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Examples I

1. Suppose that

\[ v = \sum_{j=1}^{n} \alpha_j e_j \quad \text{and} \quad v = \sum_{k=1}^{m} \beta_k f_k. \]

with \( \alpha_j, \beta_k \in \mathbb{K} \) and \( e_j, f_k \in E \). Relabel \( \beta_k \) and \( f_k \) so that \( f_j = e_j \) for \( j = 1, \ldots, n' \), and \( f_j \notin \{e_1, \ldots, e_n\} \) for \( j > n' \), i.e.

\[ v = \sum_{k=1}^{n'} \beta_k e_k + \sum_{k=n'+1}^{m} \beta_k f_k, \]

with the understanding that the first sum is zero if \( n' = 0 \), and the second zero if \( n' = m \).

It follows that

\[ \sum_{j=1}^{n} \alpha_j e_j - \sum_{k=1}^{n'} \beta_k e_k - \sum_{k=n'+1}^{m} \beta_k f_k = 0, \]

or

\[ \sum_{j=1}^{n'} (\alpha_j - \beta_j) e_j - \sum_{k=n'+1}^{n} \alpha_j e_j + \sum_{k=n'+1}^{m} \beta_k f_k = 0. \]

Since \( E \) is linearly independent, it follows that \( \alpha_j = \beta_j = 0 \) for \( j > n' \), from which \( n' = n \), and that \( \alpha_j = \beta_j \) for \( j = 1, \ldots, n \). So the expansion is unique.

2. If \( \bar{x}, \bar{y} \in \ell_{\text{lim}}(\mathbb{C}) \) then \( \bar{x} + \bar{y} \in \ell_{\text{lim}}(\mathbb{C}) \) and \( \alpha \bar{x} \in \ell_{\text{lim}}(\mathbb{C}) \), since

\[ \lim_{n \to \infty} (x_n + y_n) = \lim_{n \to \infty} x_n + \lim_{n \to \infty} y_n \quad \text{and} \quad \lim_{n \to \infty} \alpha x_n = \alpha \lim_{n \to \infty} x_n. \]
3. Write
\[ \|x\|_p^p = \sum_j |x_j|^p = \sum_j |x_j|^{p-\alpha} |x_j|^{\alpha}. \]

Hölder’s inequality with exponents \( a \) and \( b \) yields
\[ \|x\|_p^p \leq \left( \sum_j |x_j|^{(p-\alpha)a} \right)^{1/a} \left( \sum_j |x_j|^{\alpha b} \right)^{1/b} = \|x\|_r^{p-\alpha} \|x\|_r^{\alpha}. \]

We want to choose \( \alpha \), and conjugate exponents \( a, b \) \((a^{-1} + b^{-1} = 1)\) so that
\( (p-\alpha)a = q \) and \( \alpha b = r \),
so that the factors on the right are powers of \( \|x\|_r \) and \( \|x\|_r \). Thus \( b = r/\alpha \); since \( a, b \) are conjugate it follows that \( a = r/(r-\alpha) \), and then the condition that \( (p-\alpha)a = q \) gives
\[ \alpha = (p-q)r/(r-q), \quad a = (r-q)/(r-p), \quad b = (r-q)/(p-q). \]

The result follows as stated on substituting for \( \alpha \).

4.(i) If there exist \( a, b \) with \( 0 < a \leq b \) such that
\[ a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \]
then
\[ \frac{1}{b} \|x\|_2 \leq \|x\|_1 \leq \frac{1}{a} \|x\|_2, \]
while
(ii) if
\[ a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \]
and
\[ \alpha\|x\|_2 \leq \|x\|_3 \leq \beta\|x\|_2 \]
then
\[ a\alpha\|x\|_1 \leq \|x\|_3 \leq b\beta\|x\|_1. \]

5. Since (by the previous question) being equivalent is an equivalence relation (!), we only need to show that \( \| \cdot \|_2 \) and \( \| \cdot \|_\infty \) are equivalent to \( \| \cdot \|_1 \).
Examples I

We have
\[ \sum_{j=1}^{n} |x_j|^2 \leq \left( \sum_{j=1}^{n} |x_j| \right)^2 \leq n \sum_{j=1}^{n} |x_j|^2, \]
which gives \( \|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2, \) and
\[ \max_{j=1, \ldots, n} |x_j| \leq \sum_{j=1}^{n} |x_j| \leq n \max_{j=1, \ldots, n} |x_j|, \]
i.e. \( \|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty. \)

6. First we show that if \( f \) is continuous and \( K \) is closed then
\[ f^{-1}(K) = \{ x \in X : f(x) \in K \} \]
is closed. (In fact this is in the proof of Corollary 3.12 in the notes.)
Take \( x_n \in f^{-1}(K) \), and suppose that \( x_n \to x \); since \( f \) is continuous, \( f(x_n) \to f(x) \). But \( f(x_n) \in K \), and \( K \) is closed, so \( f(x) \in K \). It follows that \( x \in f^{-1}(K) \), and so \( f^{-1}(K) \) is closed.

Now suppose that whenever \( K \) is closed, \( f^{-1}(K) \) is closed. Take \( x_n \in X \) with \( x_n \to x \), and suppose that \( f(x_n) \not\to f(x) \). Then for some subsequence \( x_{n_j} \), we must have
\[ |f(x_{n_j}) - f(x)| > \epsilon. \]
So \( f(x_{n_j}) \) is contained in the closed set \( Z = Y \setminus B(f(x), \epsilon) \), where
\[ B(f(x), \epsilon) = \{ y \in Y : \|y - f(x)\|_Y < \epsilon \} \]
is open. It follows that \( f^{-1}(Z) \) is closed. Since \( f(x_{n_j}) \in Z \), \( x_{n_j} \in f^{-1}(Z) \); since \( x_{n_j} \to x \) and \( f^{-1}(Z) \) is closed, it follows that \( x \in f^{-1}(Z) \), so that
\[ |f(x) - f(x)| > \epsilon, \]
clearly a contradiction.

7. Let \( Y \) be closed subset of the compact set \( K \). If \( y_n \in Y \) then \( y_n \in K \), so there is a subsequence \( y_{n_j} \) such that \( y_{n_j} \to y \in K \). Since \( Y \) is closed, \( y \in Y \), and so \( Y \) is compact.

8. If \( \{u_n\} \) is a Cauchy sequence in \( U \) then, since \( U \subset V \) and the norm on \( U \) is simply the restriction of the map \( u \mapsto \|u\|_V \) to \( U \), it follows that \( \{u_n\} \) is also a Cauchy sequence in \( V \). Since \( V \) is complete, \( u_n \to u \) for some \( u \in V \). But since \( U \) is closed, we know that if \( u_n \to u \) then \( u \in U \). So \( U \) is complete.
As it stands the question is ambiguous, since one needs to specify a norm on \(c_0(\mathbb{K})\). To be interesting the question requires the \(\ell^\infty\) norm; any element of \(\ell^p\) with \(p < \infty\) must be an element of \(c_0(\mathbb{K})\).

