that

$$\frac{1}{q} = \frac{1}{s} + \frac{1}{\rho} \qquad \text{and} \qquad \frac{1}{q'} = \frac{1}{s'} + \frac{1}{\rho}$$

(to see this, note that $|u|^q \in L^{s/q}$ and $|\omega|^q \in L^{\rho/q}$, so for $|u|^q|\omega|^q \in L^1$ we need $1/(s/q) + 1/(\rho/q) = 1$ using Hölder's inequality).

We can now use Lemma 2.3 to show that $\omega \in L_t^r(L_x^r)$ provided that

$$3\left(\frac{1}{q} - \frac{1}{r}\right) + 2\left(\frac{1}{q'} - \frac{1}{r}\right) < 1.$$

Substituting for q and q' we need

$$3\left(\frac{1}{s} + \frac{1}{\rho} - \frac{1}{r}\right) + 2\left(\frac{1}{s'} + \frac{1}{\rho} - \frac{1}{r}\right) < 1.$$

Write $\delta = 1 - (3/s) - (2/s')$: we have $\delta > 0$ by assumption. Note that if $\rho > 5/\delta$ then in fact $\omega \in L_x^\infty L_x^\infty$ as required.

Otherwise, we can improve the regularity of ω to $\omega \in L_t^r(L_x^r)$ provided that

$$\frac{1}{\rho} - \frac{1}{r} < \frac{\delta}{5},$$

i.e.

$$r < \frac{5\rho}{5-\delta}.$$

In particular

$$\omega \in L_t^\rho(L_x^\rho) \qquad \Rightarrow \qquad \omega \in L_t^{\alpha\rho}(L_x^{\alpha\rho})$$

with $\alpha = 5/(5-\delta/2) > 1$. Thus within a finite number of steps we obtain $\omega \in L_t^\rho(L_x^\rho)$ with $\rho > 5/\delta$, from which as remarked above it follows that $\omega \in L_t^\infty(L_x^\infty)$. □

We cannot go any further using the convolution results above. Instead we need to turn to the theory of regularity for parabolic equations. We will use the theory of Hölder regularity, but in order to prove this result (in fact we will prove a parallel and easier result for elliptic problems), we will need an integral characterisation of Hölder continuity which is of interest in itself.

2.4 Integral characterisation of Hölder spaces: Campanato & Morrey Lemmas.

We say that f is α-Hölder continuous on $B_R(0)$, and write $f \in C^\alpha(B_R(0))$, if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \le C|x-y|^\alpha \qquad \text{for all} \quad x, y \in B_R(0).$$

We denote by $\langle f \rangle_{B_r(x)}$ the average of f over $B_r(x)$:

$$\langle f \rangle_{B_r(x)} = \frac{1}{\omega_n r^n} \int_{B_r(x)} f(y)\,\mathrm{d}y,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Lemma 2.5 *Let $f \in L^1(B_R(0))$ and suppose that there exist positive constants $\alpha \in (0,1]$, $M > 0$, such that*

$$\int_{B_r(x)} |f(y) - \langle f \rangle_{B_r(x)}|^2\,\mathrm{d}y \le M^2 r^{n+2\alpha} \tag{2.7}$$ CampA

for any $x \in B_{R/2}(0)$ and any $r \in (0, R/2)$. Then $f \in C^\alpha(B_{R/2}(0))$.

Note that if we consider the integrals

$$I_c = \int_{B_r(x)} |f(y) - c|^2 \, \mathrm{d}y = \int_{B_r(x)} |f(y)|^2 - 2cf(y) + c^2 \, \mathrm{d}y$$

then these are minimised by the choice $c = \langle f \rangle_{B_r(x)}$: simply observe that $I_c = \int_{B_r(x)} |f(y)|^2 - 2cf(y) + c^2 \, \mathrm{d}y$, differentiate with respect to c, and set $\mathrm{d}I_c/\mathrm{d}c = 0$.

Proof Choose $x \in B_{R/2}(0)$ and $r < R/2$. We first compare $\langle f \rangle_{B_{r/2}(x)}$ with $\langle f \rangle_{B_r}(x)$. We have

$$\begin{aligned}
\left| \langle f \rangle_{B_{r/2}(x)} - \langle f \rangle_{B_r(x)} \right|^2 &= \left| \frac{1}{\omega_n (r/2)^n} \int_{B_{r/2}(x)} f(y) - \langle f \rangle_{B_r(x)} \, \mathrm{d}y \right|^2 \\
&\leq \frac{1}{\omega_n^2 (r/2)^{2n}} \left(\int_{B_{r/2}(x)} |f(y) - \langle f \rangle_{B_r(x)}| \, \mathrm{d}y \right)^2 \\
&\leq \frac{2^{2n}}{\omega_n r^{2n}} \left(\int_{B_r(x)} |f(y) - \langle f \rangle_{B_r(x)}| \, \mathrm{d}y \right)^2 \\
&\leq \frac{2^{2n}}{\omega_n r^{2n}} \left(\int_{B_r(x)} |f(y) - \langle f \rangle_{B_r(x)}|^2 \, \mathrm{d}y \right) \omega_n r^n \\
&= \frac{2^{2n}}{\omega_n r^n} \left(\int_{B_r(x)} |f(y) - \langle f \rangle_{B_r(x)}|^2 \, \mathrm{d}y \right) \leq cM^2 r^{2\alpha}.
\end{aligned}$$

In particular,

$$\left| \langle f \rangle_{B_{r/2}(x)} - \langle f \rangle_{B_r(x)} \right| \leq cMr^\alpha.$$

Now consider $\langle f \rangle_{B_{r2^{-k}}(x)} - \langle f \rangle_{B_r(x)}$. Since

$$\langle f \rangle_{B_{r2^{-k}}(x)} - \langle f \rangle_{B_r(x)} = \sum_{j=1}^{k} \langle f \rangle_{B_{r2^{-k}}(x)} - \langle f \rangle_{B_{r2^{-(k-1)}}(x)},$$

it follows that

$$\left| \langle f \rangle_{B_{r2^{-k}}(x)} - \langle f \rangle_{B_r(x)} \right| \leq \sum_{j=1}^{k} cMr^\alpha 2^{-(j-1)\alpha} \leq \sum_{j=0}^{\infty} cMr^\alpha 2^{-j\alpha} = cMr^\alpha. \tag{2.8}$$

avav

This shows that $\langle f \rangle_{B_{r2^{-k}}(x)}$ forms a Cauchy sequence, and hence the averages converge for every $x \in B_{r/2}(0)$. By the Lebesgue Theorem, these averages converge to $f(x)$, and so if we let $k \to \infty$ in (2.8) we obtain an estimate for the difference between $f(x)$ and its average,

$$|f(x) - \langle f \rangle_{B_r(x)}| \leq c_1 M r^\alpha.$$

Now take another point $y \in B_{R/2}(0)$; we compare $\langle f \rangle_{B_r(x)}$ with $\langle f \rangle_{B_s(y)}$:

$$\begin{aligned} |\langle f \rangle_{B_r(x)} - \langle f \rangle_{B_s(y)}|^2 &= \left| \frac{1}{\omega_n r^n} \int_{B_r(x)} f(z) - \langle f \rangle_{B_s(y)} \, dz \right|^2 \\ &\leq \frac{1}{\omega_n r^n} \int_{B_r(x)} |f(z) - \langle f \rangle_{B_s(y)}|^2 \, dz, \end{aligned}$$

arguing as before. Now choose $r = |x - y|$ and $s = 2|x - y|$, so that $B_r(x) \subset B_s(y)$, and then

$$\begin{aligned} |\langle f \rangle_{B_r(x)} - \langle f \rangle_{B_s(y)}|^2 &\leq \frac{s^n}{r^n} \frac{1}{\omega_n s^n} \int_{B_s(y)} |f(z) - \langle f \rangle_{B_s(y)}|^2 \, dz \\ &\leq 2^n M^2 2^{2\alpha} |x - y|^{2\alpha}, \end{aligned}$$

i.e.

$$|\langle f \rangle_{B_r(x)} - \langle f \rangle_{B_s(y)}| \leq c_2 M |x - y|^\alpha.$$

So now (still with $r = |x - y|$ and $s = 2|x - y|$)

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - \langle f \rangle_{B_r(x)}| + |\langle f \rangle_{B_r(x)} - \langle f \rangle_{B_s(y)}| + |\langle f \rangle_{B_s(y)} - f(y)| \\ &\leq 2c_1 M |x - y|^\alpha + c_2 M |x - y|^\alpha \\ &= cM |x - y|^\alpha. \end{aligned}$$

□

We can now prove Morrey's Lemma, which follows from Campanato's Lemma and the second form of the Poincaré inequality (0.4).

Corollary 2.6 *Let $f \in H^1(B_R(0))$ and suppose that there exists a constant $M > 0$ such that*

$$\int_{B_r(x)} |Df|^2 \, dy \leq M r^{n-2+2\alpha}$$

for every $x \in B_{R/2}(0)$ and every $r \in (0, R/2)$. Then $f \in C^\alpha(B_{R/2}(0))$.

Proof Using (0.4),

$$\int_{B_r(x)} |f - \langle f\rangle_{B_r(x)}|^2 \,\mathrm{d}y \le cr^2 \int_{B_r(x)} |Df|^2 \,\mathrm{d}y \le Mr^{n+2\alpha},$$

and the result follows using the Campanato Lemma. □

2.5 Hölder regularity for parabolic problems – and an elliptic proof

In Serrin's proof we will need the following result for the parabolic problem

$$\omega_t - \Delta\omega = \operatorname{div} g.$$

We pick a point $X_0 = (x_0, t_0)$ in $\Omega \times (0, T)$, and write

$$Q_R(X) = B_R(x_0) \times (t_0 - R^2, t_0).$$

(This 'parabolic cylinder' is used, rather than a ball, since it obeys the right scaling for parabolic problems: if $u(x, t)$ solves $u_t - \Delta u = 0$ then so does $u(\lambda x, \lambda^2 t)$.)

PR **Theorem 2.7** *Suppose that $\omega \in L^\infty(Q_R(X_0))$. Then*

(i) if $g \in L^\infty(Q_R(X_0))$ then for any $\alpha \in (0, 1)$, $\omega \in C^\alpha(Q_{R/2}(X_0))$; and
(ii) if $D^k g \in C^\alpha(Q_R(X_0))$ then $D^k\omega \in C^\alpha(Q_{R/2}(X_0))$, for any $k \ge 0$.

We will prove an elliptic version of this result, using a version of the parabolic argument that can be found in Lieberman (1996). We consider for simplicity the scalar equation

$$-\Delta w = \operatorname{div} g.$$

We also prove the equivalent of (ii) only for $k = 0$, but a similar argument works for higher derivatives.

ER **Theorem 2.8** *Suppose that $w \in L^\infty(B_R(x_0))$. Then*

(i) if $g \in L^\infty(B_R(x_0))$ then for any $\alpha \in (0, 1)$, $w \in C^\alpha(B_{R/2}(x_0))$; and
(ii) if $g \in C^\alpha(Q_R(x_0))$ then $Dw \in C^\alpha(B_{R/2}(x_0))$.

We need an auxiliary lemma. Recall that if a function w is harmonic in a region U (i.e. $\Delta w = 0$ in U) then

$$w(x) = \langle w \rangle_{B_r(x)}$$

for any r such that $B_r(x) \subset U$.

L44 **Lemma 2.9** *Suppose that w is harmonic in U. Then for any $r < R$ such that $B_R(x) \subset U$,*

$$\int_{B_r(x)} |w - \langle w \rangle_{B_r(x)}|^2 = \int_{B_r(x)} |w - w(x)|^2 \leq c \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x)} |w - \langle w \rangle_{B_r(x)}|^2.$$

Proof Observe that

$$\sup_{x \in B_r(x_0)} |w(x) - w(x_0)|^2 \leq r^2 \sup_{x \in B_r(x_0)} |Dw(x)|^2.$$

Since w is harmonic in U, so is $Dw(x) = D(w(x) - w(x_0))$, from which it follows that

$$\begin{aligned} |D(w(x) - w(x_0))|^2 &= \left| \frac{1}{\omega_n R^n} \int_{B_{R/2}(x)} D(w(y) - w(x_0)) \, \mathrm{d}x \right|^2 \\ &= \left| \frac{1}{\omega_n R^n} \int_{\partial B_{R/2}} (w(y) - w(x_0)) n \, \mathrm{d}S \right|^2 \\ &\leq \frac{c}{R^2} \sup_{x \in B_{R/2(x)}} |w(x) - w(x_0)|^2. \end{aligned}$$