To show that \(c_0(\mathbb{K})\) (with the \(\ell^\infty\) norm) is complete, we will first show that \(\ell^\infty(\mathbb{K})\) is complete (this case was omitted from the proof of Proposition 4.5 in the notes, put the proof is simpler), and then that \(c_0(\mathbb{K})\) is closed, which will show that \(c_0(\mathbb{K})\) is complete by the previous question.

So first suppose that \(x^k = (x^k_1, x^k_2, \ldots)\) is a Cauchy sequence in \(\ell^\infty(\mathbb{K})\). Then for every \(\epsilon > 0\) there exists an \(N_\epsilon\) such that 

\[
\|x^n - x^m\|_\ell^\infty = \sup_j |x^n_j - x^m_j| < \epsilon \quad \text{for all } n, m \geq N_\epsilon. \quad (1.1)
\]

In particular \(\{x^k_j\}_{k=1}^{\infty}\) is a Cauchy sequence in \(\mathbb{K}\) for every fixed \(j\). Since \(\mathbb{K}\) is complete (recall \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\)) it follows that for each \(k \in \mathbb{N}\)

\[x^k_j \to a_k\]

for some \(a_k \in \mathbb{R}\).

Set \(a = (a_1, a_2, \ldots)\). We want to show that \(a \in \ell^\infty\) and also that \(\|x^k - a\|_\ell^\infty \to 0\) as \(k \to \infty\). First, since \(\{x^k\}\) is Cauchy we have from (1.1) that \(\|x^n - x^m\|_\ell^\infty < \epsilon\) for all \(n, m \geq N_\epsilon\), and so

\[
\sup_j |x^n_j - x^m_j| \leq \epsilon.
\]

Letting \(m \to \infty\) we obtain

\[
\sup_j |x^n_j - a_j| \leq \epsilon,
\]

and so \(x^k - a \in \ell^\infty\). But since \(\ell^\infty\) is a vector space and \(x^k \in \ell^\infty\), this implies that \(a \in \ell^\infty\) and \(\|x^k - a\|_\ell^\infty \leq \epsilon\).

Now take \(\bar{x}^n \in c_0(\mathbb{K})\) such that \(\bar{x}^n \to \bar{x}\) in \(\ell^\infty\). Suppose that \(\bar{x} \notin c_0(\mathbb{K})\). Then there exists a \(\delta > 0\) and a sequence \(n_j \to \infty\) such that \(|x_{n_j}| > \delta\) for every \(j\). Now choose \(N\) large enough that \(\|\bar{x}^n - \bar{x}\|_\ell^\infty < \delta/2\) for all \(n \geq N\).

In particular it follows that \(|x_{n_j}^n| > \delta/2\) for all \(n \geq N\) (and every \(j\)); but then \(\bar{x}^n \notin c_0(\mathbb{K})\), a contradiction.
Examples II

1. If \( \mathbf{x} = (x_1, x_2, \ldots) \in \ell^2(\mathbb{R}) \), then
   \[
   \sum_{j=1}^{\infty} |x_j|^2 < \infty.
   \]
   So
   \[
   \| \mathbf{x} - (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \|_{\ell^2}^2 = \sum_{j=n+1}^{\infty} |x_j|^2 \to 0
   \]
as \( n \to \infty \). So \( \ell_f(\mathbb{R}) \) is dense in \( \ell^2(\mathbb{R}) \).

2. If \( x \in X \cap Y \) then there exist \( \epsilon_X \) and \( \epsilon_Y \) such that
   \[
   \{ y \in B : \| y - x \| < \epsilon_X \} \subset X \quad \text{and} \quad \{ y \in B : \| y - x \| < \epsilon_Y \} \subset Y.
   \]
   Taking \( \epsilon = \min(\epsilon_X, \epsilon_Y) \) it follows that
   \[
   \{ y \in B : \| y - x \| < \epsilon \} \subset X \cap Y,
   \]
   and so \( X \cap Y \) is open.

Now, given \( z \in B \) and \( \epsilon > 0 \), since \( X \) is dense there exists an \( x \in X \) such that \( \| x - z \| < \epsilon/2 \). Since \( X \) is open, there is a \( \delta < \epsilon/2 \) such that
   \[
   \{ x' \in B : \| x' - x \| < \delta \} \subset X.
   \]
   Since \( Y \) is dense, there is a \( y \in Y \) such that \( \| y - x \| < \delta/2 \). By the above, it follows that we also have \( y \in X \). So we have found \( y \in X \cap Y \) such that \( \| y - z \| \leq \| y - x \| + \| x - z \| < \delta/2 + \epsilon/2 < \epsilon \), and so \( X \cap Y \) is dense.

A refinement of this argument allows one to prove the powerful Baire Category Theorem: a countable intersection of open and dense sets is dense.
3. We have to assume that \((Y, \| \cdot \|_Y)\) is a Banach space. Then if \(x_n \in X\) and \(x_n \to x\) (with \(x \in V\)) we know that \(\{x_n\}\) is Cauchy in \(V\). So, since

\[\|F(x_n) - F(x_m)\|_Y \leq L \|x_n - x_m\|\]

it follows that \(\{F(x_n)\}\) is a Cauchy sequence in \(Y\). Since \(Y\) is complete, we know that \(\lim_{n \to \infty} F(x_n)\) exists and is an element of \(Y\).

If \(x_n \to v\), and \(y_n \to v\), then

\[\|F(x_n) - F(y_n)\| \leq L \|x_n - y_n\|.
\]

Taking limits as \(n \to \infty\) on both sides implies that

\[\lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} F(y_n).
\]

That \(F\) so defined is continuous is clear: for any \(v_1, v_2 \in V\), one can find sequences in \(X\) that converge to \(v_1, v_2\), and then from the definition

\[\|F(v_1) - F(v_2)\| \leq L \|v_1 - v_2\|.
\]

4. Let \(V = \mathbb{R}\), \(X = \mathbb{Q}\), and expand each \(q \in \mathbb{Q}\) to the interval \(I_q = (q - d(q), q + d(q))\), where \(d(q) = \sqrt{q^2 - 2^q}/2\). Then \(\sqrt{2} \notin I_q\) for any \(q\), so \(\bigcup_{q \in \mathbb{Q}} I_q\) does not cover \(\mathbb{R}\).

5. Fix \(n \in \mathbb{N}\), and cover \(A_j\) with a collection of intervals \(I_{k}^{(j)}\) such that \(\sum_k |I_{k}^{(j)}| < 2^{-(n+j)}\). Then \(\bigcup_j A_j\) is covered by

\[\bigcup_{j,k} I_{k}^{(j)},\]

and \(\sum_{j,k} |I_{k}^{(j)}| < \sum_{j=n+1}^\infty 2^{-j} = 2^{-n}\).

If \(P_j\) occurs almost everywhere then it fails on a set \(A_j\) of measure zero. Since the union of the \(A_j\) still has measure zero, every \(P_j\) occurs simultaneously almost everywhere.

6. Use \(\phi_n \uparrow f\) to mean that \(\phi_n\) is an increasing sequence that converges almost everywhere to \(f\). (i) is essentially Lemma 5.5. (ii) If \(f, g \in L^{\text{inc}}(\mathbb{R})\) then there are sequences \(\{\phi_n\}, \{\psi_n\} \in L^{\text{step}}(\mathbb{R})\) such that \(\phi_n \uparrow f\) and \(\psi_n \uparrow g\). Then \(\{\phi_n + \psi_n\}\) is an increasing sequence of step functions with \(\phi_n + \psi_n \uparrow f + g\). So \(f + g \in L^{\text{inc}}(\mathbb{R})\), and since the integral is additive on \(L^{\text{step}}(\mathbb{R})\),

\[\int f + g = \lim_{n \to \infty} \int \phi_n + \psi_n = \lim_{n \to \infty} (\int \phi_n + \int \psi_n) = \int f + \int g.
\]

(iii) If \(\{\phi_n\} \in L^{\text{step}}(\mathbb{R})\) with \(\phi_n \uparrow f\) then \(\lambda \phi_n \in L^{\text{step}}(\mathbb{R})\) and \(\lambda \phi_n \uparrow f\). The result follows from the fact that \(\int \lambda \phi_n = \lambda \int \phi_n\).
(iv) Follows from the same properties for step functions, along similar lines to the above.

7. For \( x \in [n, n + 1) \) we have

\[
\int_0^x f(r) \, dr + \int_n^x f(r) \, dr = \sum_{j=1}^{n-1} \frac{(-1)^j}{j} + (x - n) \frac{(-1)^n}{n},
\]

and so

\[
\lim_{x \to \infty} \int_0^x f(x) \, dx = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} < \infty.
\]

But the same calculation with \( |f(x)| \) gives

\[
\int_0^x \geq \int_1^n |f(r)| \, dr = \sum_{j=1}^{n-1} \frac{1}{j},
\]

which diverges as \( n \to \infty \). [Functions can be 'integrable' in some sense even when they are not 'Lebesgue integrable'.]

8. Consider \( f_n = n|f| \). Then \( f_n \) is a monotonic sequence, and \( \int f_n = n \int |f| = 0 \) for every \( n \). So there exists a \( g \in L^1 \) such that \( f_n \to g \) almost everywhere. The limit as \( n \to \infty \) of \( f_n \) is zero where \( f = 0 \), and \( +\infty \) where \( f_n \neq 0 \). Since \( g \) is defined almost everywhere, it follows that \( f = 0 \) almost everywhere.

9. If \( f \in L^{\text{inc}}(\mathbb{R}) \) then there exists a sequence \( \{\phi_n\} \in L^{\text{step}}(\mathbb{R}) \) such that \( \phi_n \uparrow f \) (almost everywhere), and we define

\[
\int f = \lim_{n \to \infty} \int \phi_n.
\]

Since \( f \geq \phi_n \),

\[
\int |f - \phi_n| = \int f - \int \phi_n,
\]

which tends to zero as \( n \to \infty \) by the definition of \( \int f \).

Now take a step function

\[
\phi(x) = \sum_{j=1}^{n} c_j \chi_{I_j}(x)
\]

with the endpoints of \( I_j \) being \( a_j \) and \( b_j \). Consider the ‘continuous version’
of $\chi_I(x)$, for an interval $I$ with endpoints $a, b$,

$$X_I(x; \delta) = \begin{cases} 
0 & x < a - \delta \\
(x - a + \delta)/\delta & a - \delta \leq x \leq a \\
1 & a < x < b \\
(b + \delta - x)/\delta & b \leq x \leq b + \delta \\
0 & x > b + \delta.
\end{cases}$$

Now if $\delta < \epsilon/(n \max_j |c_j|)$ and

$$\phi_{\epsilon}(x) = \sum_{j=1}^{n} c_j X_{I_j}(x; \delta),$$

then

$$\int |\phi_{\epsilon}(x) - \phi(x)| \, dx < \epsilon.$$

So if $g \in L^1(\mathbb{R})$, we have $g = f_1 - f_2 \in L^{\text{inc}}(\mathbb{R})$. Approximate $f_1$ and $f_2$ to within $\epsilon/4$ (wrt the $L^1$ norm) by step functions $h_1$ and $h_2$, and then approximate $h_1$ and $h_2$ by continuous functions $c_1$ and $c_2$ to within $\epsilon/4$ (again wrt the $L^1$ norm). So $c_1 - c_2 \in C^0(\mathbb{R})$ approximates $g$ to within $\epsilon$, i.e. $C^0(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.

10. With $g_n = (\sum_{k=1}^{n} |f_k|)^2$ we have

$$\int g_n = \int \left(\sum_{k=1}^{n} |f_k|\right)^2 = \left(\sum_{k=1}^{n} \|f_k\|_{L^2}\right)^2 \leq \left(\sum_{k=1}^{\infty} \|f_k\|_{L^2}\right)^2$$

using the definition of the $L^2$ norm and the triangle inequality. It follows that $g_n \to g$ almost everywhere for some $g \in L^1$. It follows that $\sum_{k=1}^{\infty} f_k$ is absolutely convergent almost everywhere to some $f \in L^2$.

11. Note that we have

$$\left| f - \sum_{k=1}^{n} f_k \right|^2 = \left(\sum_{k=1}^{\infty} f_k \right)^2 \leq \left(\sum_{k=1}^{\infty} |u_k| \right)^2 \leq |f|^2.$$

So we can use the DCR to deduce that

$$\lim_{n \to \infty} |f - \sum_{k=1}^{n} f_k|^2 = \int \lim_{n \to \infty} |f - \sum_{k=1}^{n} u_k|^2 = 0.$$
Examples III

1. Expanding the right-hand side gives

\[(x + y, x + y) - (x - y, x - y) + i(x + iy, x + iy) - i(x - iy, x - iy)\]

\[= \|x\|^2 + (x, y) + \|y\|^2 - \|x\|^2 + (x, y) + (y, x) - \|y\|^2\]

\[+ i(\|x\|^2 + iy, x - iy) - i(y, x) + (y, x) - \|y\|^2\]

\[= 4 \text{real}(x, y) + 4 \text{imag}(x, y) = 4(x, y)\]

2. Write

\[\left(\int_s^t |f(r)| \, dt \right)^2 = \left(\int_s^t r^{-1/2} r^{1/2} |f(r)| \, dr \right)^2\]

\[\leq \left(\int_s^t r^{-1} \, dr \right) \left(\int_s^t r |f(r)|^2 \, dr \right)\]

\[\leq K (\log t - \log s)^{1/2}\]

using the Cauchy-Schwarz inequality. [\(K^2\) in examples sheet is a misprint.]