Now, the supremum of $|w(x) - w(x_0)|^2$ is attained at some $x^* \in B_{R/2}(x)$, and since $w(x) - w(x_0)$ is harmonic,

$$w(x^*) - w(x_0) = \frac{2^n}{\omega_n R^n} \int_{B_{R/2}(x^*)} w(z) - w(x_0) \, \mathrm{d}z,$$

from whence

$$\begin{aligned} |w(x^*) - w(x_0)|^2 &\leq \frac{2^{2n}}{\omega_n^2 R^{2n}} \left| \int_{B_{R/2}(x^*)} |w(z) - w(x_0)| \, \mathrm{d}z \right|^2 \\ &\leq \frac{2^n}{\omega_n R^n} \int_{B_{R/2}(x^*)} |w(z) - w(x_0)|^2 \, \mathrm{d}z \\ &\leq \frac{2^n}{\omega_n R^n} \int_{B_R(x_0)} |w(z) - w(x_0)|^2 \, \mathrm{d}z. \end{aligned}$$

Putting these ingredients together we obtain

$$\begin{aligned}\int_{B_r(x_0)} |w(x) - w(x_0)|^2 \, \mathrm{d}x &\leq \omega_n r^n \sup_{x \in B_r(x_0)} |w(x) - w(x_0)|^2 \\ &\leq \omega_n r^{n+2} \sup_{x \in B_r(x_0)} |Dw(x)|^2 \\ &\leq \omega_n r^{n+2} \frac{c}{R^2} \sup_{x \in B_{R/2(x)}} |w(x) - w(x_0)|^2 \\ &\leq \omega_n r^{n+2} \frac{c}{R^2} \frac{2^n}{\omega_n R^n} \int_{B_R(x_0)} |w(z) - w(x_0)|^2 \; dz \\ &= c \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |w(z) - w(x_0)|^2 \, \mathrm{d}z.\end{aligned}$$

□

We now prove Theorem 2.8

Proof Choose $x \in B_{R/2}(x_0)$ and $r \in (0, R/2)$. Let v be the solution of

$$-\Delta u = 0 \quad \text{in} \quad B_r(x) \quad \text{with} \quad u = w \quad \text{on } \partial B_r(x).$$

Then the difference $v = w - u$ satisfies

$$-\Delta v = \operatorname{div} g \quad \text{in} \quad B_r(x) \quad \text{with} \quad v = 0 \quad \text{on } \partial B_r(x).$$

If we take the inner product of this with v, then we obtain

$$\int_{B_r(x)} |Dv|^2 = \int_{B_r(x)} g \cdot Dv, \tag{2.9}$$ backto

since $v = 0$ on ∂B_r.

For case (i), we estimate this as

$$\int_{B_r(x)} |Dv|^2 \leq \left(\int_{B_r(x)} \|g\|_\infty^2 \right)^{1/2} \left(\int_{B_r(x)} |Dv|^2 \right)^{1/2},$$

so that

$$\int_{B_r(x)} |Dv|^2 \leq \|g\|_\infty \omega_n r^n.$$

At this stage it is tempting to appeal to Morrey's Lemma to deduce that $v \in C^\alpha$; but the function v is defined differently for each x and r, so we cannot do this. Instead, we proceed as follows. First, we use the Poincaré inequality (remember that we have $v = 0$ on ∂B_r) to deduce that

$$\int_{B_r(x)} |v|^2 \leq cr^2 \int_{B_r(x)} |Dv|^2 \leq c\|g\|_\infty \omega_n r^{n+2}. \tag{2.10}$$ vor

We now use the triangle inequality to write

$$\int_{B_r(x)} |w(y) - u(x)|^2 \leq \int_{B_r(x)} |w(y) - u(y)|^2 \, \mathrm{d}y + \int_{B_r(x)} |u(y) - u(x)|^2 \, \mathrm{d}y.$$

Appealing to Lemma 2.9 we have

$$\int_{B_r(x)} |u(y) - u(x)|^2 \, \mathrm{d}y \leq c \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x)} |u(y) - u(x)|^2 \, \mathrm{d}y,$$

and hence

$$\begin{aligned} \int_{B_r(x)} |w(y) - u(x)|^2 &\leq \int_{B_r(x)} |v(y)|^2 \, \mathrm{d}y + c \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x)} |u(y) - u(x)|^2 \, \mathrm{d}y \\ &\leq \int_{B_r(x)} |v(y)|^2 \, \mathrm{d}y + c \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x)} |w(y) - u(y)|^2 \, \mathrm{d}y \\ &\qquad + \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x)} |w(y) - u(x)|^2 \, \mathrm{d}y \\ &\leq c \int_{B_r(x)} |v(y)|^2 \, \mathrm{d}y + c \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x)} |w(y) - u(x)|^2 \, \mathrm{d}y. \end{aligned}$$

From (2.10) we have

$$\begin{aligned} \int_{B_r(x)} |w(y) - u(x)|^2 &\leq \|g\|_\infty r^{n+2} + c \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x)} |w(y) - u(x)|^2 \, \mathrm{d}y \\ &= \left[c\|g\|_\infty + c \frac{1}{R^{n+2}} \int_{B_R(x)} |w(y)|^2 \, \mathrm{d}y \right] r^{n+2}. \end{aligned}$$

It is now safe to use the Campanto Lemma to deduce that $w \in C^\alpha(B_{R/2}(x_0))$.

For (ii) we proceed similarly, but noting that for any x we can replace the right-hand side of our equation, $\operatorname{div} g$, by $\operatorname{div}(g(\cdot) - g(x))$. So we can write the right-hand side of (2.9) as

$$\int [g_i(y) - g_i(x)] D_i v(y) \, \mathrm{d}y$$

and use the fact that g is Hölder continuous to estimate it as

$$\int [g(y) - g(x)] \cdot Dv \, \mathrm{d}X \leq \int |g(y) - g(x)|^2 \, \mathrm{d}y + \frac{1}{2} \int |Dv|^2$$
$$\leq c[g]_\alpha^2 r^{n+2\alpha} + \frac{1}{2} \int |Dz|^2.$$

Thus we now obtain

$$\int |Dz|^2 \leq c r^{n+2\alpha} [g]_\alpha^2.$$

We now proceed exactly as before, but replacing v, w, and z by Dv, Dw, and Dz throughout the argument to conclude that $Dw \in C^\alpha(B_{R/2}(0))$. □

2.6 Serrin III: The Biot–Savart law and parabolic regularity

In the final part of the proof we will need to use the regularity we have for ω to deduce further regularity of u. Given $\omega = \nabla \wedge u$, we can recover u using the Biot–Savart law:

$$u = \frac{1}{4\pi} \int_G \frac{x - \xi}{|x - \xi|^3} \wedge \omega(\xi) \, \mathrm{d}\xi + A(x),$$

where $A(x)$ is harmonic in G.