3. Suppose that \(E\) is orthonormal and that

\[\sum_{j=1}^n \alpha_j e_j = 0\]

for some \(\alpha_j \in \mathbb{K}\) and \(e_j \in E\). Then taking the inner product of both sides with \(e_k\) (with \(k = 1, \ldots, n\)) gives

\[\sum_{j=1}^n \alpha_j e_j, e_k\]

\[\Rightarrow \alpha_k = 0,\]

so \(\alpha_k = 0\) for each \(k\), i.e. \(E\) is linearly independent.
4.(i) Suppose that \( d = 0 \). Then there exists a sequence \( y_n \in Y \) such that
\[
\lim_{n \to \infty} \|x - y_n\| = 0,
\]
i.e. \( y_n \to x \). Since \( Y \) is a closed linear subspace, it follows that \( x \in Y \) but this contradicts the fact that \( x \notin Y \).

(ii) By definition there exists a sequence \( y_n \in Y \) such that \( d = \lim_{n \to \infty} \|x - y_n\| \). So for \( n \) sufficiently large, clearly \( \|x - y_n\| < 2d \). Let \( z \) be one such \( y_n \).

(iii) We have
\[
\|\hat{x} - y\| = \left\| \frac{x - z}{\|x - z\|} - y \right\| = \frac{1}{\|x - z\|} \left\| x - (z + \|x - z\|y) \right\| > \frac{1}{2d} d = \frac{1}{2},
\]
since \( z + \|x - z\|y \in Y \) and \( d = \inf \{ \|x - y\| : y \in Y \} \).

5. Take \( x_1 \in \mathcal{B} \) with \( \|x_1\| = 1 \). Let \( Y_1 \) be the linear subspace \{\( \alpha x_1 : \alpha \in \mathbb{K} \)\}. Using the result of question 7 there exists an \( x_2 \in \mathcal{B} \) with \( \|x_2\| = 1 \) such that \( \|x_2 - x_1\| > \frac{1}{2} \). Now let \( Y_2 = \text{Span}(x_1, x_2) \), and use question 7 to find \( x_3 \in \mathcal{B} \) with \( \|x_3\| = 1 \) with \( \|x_3 - y\| > \frac{1}{2} \) for every \( y \in \text{Span}(x_1, x_2) \) – in particular
\[
\|x_3 - x_1\| > \frac{1}{2} \quad \text{and} \quad \|x_3 - x_2\| > \frac{1}{2}.
\]
Continuing in this way we obtain a sequence \( \{x_n\} \in \mathcal{B} \) such that \( \|x_n\| = 1 \) and
\[
\|x_n - x_m\| > \frac{1}{2}
\]
for all \( n \neq m \). It follows that the unit ball is not compact, since no subsequence of the \( \{x_n\} \) can be a Cauchy sequence.

6. The unit ball is closed and bounded. If \( \mathcal{B} \) is finite-dimensional then this implies that the unit ball is compact. We have just shown, conversely, that if \( \mathcal{B} \) is infinite-dimensional then the unit ball is not compact.

7. Take \( x, y \in A \) and suppose that \( \lambda x + (1 - \lambda)y \in A \) for all \( \lambda \in [0, 1] \) such that \( 2^k \lambda \in \mathbb{N} \). Then given \( \mu \) with \( 2^{k+1} \mu \in \mathbb{N} \), \( 2^k \mu \notin \mathbb{N} \) we have
\[
\mu = l 2^{-(k+1)}
\]
with \( l \) odd, so that
\[
\mu = \left( \frac{l - 1}{2} \right) 2^{-(k+1)} + \left( \frac{l + 1}{2} \right) 2^{-(k+1)}.
\]
With \( \lambda_1 = (l - 1)2^{-(k+1)} \) and \( \lambda_2 = (l + 1)2^{-(k+1)} \) we have \( 2^k \lambda_1, 2^k \lambda_2 \in \mathbb{N} \), and so
\[
\mu x + (1 - \mu)y = \frac{1}{2} \left[ (\lambda_1 x + (1 - \lambda_1)y) + (\lambda_2 x + (1 - \lambda_2)y) \right],
\]
with $\lambda_j x + (1 - \lambda_j)y \in A$, i.e. $\mu x + (1 - \mu)y \in A$.

It follows by induction that $\lambda x + (1 - \lambda)y \in A$ for any $\lambda \in [0, 1]$ such that $2^k \lambda \in \mathbb{N}$ for some $k \in \mathbb{N}$. Since any $\lambda \in [0, 1]$ can be approximated by a sequence $\lambda_j$ such that $2^j \lambda_j \in \mathbb{N}$, it follows since $A$ is closed that $\lambda x + (1 - \lambda)y \in A$ for all $\lambda \in [0, 1]$, i.e. $A$ is convex.

8. We have $x \in (X^\perp)^\perp$ if for every $y \in X^\perp$, $(x, y) = 0$. So clearly $X \subseteq (X^\perp)^\perp$. If we do not have equality here then there exists a $z \notin X$ with $(z, y) = 0$ for every $y \in X^\perp$. But we know that $z = x + \xi$ with $x \in X$ and $\xi \in X^\perp$; but then $(x + \xi, \xi) = 0$, which implies that $\|x\|^2 = 0$, i.e. $x \in X$. So $(X^\perp)^\perp = X$ as claimed.

9. We have $e'_4 = x^3 - \left(x^3, \sqrt{\frac{5}{8}}(3x^2 - 1)\right) \sqrt{\frac{5}{8}}(3x^2 - 1) - \left(x^3, \sqrt{\frac{3}{2}}x\right) \sqrt{\frac{3}{2}}x$.

Then
\[ \|e'_4\|^2 = \int_{-1}^{1} \left(t^3 - \frac{3t^5}{5}\right)^2 dt = \left[\frac{t^7}{7} - \frac{6t^5}{25} + \frac{3t^3}{25}\right]_{-1}^{1} = \frac{8}{7 \times 25} \]
and so
\[ e_4 = \sqrt{\frac{7}{8}} (5x^3 - 3x). \]

10. Approximation of $\sin x$ by a third degree polynomial $f_3(x)$: note that $(\sin x, e_1)$ and $(\sin x, e_3)$ will be zero since $\sin x$ is odd and $e_1$ and $e_3$ are even, so
\[ f_3(x) = \frac{3x}{2} \int_{-1}^{1} t \sin t dt + \frac{7(5x^3 - 3x)}{8} \int_{-1}^{1} (5t^3 - 3t) \sin t dt. \]