We will not study this here, but merely state the following – for a proof see Chapter 4 of Majda & Bertozzi (2002):

(i) if $\omega \in L^\infty$ then $u \in L^\infty$;
(ii) if $D^k\omega \in C^\alpha$ then $D^k u \in C^\alpha$.

We can now finish Serrin's proof.

Theorem 2.10 *Suppose that* $\omega \in L_t^2(L_x^2)$ *and* $u \in L_t^{s'}(L_x^s)$ *with*

$$\frac{3}{s} + \frac{2}{s'} < 1.$$

Then $u(t) \in C^\infty(U)$ *for all* $t \in (T_1, T_2)$.

Proof We know from above that $\omega \in L^\infty(L^\infty)$.

Using the Biot–Savart representation (BSL), it follows that $u \in L^\infty(L^\infty)$. Hence $g \in L^\infty(L^\infty)$.

Parabolic regularity now guarantees that $\omega \in C^\alpha$.
BSL implies that therefore $Du \in C^\alpha$.
It follows that $g \in C^\alpha$. Parabolic regularity guarantees that $D\omega \in C^\alpha$.
BSL implies that therefore $D^2u \in C^\alpha$.
It follows that $Dg \in C^\alpha$. Parabolic regularity guarantees that $D^2\omega \in C^\alpha$.
BSL... and so on, and therefore $u \in C^\infty$. □

This regularity result has been improved since Serrin's paper in a number of ways.

Fabes, Jones, & Rivière (1972) showed that the same result holds if

$$\frac{3}{s} + \frac{2}{s'} = 1$$

for $3 < s < \infty$. Struwe (1988) gave an alternative proof of this, also allowing the case $s = \infty$.

Takahashi (1990) showed that one can replace the condition that the L^s_x norm is $L^{s'}$ in time with the condition that it is weakly $L^{s'}$ in time. A function $f \in L^{s'}_w$ if

$$\sup_{r>0} r^{s'} \mu\{x : \ |f(x)| > r\} < \infty.$$

Note that $1/x \notin L^1$, but $1/x \in L^1_w$. Takahashi shows that if

$$\frac{3}{s} + \frac{2}{s'} = 1 \qquad 3 \le s \le \infty$$

and $\|u\|_{L^{s'}_w(L^s_x)}$ is sufficiently small then u is C^∞ in space. He also proves the same result if $\|u\|_{L^\infty_t(L^3_x)}$ is sufficiently small.

Kim & Kozono (2004) extend Takahashi's result by showing that it is in fact sufficient for u to be small in $L^{s'}_w(L^s_w)$, i.e. in the weak Lebesgue spaces in both space and time.

Finally, Escauriaza, Seregin, & Sverak (2002) have recently shown that the result does still hold in the boundary case $L^\infty_t(L^3_x)$.

3

Partial regularity

Caffarelli, Kohn, & Nirenberg (1982) – hereafter CKN – proved that the one-dimensional parabolic measure of the set of space-time singularities of a 'suitable weak solution' is zero.

We now define all these terms.

A point (x,t) is regular if u is essentially bounded in a neighbourhood of (x,t), i.e. if $u \in L^\infty(U)$ where U is a neighbourhood of (x,t). A point (x,t) is singular if it is not regular.

For a set $X \subset \mathbb{R}^3 \times \mathbb{R}$ and $k \geq 0$, we define $P^k(X) = \lim_{\delta \to 0^+} P^k_\delta(X)$, where

$$P^k_\delta(X) = \inf \left\{ \sum_{i=1}^{\infty} r_i^k : \ X \subset \bigcup_i Q^*_{r_i}(x_i,t_i) : \ r_i < \delta \right\},$$

and $Q^*_r(x,t) = B_r(x) \times (t - r^2, t + r^2)$. We have $P^k(X) = 0$ if and only if for every $\delta > 0$ X can be covered by a collection $\{Q^*_{r_i}\}$ such that $\sum_i r_i^k < \delta$.

A suitable weak solution is a weak solution that has a corresponding pressure $p \in L^{5/3}((0,T) \times \Omega)$ and satisfies a local form of the energy inequality. If assume that all terms are smooth and take the inner product of the Navier–Stokes equations with $u\varphi$, where φ is some C^∞ scalar cut-off function, we obtain (after an integration by parts)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\varphi - \frac{1}{2}\int |u|^2\varphi_t + \int |\nabla u|^2\varphi - \frac{1}{2}\int |u|^2\Delta\varphi \\ - \frac{1}{2}\int |u|^2(u\cdot\nabla)\varphi - \int p(u\cdot\nabla)\varphi = 0.$$

If we integrate in time we obtain

$$\int |u|^2\varphi|_t + 2\iint |\nabla u|^2\varphi \leq \iint |u|^2[\varphi_t + \Delta\varphi] + (|u|^2 + 2p)(u\cdot\nabla)\varphi. \quad (3.1)$$ LEI

A suitable weak solution has to satisfy (3.1) for every φ with compact support in $(0, T) \times \Omega$.

Note that just as we do not know whether every weak solution satisfies an energy inequality, we do not know if every weak solution is 'suitable'. In fact, we do not know whether our Leray–Hopf weak solutions are suitable. So we are required to look at a subclass of all possible weak solutions. Caffarelli, Kohn, & Nirenberg give a proof that suitable weak solutions exist in an appendix to their paper.

Although "$P^1(S) = 0$" is CKN's headline result, they in fact prove two powerful theorems about local conditions for regularity of u. In what follows, we set

$$Q_r(x,t) = B_r(x) \times [t - r^2, t]$$

(recall that $Q_r^*(x,t) = B_r(x) \times [t - r^2, t + r^2]$).

CKN1 **Theorem 3.1** *There exists a constant $\epsilon_* > 0$ such that if u is a suitable weak solution and*[1] *for some $r > 0$*

$$\frac{1}{r^2}\iint_{Q_r(x_0,t_0)} |u|^3 + |p|^{3/2} < \epsilon_* \quad (3.2)$$ CKN1cond

then u is essentially bounded on $Q_{r/2}(x_0, t_0)$.