Now,
\[ \int_{-1}^{1} t \sin t dt = [-t \cos t]_{-1}^{1} + \int_{-1}^{1} \cos t dt = [\sin t]_{-1}^{1} = 2 \sin 1. \]
and
\[
\int_{-1}^{1} t^3 \sin t \, dt = \left[ -t^3 \cos t \right]_{-1}^{1} + \int_{-1}^{1} 3t^2 \cos t \, dt
\]
\[
= \left[ 3t^2 \sin t \right]_{-1}^{1} - 6 \int_{-1}^{1} t \sin t \, dt
\]
\[
= -6 \sin 1,
\]
giving
\[
\int_{-1}^{1} (5t^3 - 3t) \sin t \, dt = -36 \sin 1;
\]
it follows that
\[
f_3(x) = \sin 1 \left[ 3x + \frac{7(27x - 45x^3)}{2} \right] = \sin 1 \left[ \frac{195x - 315x^3}{2} \right].
\]

11. First way: given orthonormal polynomials \( \tilde{e}_j (x) \) on \([-1, 1]\) put
\[
e_j (x) = \sqrt{2} \tilde{e}_j (2x - 1),
\]
since then
\[
\int_{0}^{1} e_j (x) e_k (x) \, dx = 2 \int_{0}^{1} \tilde{e}_j (2x - 1) \tilde{e}_k (2x - 1) \, dx = \int_{-1}^{1} \tilde{e}_j (y) \tilde{e}_k (y) \, dy.
\]
Doing this gives
\[
e_1 (x) = 1, \quad e_2 (x) = \sqrt{3} (2x - 1),
\]
\[
e_3 (x) = \frac{\sqrt{5}}{2} \left[ 3(4x^2 - 4x + 1) - 1 \right] = \sqrt{5} (6x^2 - 6x + 1),
\]
and
\[
e_4 (x) = \frac{\sqrt{7}}{2} \left[ 5(8x^3 - 12x^2 + 6x - 1) - 3(2x - 1) \right]
\]
\[
= \sqrt{7} [20x^3 - 30x^2 + 12x - 1].
\]

Or the painful way: \( e_1 (x) = 1 \), then
\[
e_2' (x) = x - \int_{0}^{1} t \, dt = x - \frac{1}{2},
\]
and then
\[
\|e_2'\|^2 = \int_{0}^{1} (x - \frac{1}{2})^2 \, dx = \left[ \frac{(x - \frac{1}{2})^3}{3} \right]_{0}^{1} = \frac{1}{12}.
which gives \( e_2(x) = \sqrt{3}(2x - 1) \). Now set
\[
e'_3(x) = x^2 - 3(2x - 1) \left( \int_0^1 t^2(2t - 1) \, dt \right) - \int_0^1 t^2 \, dt
\]
\[
= x^2 - 3(2x - 1) \left[ \frac{t^4}{2} - \frac{t^3}{3} \right]_0^1 - \frac{1}{3}
\]
\[
= x^2 - \frac{2x - 1}{2} - \frac{1}{3} = x^2 - x + \frac{1}{6},
\]
and then
\[
\|e'_3\|^2 = \int_0^1 (t^2 - t + \frac{1}{6})^2 \, dt = \int_0^1 t^4 - 2t^3 + \frac{4t^2}{3} - \frac{t}{3} + \frac{1}{36} \, dt
\]
\[
= \left[ \frac{t^5}{5} - \frac{t^4}{2} + \frac{4t^3}{9} - \frac{t^2}{6} + \frac{t}{36} \right]_0^1
\]
\[
= \frac{1}{180},
\]
and so \( e_3(x) = \sqrt{5}(6x^2 - 6x + 1) \).

Finally, we have
\[
e'_4(x) = x^3 - 5(6x^2 - 6x + 1) \left( \int_0^1 t^3(6t^2 - 6t + 1) \, dt \right) - 3(2x - 1) \left( \int_0^1 t^3(2t - 1) \, dt \right) - \int_0^1 t^3 \, dt
\]
\[
= x^3 - 6x^2 - 6x + 1 - \frac{9}{20}(2x - 1) - \frac{1}{4}
\]
\[
= \frac{20x^3 - 30x^2 + 12x - 1}{20}
\]
with
\[
\|e'_4\|^2 = \frac{1}{400} \int_0^1 (20t^3 - 30t^2 + 12t - 1)^2 \, dt
\]
\[
= \frac{1}{400} \int_0^1 400t^6 + 900t^4 + 144t^2 + 1 - 1200t^5 + 480t^4 - 40t^3 - 720t^3
\]
\[
+ 60t^2 - 24t \, dt
\]
\[
= \frac{1}{400} \left[ \frac{400}{7} + \frac{900}{5} + \frac{144}{3} + 1 - \frac{1200}{6} + \frac{480}{5} - \frac{40}{4} - \frac{720}{4} + \frac{60}{3} - \frac{24}{2} \right]
\]
\[
= \frac{1}{400} \times 7,
\]
and so \( e_4(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1) \) as above.
12. Given \( x, y \in \ell^2_w \), set \( \hat{x}_j = w_j^{1/2} x_j \) and \( \hat{y}_j = w_j^{1/2} y_j \). Then
\[
(\tilde{x}, y)_{\ell^2_w} = (\hat{x}, \hat{y})_{\ell^2},
\]
so the fact that \((\cdot, \cdot)_{\ell^2_w}\) is an inner product on \( \ell^2_w \) follows from the fact that \((\cdot, \cdot)_{\ell^2}\) is an inner product on \( \ell^2 \). Similarly the completeness of \( \ell^2_w \) follows from the completeness of \( \ell^2 \) — if \( \{x_n\} \) is a Cauchy sequence in \( \ell^2_w \), then \( \{\hat{x}_n\} \) is a Cauchy sequence in \( \ell^2 \), so there exists a \( y \in \ell^2 \) such that \( \hat{x}_n \to y \). Setting \( z_j = y_j/w_j^{1/2} \) gives a \( z \in \ell^2_w \) such that \( x_n \to z \) in \( \ell^2_w \).
Examples IV

1. If \( \|x\|_X \leq 1 \) then
\[
\|Ax\|_Y = \|x\|_X \left\| \frac{x}{\|x\|_X} \right\| \leq \left\| A \frac{x}{\|x\|_X} \right\|,
\]
and so since \( \|(x/\|x\|_X)\|_X = 1 \) we have
\[
\sup_{\|x\|_X \leq 1} \|Ax\|_Y \leq \sup_{\|z\|_X = 1} \|Az\|_Y.
\]
Rearranging the above equality we have
\[
\frac{\|Ax\|_Y}{\|x\|_X} = \left\| A \frac{x}{\|x\|_X} \right\|_Y,
\]
and so, again since \( \|(x/\|x\|_X)\|_X = 1 \),
\[
\sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|z\|_X = 1} \|Az\|_Y.
\]

2. Given \( x \in \ell^2 \), we have
\[
\|Tx\|_{\ell^2}^2 = \sum_j \frac{|x_j|^2}{j^2} \leq \sum_j |x_j|^2,
\]
so \( \|T\| \leq 1 \). Now, for any fixed \( n \) one can take \( x_n = (1, \cdots, 1, 0, \cdots) \), an element of \( \ell^2 \) whose first \( n \) entries are 1 and whose others are zero. Then
\[
T_x = (1, 1/2, 1/3, \ldots, 1/n, 0, \cdots)
\]
which is in \( \ell^2 \). \( \{T_x\} \) forms a sequence in \( \ell^2 \) that converges to
\[
(1, 1/2, 1/3, 1/4, \ldots),
\]
which is an element of \( \ell^2 \) since \( \sum_n (1/n^2) \) is finite. However, the ‘preimage’
of this would be the sequence consisting all of 1s, which is not an element of \( \ell^2 \). So the range of \( T \) is not closed.