CKN's first theorem is not quite in this form; but you can find this in Lin (1998) and Ladyzhenskaya & Seregin (1999). CKN deduce from (3.5) that

$$\frac{1}{r^3}\int_{|x-a|} |u(x,s)|^2 \,\mathrm{d}s \leq C$$

[1] Recall from our discussion at the end of Chapter 1 that any sensible 'smallness condition' should be 'scale invariant', i.e. invariant under the scaling transformation $u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)$ and $p(x,t) \mapsto \lambda^2 p(\lambda x, \lambda^2 t)$. This is such a condition: if we consider

$$\frac{1}{r^2}\iint_{Q_r} |u|^3 < \epsilon_*$$

and rescale u so that the domain of integration is Q_1 (i.e. consider $u_\lambda(x,t)$ with $\lambda = 1/r$) we obtain

$$\frac{1}{r^2}\iint_{Q_r} \left|\frac{1}{r}u(x/r, t/r^2)\right|^3 \mathrm{d}x\,\mathrm{d}t = \frac{1}{r^5}\iint_{Q_1} |u(y,\tau)|^3\, r^3\mathrm{d}y\, r^2\mathrm{d}\tau = \iint_{Q_1} |u|^3 < \epsilon_*.$$

for every $(a,s) \in Q_{r/2}(0,0)$, and hence that $|u(a,s)|^2 \le C$ at any Lebesgue point in $Q_{r/2}(0,0)$, i.e. almost everywhere in $Q_{r/2}(0,0)$. Lin (1998) and Ladyzhenskaya & Seregin (1999) show that away from the singular set u is in fact $C^\alpha(Q_{r/2}(0,0))$ (i.e. Hölder continuous in space and time) by showing that (3.5) implies that

$$\frac{1}{r^5}\int_{Q_r} |v - \langle v\rangle_{Q_r}|^3 \,\mathrm{d}x\,\mathrm{d}t \le Cr^\alpha$$

for every $r \in (0, r/2)$, $x \in Q_{r/2}(0,0)$ and then using a parabolic version of the Campanato Lemma. (Lin is not explicit about the fact that this has to hold for a range of x and r; Ladyzhenskaya & Seregin are explicit about this step.)

This theorem only allows one to show that $P^{5/3}(S) = 0$. In fact it is also of the right form (the condition need only hold for *some* $r > 0$) to allow one to show that $d_{\rm box}(S) \le 5/3$, using an argument similar to that used in Chapter 1 for the set of singular times (the proof here is due to Robinson & Sadowski, 2009). The argument relies in addition on the fact that the pressure associated with any weak solution satisfies $p \in L^{5/3}((0,T)\times\Omega)$, a result to due Sohr & von Wahl (1986), and that any weak solution has $u \in L^{10/3}((0,T)\times\Omega)$. This follows using Hölder's inequality and the embedding $H^1 \subset L^6$: indeed, we know that for any weak solution $u \in L^\infty(0,T;L^2)$ and $u \in L^2(0,T;H^1)$, so

$$\int_\Omega |u|^{10/3} \le \int_\Omega |u|^{4/3}|u|^2 \le \left(\int_\Omega |u|^2\right)^{2/3}\left(\int_\Omega |u|^6\right)^{1/3} = \|u\|^{4/3}\|u\|_{L^6}^2,$$

and so

$$\int_\Omega |u|^{10/3} \le c\|u\|^{4/3}\|u\|_{H^1}^2,$$

from whence

$$\int_0^T\int_\Omega |u|^{10/3} \le c\left(\sup_{0<t<T}\|u(t)\|^{4/3}\right)\left(\int_0^T \|u\|_{H^1}^2\right) < \infty. \qquad (3.3)$$ u103

Theorem 3.2 *If u is a suitable weak solution of the Navier–Stokes equations, then $d_{\rm box}(S) \le 5/3$.*

(Actually we should consider $S \cap K$ for some compact subset K of $(0,T)\times\Omega$, since the parabolic cylinders must lie within $(0,T)\times\Omega$ for Theorem 3.1 to be valid, a subtlety that we have skipped in its statement.)

Proof Note that, using Hölder's inequality, we can bound the LHS of (3.5) by

$$\left(\frac{1}{r^{5/3}}\iint |u|^{10/3}\right)^{9/10} + \left(\frac{1}{r^{5/3}}\iint |p|^{5/3}\right)^{9/10}.$$

So there is a constant $\epsilon_\star$ such that if

$$\frac{1}{r^{5/3}}\iint |u|^{10/3} + |p|^{5/3} < \epsilon_\star$$

the conclusions of Theorem 3.1 hold.

Now suppose that $d_{\text{box}}(S) > 5/3$. Pick d with $5/3 < d < d_{\text{box}}(S)$: then there exists a sequence $\epsilon_j \to 0$ such that $N(S, \epsilon_j) > \epsilon_j^{-d}$. As before, we use $N(S, \epsilon)$ to denote the maximum number of disjoint balls of radius ϵ with centres in S. It follows that on every ball

$$\int_{\mathscr{B}_{\epsilon_j}} |u|^{10/3} + |p|^{5/3} > \epsilon_\star \epsilon_j^2,$$

and since these $N(S, \epsilon_j) \geq \epsilon_j^{-d}$ balls are disjoint,

$$\iint |u|^{10/3} + |p|^{5/3} \geq \epsilon_\star \epsilon_j^{-d+(5/3)}.$$

Now if we let $j \to \infty$, since $d > 5/3$ it follows that the LHS is infinite. But this is a contradiction, since we know that it is finite. It follows that $d_{\text{box}}(S) \leq 5/3$ as claimed. □

Kukavica (2009b) has improved this bound to $d_{\text{box}}(S) \leq 135/82$. It is an open problem whether in fact $d_{\text{box}}(S) \leq 1$.

CKN obtain $P^1(S) = 0$ by proving a second theorem:

CKN2 **Theorem 3.3** *There exists a constant $\delta_* > 0$ such that if u is a suitable weak solution and*

$$\limsup_{r\to 0} \frac{1}{r} \iint_{Q_r^*(x_0,t_0)} |\nabla u|^2 < \delta^* \qquad (3.4) \quad \text{H1lim}$$

then

$$\frac{1}{r^2} \iint_{Q_r^*(x_0,t_0)} |u|^3 + |p|^{3/2} < \epsilon_* \qquad (3.5) \quad \text{CKN1cond}$$

holds for some $r > 0$, and hence (x_0, t_0) is regular.

Note that the "and hence (x_0, t_0) is regular" here is in fact Theorem 3.1. The real content of Theorem 3.3 is that the condition (3.4) implies that (3.5) holds for some $r > 0$.

We will show at the end of this chapter that Theorem 3.3 implies that $P^1(S) = 0$ – the proof is essentially straightforward. First we will give an outline of a result along the lines of Theorem 3.1 due to Kukavica (2009a).

We will not prove any version of Theorem 3.3 here, but instead quote the following related result (also due to Kukavica, 2009a). Note that he obtains a much stronger conclusion from (3.4) than CKN: essentially (3.5) holds uniformly over all points $(x,t) \in Q^*_{r_0}(0,0)$ and all radii $0 < r < r_0$. This uniformity makes it much easier to deduce the regularity of u at $(0,0)$.

In other words, Kukavica proves a stronger result than Theorem 3.3, which allows him to prove a weaker result than Theorem 3.1 and still obtain $P^1(S) = 0$.

Theorem 3.4 *Given $\mu > 0$, there exists a constant $\delta_*(\mu) > 0$ such that*

$$\limsup_{r \to 0} \frac{1}{r} \iint_{Q_r(0,0)} |\nabla u|^2 < \delta^*$$

implies that there exists an $r_0 > 0$ such that

$$\frac{1}{r^2} \iint_{Q_r(x,t)} |u|^3 + |p|^{3/2} \le \mu^3$$

for all $(x,t) \in Q^*_{r_0}(0,0)$ and all $r \in (0, r_0]$

We will therefore prove a theorem that starts with such a uniform assumption (unlike (3.5)). The main aim of the argument will be to use this uniform assumption to prove one that appears to be only minimally stronger, namely that for some $C > 0$, $\epsilon > 0$, and some $r_1 > 0$,

$$\frac{1}{r^{2+\epsilon}} \iint_{Q_r(x,t)} |u|^3 + |p|^{3/2} \le C$$

for all $(x,t) \in Q^*_{r_1}(0,0)$ and all $r \in (0, r_1]$. However, this 'minimal' improvement is sufficient to obtain regularity for u, as we will see by applying the following parabolic regularity result.

Theorem 3.5 *Suppose that* $g \in L^p(Q_R^*)$ *and*

$$\sup_{x,t} \in Q_R^*(x_0,t_0) \sup_{0<r<R} \frac{1}{r^\lambda} \int_{Q_r^*(x,t)} |g|^q < \infty$$

(if this holds we will say that $g \in \mathscr{L}_\lambda^q(Q_R^*(x_0,t_0))$*). Then the solution of*

$$u_t - \Delta u = \operatorname{div} g$$

satisfies $u \in L^{\tilde{p}}(Q_R^*)$ *for*

$$\tilde{p} = \frac{p}{1 - \frac{q}{5-\lambda}}.$$

Now note that we can write the Navier–Stokes equations as

$$u_t - \Delta u = -\operatorname{div}(u \otimes u) - \nabla p.$$

We'll neglect the pressure term in our discussion here, although this can be dealt with appropriately in order to make what we are about to say watertight. (We will soon come back to the pressure in a different context.)

Suppose we know that $u \in L^p(Q_R^*)$ (according to (3.3) we always have $u \in L^{10/3}(Q_R^*)$) and also that $u \in \mathscr{L}_{2+\epsilon}^3(Q_R^*)$. The function g on the right-hand side (remember that we are neglecting p) satisfies $|g| \le |u|^2$, and so we have

$$g \in L^{p/2}(Q_R^*) \qquad \text{and} \qquad g \in \mathscr{L}_{2+\epsilon}^{3/2}.$$

It follows therefore that $u \in L^{\tilde{p}}(Q_R^*)$ where

$$\tilde{p} = \frac{p/2}{1 - \frac{3/2}{3-\epsilon}} = \frac{p}{2 - \frac{3}{3-\epsilon}} = p\left(\frac{3-\epsilon}{3-2\epsilon}\right).$$

So if $\epsilon > 0$ (which we will obtain) we can increase the regularity of u until $u \in L^s(Q_R^*)$ with $s > 5$, which then shows that $u \in L^\infty$ (and in fact C^∞ in space) using Serrin's argument. If $\epsilon = 0$ (which we start with) we cannot increase the regularity of u this way.

We therefore concentrate on proving the following theorem, outlining Kukavica's (relatively) simple argument.

IK **Theorem 3.6** *There exists a constant* $\mu_* > 0$ *such that if there exists an* $r_0 > 0$ *such that*

$$\frac{1}{r^2} \iint_{Q_r(x,t)} |u|^3 + |p|^{3/2} \le \mu_*^3$$

for all $(x,t) \in Q^*_{r_0}(0,0)$ *and all* $r \in (0, r_0]$ *then for some* $r_1 > 0$ *and some* $\epsilon > 0$,

$$\frac{1}{r^{2+\epsilon}} \iint_{Q_r(x,t)} |u|^3 + |p|^{3/2} \leq C$$

for all $(x,t) \in Q^*_{r_1}(0,0)$ *and all* $r \in (0, r_1]$.

Proof A key step is to choose a test function φ that allows us to deduce useful estimates from the local energy inequality. We want to choose a φ such that $\varphi_t + \Delta\varphi = 0$ (a solution of the backwards heat equation); but the heat kernel has too sharp a singularity to enable us to estimate the final term of the LEI. So we use a test function that is 'almost' the backwards heat kernel. Set

$$\psi(x,t) = r^2 K(x, r^2 - t),$$

where $K(x,t)$ is the heat kernel (we saw this in the previous chapter). We have shifted the time so that the singular now lies 'in the future', outside Q_r. Now multiply ψ by a cutoff function $\chi(x,t)$, to obtain our test function $\varphi(x,t) = \xi(x,t)\psi(x,t)$. We choose r and R with $0 < r \leq R/2$; then we can choose χ (depending on both r and R), such that φ satisfies the following properties: $\operatorname{supp}\varphi \subset Q_R$,

$$\varphi(x,t) \geq \frac{1}{Cr} \qquad (x,t) \in Q_r,$$

and for $x \in Q_R$

$$\varphi(x,t) \leq \frac{c}{r}, \qquad |\nabla\varphi(x,t)| \leq \frac{c}{r^2}, \qquad \text{and} \qquad |\varphi_t + \Delta\varphi| \leq \frac{Cr^2}{R^5}.$$

We now use this function φ in the local energy inequality (3.1) to deduce that

$$\frac{1}{r}\int_{B_r} |u(t)|^2 + \frac{1}{r}\iint_{Q_r} |\nabla u|^2 \leq C\frac{r^2}{R^5}\iint_{Q_R} |u|^2 + \frac{C}{r^5}\iint_{Q_R} |u|^3 + \frac{c}{r^2}\iint_{Q_R} |u||p|.$$

Using Hölder's inequality on the first and third terms on the right-hand side, we can replace the right-hand side by

$$c\frac{r^2}{R^{10/3}}\left(\iint_{Q_R} |u|^3\right)^{2/3} + \frac{C}{r^2}\iint_{Q_R} |u|^3 + \frac{c}{r^2}\left(\iint_{Q_R} |u|^3\right)^{1/3}\left(\iint |p|^{3/2}\right)^{2/3}.$$