3. Clearly \( f_\phi \) is linear. Given \( u \in C^0([a, b]) \) we have

\[
|f_\phi(u)| = \left| \int_a^b \phi(t)u(t) \, dt \right| \\
\leq \left( \max_{t \in [a, b]} |u(t)| \right) \int_a^b |\phi(t)| \, dt \\
= \|u\|_\infty \int_a^b |\phi(t)| \, dt,
\]

(4.1)

and so \( \|f_\phi\| \leq \int_a^b |\phi(t)| \, dt \). Now since \( \phi \) is continuous we can find a sequence of continuous functions \( u_n \) that approximate the sign of \( \phi \) (+1 if \( \phi > 0 \), −1 if \( \phi < 0 \)) increasingly closely in the \( L^2 \) norm: if \( \phi(s) \neq 0 \) then there is an interval around \( s \) on which \( \phi \) has the same sign. Since this interval has non-zero length, and the union of these intervals lies within \([a, b]\), there are at most a countable number, \( \{I_j\}_{j=1}^\infty \). Let \( u_n \) be the continuous function defined on \( I_j = (l_j, r_j) \) as

\[
u_n(t) = \text{sign}(\phi(t)) \times \begin{cases} 
(t - l_j)/\delta_{j,n} & l_j \leq t \leq l_j + \delta_{j,n} \\
1 & l_j + \delta_{j,n} < t < r_j - \delta_{j,n} \\
(r_j - t)/\delta_{j,n} & r_j - \delta_{j,n} \leq t \leq r_j 
\end{cases}
\]

where \( \delta_{j,n} = \min(2^{-(n+j)}, (r_j - l_j)/2) \), and zero when \( \phi(t) = 0 \). Then

\[
\int_a^b |u_n(t) - \text{sign}(\phi(t))|^2 \, dt \leq \sum_{j=1}^\infty 2^{-(n+j)} = 2^{-(n-1)},
\]

i.e. \( u_n \to \text{sign}(\phi) \) in \( L^2(a, b) \). Note that \( \|u_n\|_\infty = 1 \). It follows that

\[
|f_\phi(u_n)| = \left| \int_a^b \phi(t)u_n(t) \, dt \right| \\
\geq \left| \int_a^b |\phi(t)| \, dt - \int_a^b \phi(t)(u_n(t) - \text{sign}(\phi(t))) \, dt \right| \\
\geq \int_a^b |\phi(t)| \, dt - \left| \int_a^b \phi(t)(u_n(t) - \text{sign}(\phi(t))) \, dt \right| \\
\geq \int_a^b |\phi(t)| \, dt - \|\phi\|_{L^2} \|u_n - \text{sign}(\phi)\|_{L^2}.
\]

Since \( \phi \in C^0([a, b]) \) we know that \( \|\phi\|_{L^2} < \infty \), and we have just shown
that $\|u_n - \text{sign}(\phi)\|_{L^2} \to 0$ as $n \to \infty$. It follows since $\|u_n\|_{\infty} = 1$ that we cannot improve the bound in (4.1) and therefore

$$\|f_\phi\| = \int_a^b |\phi(t)| \, dt.$$  

4. Take $x, y \in L^2(0, 1)$, then

$$(Tx, y) = \int_0^1 \left( \int_0^t K(t, s)x(s) \, ds \right) y(t) \, dt$$

$$= \int_0^1 \int_0^t K(t, s)x(s)y(t) \, ds \, dt$$

$$= \int_0^1 \int_t^1 K(t, s)x(s)y(t) \, dt \, ds$$

$$= \int_0^1 \left( \int_t^1 K(t, s)y(t) \, dt \right) x(s) \, ds$$

$$= (x, T^*y)$$

(draw a picture to see how the limits of integration change) where

$$T^*(y)(s) = \int_t^1 K(t, s)y(t) \, dt.$$  

5. If $x \in (\text{range}(T)) ^\perp$ then

$$(x, y) = 0 \quad \text{for all} \quad y \in \text{range}(T),$$

i.e.

$$(x, Tz) = 0 \quad \text{for all} \quad z \in H.$$  

Since $(T^*)^* = T$, this is the same as $(T^*x, z) = 0$ for all $z \in H$. This implies that $T^*x = 0$, i.e. that $x \in \text{Ker}(T^*)$. This argument can be reversed, which gives the required equality.

Now – with apologies for the mistake in the question: we will show that $\bar{\lambda}$ is an eigenvalue of $T^*$ – suppose that $\text{range}(T - \lambda I) \neq H$. Then there exists some non-zero $u \in (\text{range}(T - \lambda I))^\perp$. By the equality we have just proved, this gives a non-zero $u$ in $\text{Ker}((T - \lambda I)^*)$, i.e. a non-zero $u$ such that

$$(T - \lambda I)^*u = 0.$$  

Since $(T - \lambda I)^* = T^* - \bar{\lambda}I$, this gives a non-zero $u$ with $T^*u = \bar{\lambda}u$. So $\bar{\lambda}$ is an eigenvalue of $T^*$.  

6.(i) If $T \in B(H, H)$ then $T^* \in B(H, H)$ and $\|T^*\|_{\text{op}} = \|T\|_{\text{op}}$. So if $\{x_n\}$ is a bounded sequence in $H$, with $\|x_n\| \leq M$, say, it follows that

$$\|T^*x_n\| \leq \|T^*\|_{\text{op}} \|x_n\| \leq M \|T^*\|_{\text{op}}.$$ 

So $\{T^*x_n\}$ is also a bounded sequence in $H$. Since $T$ is compact, it follows that $\{T(T^*x_n)\} = \{(TT^*)x_n\}$ has a convergent subsequence, which shows that $TT^*$ is compact.