As discussed earlier, it is natural to work in terms of scale-invariant quantities, so we define

$$\begin{aligned}
\alpha_r^2 &= \frac{1}{r} \sup_{-r^2<t<0} \int_{B_r} |u(\cdot,t)|^2, \\
\beta_r^2 &= \frac{1}{r} \iint_{Q_r} \iint |\nabla u|^2, \\
\gamma_r^3 &= \frac{1}{r^2} \iint_{Q_r} |u|^3, \qquad \text{and} \\
\delta_r^3 &= \frac{1}{r^2} \iint_{Q_r} |p|^{3/2}.
\end{aligned}$$

We can rewrite our LEI estimate in terms of these quantities as

$$\alpha_r^2 + \beta_r^2 \le C\kappa^2\gamma_R^2 + C\kappa^{-2}\gamma_R^3 + C\kappa^{-2}\delta_R^2\gamma_R. \tag{3.6}$$

abc

Now, the main assumption of Theorem 3.6 expressed in terms of these quantities is that $\gamma_r \le \mu$ and $\delta_r \le \mu$ (for all $0 < r < r_0$ and over a range of centres of the cylinders). If we use these bounds in (3.6) we obtain

$$\alpha_r^2 + \beta_r^2 \le C\kappa^2\mu^2 + C\kappa^{-2}\mu^3.$$

Choosing κ then μ such that

$$C\kappa^2 < 1 \qquad \text{and then} \qquad C\kappa^{-2}\mu < 1 \tag{3.7}$$

firstchoice

it follows that $\alpha_r^2 + \beta_r^2 \le \mu^2$, so that we also have $\alpha_r \le \mu$ and $\beta_r \le \mu$.

Having obtained these bounds, we return to (3.6) and try to ensure that the terms on the right-hand side only involve the same quantities as the left-hand side. L^p interpolation and the Sobolev embedding theorem guarantee that $\gamma_R \le C(\alpha_R + \beta_R)$, and so

$$\begin{aligned}
\alpha_r + \beta_r &\le C\kappa(\alpha_R + \beta_R) + C\kappa^{-1}(\alpha_R + \beta_R)^{3/2} + C\kappa^{-2}\delta_R(\alpha_R + \beta_R)^{1/2} \\
&\le C\kappa(\alpha_R + \beta_R) + C\kappa^{-1}(\alpha_R + \beta_R)^{3/2} + C\kappa(\alpha_R + \beta_R) + C\kappa^{-3}\delta_R^2 \\
&\le C\kappa(\alpha_R + \beta_R) + C\kappa^{-3}\delta_R^2,
\end{aligned}$$

since $\kappa^{-1}\mu^{1/2} < C$ by (3.7). So we have obtained

$$\alpha_r + \beta_r \le C\kappa[\alpha_R + \beta_R + \kappa^{-4}\delta_R^2]. \tag{3.8}$$

abcd

We now have to estimate on δ_R, i.e. the pressure p. We give a very brief idea how this is obtained, but do not give the derivation. Note that if we take the divergence of the Navier–Stokes equations we obtain

$$\Delta p = -\partial_i\partial_j u_i u_j.$$

This tells us (heuristically) to expect that p has the same regularity as $|u|^2$

– we expect to estimate δ_r (the $L^{3/2}$ norm of p) in terms of γ_R (the L^3 norm of u) plus some boundary terms.

Since we want a local estimate, the idea is to choose a cut-off function η, and write

$$\Delta(\eta p) = (\Delta\eta)p + 2\nabla p \cdot \nabla\eta - \eta\partial_i\partial_j u_i u_j$$

(the last term is $-\eta\Delta p$). We can now invert the Laplacian to give

$$\eta p = \Delta^{-1}[(\Delta\eta)p + 2\nabla p \cdot \nabla\eta - \eta\partial_i\partial_j u_i u_j].$$

The Laplacian can be inverted explicitly, and there are many methods from harmonic analysis that allow us to estimate the resulting terms on the right-hand side. One can also choose a variety of ways to split up the terms on the right-hand side, resulting in a selection of different possible estimates; different splittings have been exploited by different authors.

The bound obtained by Kukavica can be expressed in terms of the scale-invariant quantities we are using, as

$$\delta_r \le C\kappa^{-1/2}\alpha_R^{1/2}\beta_R^{1/2} + C\kappa^{1/3}\delta_R, \tag{3.9}$$

pbound

where $0 < r \le R/2$ as before. (Note that the first term is just an interpolation of the L^3 norm of u - we have essentially $\|p\|_{L^{3/2}} \le c\|u\|_{L^3}+$ boundary terms, as our heuristics would suggest.)

Now, if we return to (3.8) note that the quantity $\alpha_R + \beta_R + \kappa^{-4}\delta_R^2$ occurs on the right-hand side. So we add $\kappa^{-4}\delta_r^2$ to the left-hand side and use (3.9) to write

$$\begin{aligned}\alpha_r + \beta_r + \kappa^{-4}\delta_r^2 &\le C\kappa(\alpha_R + \beta_R + \kappa^{-4}\delta_R^2) + C\kappa^{-5}\alpha_R\beta_R + C\kappa^{-10/3}\delta_R^2 \\ &\le C\kappa(\alpha_R + \beta_R + \kappa^{-4}\delta_R^2) + C\kappa^{-5}(\alpha_R + \beta_R)^2 + C\kappa^{2/3}\frac{\delta_R^2}{\kappa^4}.\end{aligned}$$

Now choose κ then μ such that

$$C\kappa^{2/3} < 1/8 \qquad \text{and then} \qquad C\kappa^{-5}\mu < 1/4. \tag{3.10}$$

choice2

With this choice it follows that

$$\alpha_r + \beta_r + \kappa^{-4}\delta_r^2 \le \frac{1}{2}(\alpha_R + \beta_R + \kappa^{-4}\delta_R^2).$$

Since $r = \kappa R$, we can iterate this inequality to deduce that

$$\alpha_{\kappa^n R} + \beta_{\kappa^n R} + \kappa^{-4}\delta_{\kappa^n R} \le 2^{-n}(\alpha_R + \beta_R + \kappa^{-4}\delta_R^2).$$

This result enhances the decay rate of α_r: since

$$\alpha_{\kappa^n R} \le C2^{-n},$$

it follows that

$$\alpha_r \leq C2^{-(\log r - \log R)/\log \kappa} = Cr^{-\log 2/\log \kappa};$$

and similarly for β_r and δ_r^2. Therefore, since $\gamma_r \leq C(\alpha_r + \beta_r)$, we have (with $\epsilon = -\log 2/\log \kappa > 0$)

$$\frac{1}{r^2} \iint_{Q_r} |u|^3 \leq Cr^{\epsilon}$$

and

$$\frac{1}{r^2} \iint_{Q_r} |p|^{3/2} \leq Cr^{\epsilon/2},$$

as required. □

Theorem 3.7 *Let S denote the singular set of a suitable weak solution of the Navier–Stokes equations. Then $P^1(S) = 0$.*

We will need the following fact: given a family of parabolic cylinders $Q_r^*(x,t)$, there exists a finite or countable disjoint subfamily $\{Q_{r_i}^*(x_i,t_i)\}$ such that for any cylinder $Q_r^*(x,t)$ in the original family there exists an i such that $Q_r^*(x,t) \subset Q_{5r_i}^*(x_i,t_i)$. (For a proof see CKN.)

Proof Let V be any neighbourhood of S, and choose $\delta > 0$.

For each $(x,t) \in S$, choose a cylinder $Q_r^*(x,t)$ such that $Q_r^*(x,t) \subset V$, $r < \delta$, and

$$\frac{1}{r} \iint_{Q_r^*(x,t)} |\nabla u|^2 > \delta_*.$$

(This must be possible, for otherwise by Theorem 3.3 the point (x,t) would be regular.) We now find a disjoint subcollection of these cylinders $\{Q_{r_i}^*(x_i,t_i)\}$ such that the singular set is still covered by $\{Q_{5r_i}^*(x_i,t_i)\}$. Since these cylinders are disjoint,

$$\iint_V |\nabla u|^2 \geq \sum_i \iint_{Q_{r_i}^*(x_i,t_i)} |\nabla u|^2 \geq \delta^* \sum_i r_i.$$

Since $\nabla u \in L^2((0,T)\times\Omega)$, the left-hand side is finite, so $\sum_i r_i \leq C$. Since S is contained in the union of $\{Q_{5r_i}^*(x_i,t_i)\}$, and $r_i < \delta$ for every i, we must have

$$\mu(S) \leq c \sum_i (5r_i)^5 \leq c\delta^4 \sum_i r_i \leq K\delta^4.$$

Since $\delta > 0$ we arbitrary, it follows that $\mu(S) = 0$.

Since $|\nabla u|^2$ is integrable and V is an arbitrary neighbourhood of S (which has zero measure), we can make

$$\frac{1}{\delta^*} \iint_V |\nabla u|^2$$

as small as we wish by choosing V suitably. The above construction then furnishes a cover with $\sum_i (5r_i)$ arbitrarily small, and so $P^1(S) = 0$ as claimed. □

References

Caffarelli, L., Kohn, R., & Nirenberg, L. (1982) Partial regularity of suitable weak solutions of the Navier–Stokes equations. *Comm. Pure. Appl. Math.* **35**, 771–831.

Constantin, P., & Foias, C. (1988) *Navier-Stokes equations* (University of Chicago Press, Chicago).

Escauriaza, Seregin, & Sverak (2002) $L^{3,\infty}$-solutions to the Navier–Stokes equations and backward uniqueness. *Uspekhi Matematicheskih Nauk* **vol. 58, 2(350)**, 3-44. English translation in *Russian Mathematical Surveys* **58** (2003) 2, 211–250.

Fabes, E.B., Jones, B.F., & Rivière, N.M. (1972) The initial value problem for the Navier–Stokes equations with data in L^p. *Arch. Rat. Mech. Anal.* **45**, 222–240.

Galdi, G.P. (2000) An introduction to the Navier–Stokes initial-boundary value problem. In: Galdi, G.P., Heywood, J.G., & Rannacher, R. (eds.) *Fundamental Directions in Mathematical Fluid Dynamics.* (Birkhuser-Verlag, Basel, 2000, pp. 170). Also available online:
`http://www.numerik.uni-hd.de/Oberwolfach-Seminar/Galdi_Navier_Stokes_Notes.pdf`

Grafakos, L. (2008) *Classical Fourier analysis.* (Springer).

Kim, H. & Kozono, H. (2004) Interior regularity criteria in weak spaces for the Navier–Stokes equations. *Manuscripta Math.* **115**, 85-100.

Kukavica, I. (2009a) Partial regularity results for solutions of the Navier-

Stokes system. pp. 121–145 in: Robinson, J.C. & Rodrigo, J.L. (eds.) *Partial Differential Equations and Fluid Mechanics* (Cambridge University Press, Cambridge).

Kukavica, I. (2009b) The fractal dimension of the singular set for solutions of the Navier–Stokes system. *Nonlinearity* **22**, 2889–2900.

Ladyzhenskaya, O.A. & Seregin, G.A. (1999) On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations. *J. Math. Fluid Mech.* **1**, 356-387.

Leray, J. (1934) Sur le mouvement d'un liquide visqueux emplissant l'esapce. *Acta Math* **63**, 193–248.

Lieberman, G.M. (1996) *Second order parabolic differential equations.* (World Scientific, Singapore).

Lin, F.H. (1998) A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure. Appl. Math.* **51**, 241–257.

Majda, A.J. & Bertozzi, A.L. (2002) *Vorticity and incompressible flow.* (Cambridge University Press, Cambridge).

Robinson, J.C. & Sadowski, W. (2007) Decay of weak solutions and the singular set of the three-dimensional Navier-Stokes equations. *Nonlinearity* **20**, 1185-1191.

Robinson, J.C. & Sadowski, W. (2009) Almost everywhere uniqueness of Lagrangian trajectories for suitable weak solutions of the three-dimensional Navier–Stokes equations. *Nonlinearity* **22**, 2093–2099.

Scheffer, V. (1976) Turbulence and Hausdorff dimension. pp. 174-183 in: *Turbulence and Navier–Stokes Equation, Orsay 1975*, Springer Lecture Notes in Mathematics vol 565 (Springer, Berlin)

Serrin, J. (1962) On the interior regularity of weak solutions of the Navier–Stokes equations. *Arch. Rat. Mech. Anal.* **9**, 187–195.

Sohr, H., & von Wahl, W. (1986) On the regularity of the pressure for weak solutions of the Navier–Stokes equations. *Arch. Math.* **36**, 428–439.

Struwe, M. (1988) On partial regularity results for the Navier–Stokes equations. *Comm. Pure. Appl. Math.* **41**, 437–458.

Takahashi, S. (1990) On interior regularity criertia for weak solutions of the Navier–Stokes equations. *Manuscripta Math.* **69**, 237–254.

Temam, R. (2001) *Navier-Stokes equations: Theory and numerical analysis.* AMS Chelsea Publishing, Providence, RI. Reprint of the 1984 edition.