(ii) Following the hint, we have

$$\|T^*x\|^2 = (T^*x, T^*x) = (TT^*x, x) \leq \|TT^*x\| \|x\|.$$ 

Now, if $\{TT^*x_n\}$ is Cauchy then given any $\epsilon > 0$, there exists a $N$ such that for all $n, m \geq N$,

$$\|TT^*x_n - TT^*x_m\| = \|TT^*(x_n - x_m)\| \leq \epsilon.$$ 

It follows that for all $n, m \geq N$,

$$\|T^*x_n - T^*x_m\|^2 = \|T^*(x_n - x_m)\|^2 \leq \|TT^*(x_n - x_m)\| \|x_n - x_m\|.$$ 

Since $\{x_n\}$ is a bounded sequence, with $\|x_n\| \leq M$, say, we have

$$\|T^*x_n - T^*x_m\|^2 \leq M \epsilon \quad \text{for all} \quad n, m \geq N.$$ 

It follows that $\{T^*x_n\}$ is Cauchy.

(iii) So suppose that $\{x_n\}$ is a bounded sequence in $H$. Part (i) shows that $TT^*$ is compact, so $\{TT^*x_n\}$ has a subsequence $\{TT^*x_{n_j}\}$ that is Cauchy. But part (ii) shows that this implies that $\{T^*x_{n_j}\}$ is Cauchy too. So $T^*$ is compact.

7.(i) If for every $z \in H$ we have

$$(x, z) \to (x, z) \quad \text{and} \quad (x_n, z) \to (y, z)$$

then clearly $(x, z) = (y, z)$ for every $z \in H$. But then $(x - y, z) = 0$ for every $z \in H$; in particular we can take $z = x - y$, which shows that $\|x - y\|^2 = 0$, i.e. that $x = y$.

(ii) Take $y \in H$, and suppose that $(e_n, y)$ does not tend to zero as $n \to \infty$. Then for some $\epsilon > 0$, there exists a sequence $n_j \to \infty$ such that $|(e_{n_j}, y)| > \epsilon$. But then

$$\|y\|^2 = \sum_n |(y, e_j)|^2 \geq \sum_j |(y, e_{n_j})|^2 = \infty,$$

contradicting the fact that $y \in H$ with $\{e_j\}$ an orthonormal basis.
(iii) For some fixed $z \in H$, consider the map $f : H \to K$ given by 
$$u \mapsto (Tu, z).$$
This map is clearly linear,
$$u + \lambda v \mapsto (T(u + \lambda v), z) = (Tu, z) + \lambda(Tv, z),$$
and it is bounded,
$$|(Tu, z)| \leq \|Tu\|\|z\| \leq \left( \|T\|_{\text{op}} \|z\| \right) \|u\|.$$
So $f \in H^*$. It follows that there exists a $y \in H$ such that 
$$(Tu, z) = f(u) = (u, y)$$
for every $u \in H$. So if $u_n \to u$, in particularly for this choice of $y$ we have 
$$(u_n, y) \to (u, y) \Rightarrow (Tu_n, z) \to (Tu, z).$$
Since this holds whatever our choice of $z$, it follows that $Tu_n \to Tu$.

(iv) **This part requires you to know that any weakly convergence sequence is bounded, which is not at all obvious (but true).**

Suppose that $Tx_n \not\to Tx$. Then there exists a $\delta > 0$ and a subsequence $Tx_{n_j}$ such that 
$$\|Tx_{n_j} - Tx\| > \delta \quad \text{for all} \quad j = 1, 2, 3, \ldots \quad (4.2)$$
Since $T$ is compact and $\{x_{n_j}\}$ is bounded there exists a further subsequence such that $Tx_{n_{j_k}} \to w$ for some $w \in H$. But since convergence implies weak convergence, we must have $Tx_{n_{j_k}} \to w$. But we already know that $Tx_{n_{j_k}} \to Tx$, so it follows from the uniqueness of weak limits that $w = Tx$, and so $Tx_{n_{j_k}} \to Tx$, contradicting (4.2).

8. (i) For an $x$ with $K(x, \cdot) \in L^2(a, b)$, use the fact that $\{\phi_j\}$ is a basis for $L^2$ to write 
$$K(x, y) = \sum_{i=1}^{\infty} k_i(x)\phi_i(y).$$
Since 
$$k_i(x) = \int_{a}^{b} K(x, y)\phi_i(y) \, dy,$$
we have
\[\int_a^b |k_i(x)|^2 \, dx = \int_a^b \left| \int_a^b K(x, y) \phi_i(y) \, dy \right|^2 \, dx \]
\[\leq \int_a^b \left( \int_a^b |K(x, y)|^2 \, dy \right) \left( \int_a^b |\phi_i(y)|^2 \, dy \right) \, dx \]
\[= \int_a^b |\phi_i(y)|^2 \, dy \times \int_a^b |K(x, y)|^2 \, dy \, dx,\]
and so \(k_i \in L^2(a,b)\).

(ii) Since \(k_i \in L^2(a,b)\), we can write \(k_i(x) = \sum_j \kappa_{ij} \phi_j(x)\), where \(\kappa_{ij} = \int_a^b k_i(x) \phi_j(x) \, dx\), i.e.
\[K(x, y) = \sum_{i,j} \kappa_{ij} \phi_i(y) \phi_j(x),\]
and so \(\{\phi_i(y) \phi_j(x)\}\) is a basis for \(L^2((a,b) \times (a,b))\).

9. Clearly
\[T^n x = \sum_{j=1}^n \lambda^n_j (x, e_j) e_j.\]
If \(|\lambda_j| < 1\) for all \(j = 1, \ldots, n\) then \(\|T\| < 1\), and so \((I - T)^{-1} = I + T + T^2 + \cdots\), i.e.
\[(I - T)^{-1} = I + \sum_{j=1}^n \lambda_j (x, e_j) e_j + \sum_{j=1}^n \lambda_j^2 (x, e_j) e_j + \cdots \]
\[= I + \sum_{j=1}^n (\lambda_j + \lambda_j^2 + \cdots)(x, e_j) e_j \]
\[= I + \sum_{j=1}^n \frac{\lambda_j}{1 - \lambda_j} (x, e_j) e_j.\]

10.(i) Since \(x = f + \alpha Tx\), the solution is given by \(x = (I - \alpha T)^{-1} f\). So if \(\|\alpha T\|_{\text{op}} < 1\) we can write
\[x = (I + \alpha T + \alpha^2 T^2 + \alpha^3 T^3 + \cdots) f = \sum_{j=0}^\infty (\alpha T)^j f.\]
So this expansion is valid for \(|\alpha| < 1/\|T\|_{\text{op}}\).

(ii) Suppose that
\[(T^{n-1} x)(t) = \int_a^b K_{n-1}(t, s)x(s) \, ds.\]
Then

\[(T^n x)(t) = \int_a^b K(t, s)(T^{n-1}x)(s) \, ds\]
\[= \int_a^b K(t, s) \left( \int_a^b K_{n-1}(s, r)x(r) \, dr \right) \, ds\]
\[= \int_a^b \int_a^b K(t, s)K_{n-1}(s, r)x(r) \, dr \, ds\]
\[= \int_a^b \left( \int_a^b K(t, s)K_{n-1}(s, r) \, ds \right) x(r) \, dr,
\]
i.e.

\[(T^n x)(t) = \int_a^b K_n(t, r)x(r) \, dr.\]

with

\[K_n(t, r) = \int_a^b K(t, s)K_{n-1}(r, s) \, ds\]
as claimed.

11. We have

\[\langle Tu, v \rangle = \left( \sum_{j=1}^{\infty} \lambda_j(u, e_j)v \right) = \sum_{j=1}^{\infty} \lambda_j(u, e_j)(e_j, v)\]
\[= \left( u, \sum_{j=1}^{\infty} \lambda_j(e_j, v)e_j \right) = \left( u, \sum_{j=1}^{\infty} \lambda_j(v, e_j)e_j \right)\]
\[= (u, Tv)\]
and so \(T\) is self-adjoint. Now consider \(T_nu\) defined by

\[T_nu = \sum_{j=1}^{n} \lambda_j(u, e_j)e_j.\]

Since the range of \(T\) is contained in the span of \(\{e_j\}_{j=1}^{n}\) it is finite-dimensional, and so \(T_n\) is compact. Since

\[\|T_n u - Tu\|^2 = \left\| \sum_{j=n+1}^{\infty} \lambda_j(u, e_j)e_j \right\|^2\]
\[= \sum_{j=n+1}^{\infty} |\lambda_j|^2|(u, e_j)|^2,\]
we have from the fact that $\lambda_n \to 0$ that given any $\epsilon > 0$ there exists an $N$ such that for all $n \geq N$ we have $|\lambda_j| < \epsilon$, and hence

$$\|T_n u - Tu\|^2 \leq \epsilon^2 \sum_{j=n+1}^{\infty} |(u, e_j)|^2 \leq \epsilon^2 \sum_{j=1}^{\infty} |(u, e_j)|^2 = \epsilon^2 \|u\|^2,$$

i.e.

$$\|T_n - T\|_{\text{op}} \leq \epsilon,$$

and so $T_n \to T$. It follows from Theorem 13.8 that $T$ is compact.

12. We have

$$Tu = \int_a^b K(x, y)u(y)\,dy = \int_a^b \sum_{j=1}^{\infty} \lambda_j e_j(x)e_j(y)u(y)\,dy = \sum_{j=1}^{\infty} \lambda_j e_j(e_j, u),$$

and so $Te_k = \lambda_k e_k$.

To show that these are the only eigenvalues and eigenvectors, if $u \in L^2(a, b)$ with $u = w + \sum_{k=1}^{\infty} (u, e_k)e_k$ and $Tu = \lambda u$ then, since $w \perp e_j$ for all $j$,

$$Tu = Tw + \sum_{k=1}^{\infty} (u, e_k)Te_k = \sum_{j,k=1}^{\infty} \int_a^b \lambda_j e_j(x)e_j(y)(u, e_k)e_k(y)\,dy = \sum_{j=1}^{\infty} \lambda_j (u, e_j)e_j(x)$$

and

$$\lambda u = \sum_{j=1}^{\infty} \lambda(u, e_j)e_j(x).$$

Taking the inner product with each $e_k$ yields

$$\lambda(u, e_k) = \lambda_k(u, e_k),$$

so either $(u, e_k) = 0$ or $\lambda = \lambda_k$. 
(i) Let \((a, b) = (-\pi, \pi)\) and consider \(K(t, s) = \cos(t - s)\). Then \(K\) is clearly symmetric, so the corresponding \(T\) is self-adjoint. We have
\[
\cos(t - s) = \cos t \cos s - \sin t \sin s;
\]
recall that \(\cos t\) and \(\sin t\) are orthogonal in \(L^2(-\pi, \pi)\). We have
\[
T(\sin t) = \int_{-\pi}^{\pi} K(t, s) \sin s \, ds = \int_{-\pi}^{\pi} \cos t \cos s \sin s - \sin t \sin^2 s \, ds = 2\pi \sin t
\]
and
\[
T(\cos t) = \int_{-\pi}^{\pi} K(t, s) \cos s \, ds = \int_{-\pi}^{\pi} \cos t \cos^2 s - \sin t \sin s \cos s \, ds = 2\pi \cos t.
\]

(ii) Let \((a, b) = (-1, 1)\) and let
\[
K(t, s) = 1 - 3(t - s)^2 + 9t^2 s^2.
\]
Then in fact
\[
K(t, s) = 4 \left( \sqrt{\frac{3}{2}} t \right) \left( \sqrt{\frac{3}{2}} s \right) + \frac{8}{5} \left( \sqrt{\frac{5}{8}} (3t^2 - 1) \right) \left( \sqrt{\frac{5}{8}} (3s^2 - 1) \right),
\]
and so, since \(\{ \sqrt{\frac{3}{2}} t, \sqrt{\frac{5}{8}} (3t^2 - 1) \}\) are orthonormal (they are some of the Legendre polynomials from Chapter 6) the integral operator associated with \(K(t, s)\) has
\[
T(t) = 4t \quad \text{and} \quad T(3t^2 - 1) = \frac{8}{5}(3t^2 - 1).
\]

13. This equation is the equation in Theorem 14.2 with \(p = 1\) and \(q = 0\). So it suffices to show that (i) \(u_1(x) = x\) satisfies the equation \(-d^2u/dx^2 = 0\) with \(u(0) = 0\) (which is clear), (ii) that \(u_2(x) = (1 - x)\) satisfies the same equation with \(u(1) = 0\) (which is also clear), and (iii) that \(W_p(u_1, u_2) = 1\):
\[
W_p(u_1, u_2) = [u'_1 u_2 - u'_2 u_1] = 1(1 - x) - (-1)x = 1.
\]
Or one could follow the proof of Theorem 4.12 with this particular choice of \(G\) to show that \(u(x)\) as defined does indeed satisfy \(-d^2u/dx^2 = f\).
14.(i) If $x \neq 0$ then $\|x\| > 0$, from which it follows that $\|Tx\| > 0$, and so $x \notin \text{Ker}(T)$. So Ker$(T) = \{0\}$ and $T^{-1}$ exists for all $y \in \text{range}(T)$. Then we have

$$\|T(T^{-1}x)\|^2 \geq \alpha \|T^{-1}x\|^2,$$

i.e. $\|T^{-1}x\|^2 \leq \alpha^{-1}\|x\|^2$, so $T^{-1}$ is bounded.

(ii) Let $\lambda = \alpha + i\beta$. Then

$$\|(T - \lambda I)x\|^2 = (Tx - \alpha x - i\beta x, Tx - \alpha x - i\beta x)$$

$$= \|(Tx - \alpha x)\|^2 - (Tx - \alpha x, i\beta x) - (i\beta x, Tx - \alpha x) + \beta iK x$$

$$= \|Tx - \alpha x\|^2 + i(Tx - \alpha x, \beta x) + i(\beta x, Tx - \alpha x) + |\beta|^2\|x\|^2.$$

Now, since $T$ is self-adjoint and $\alpha$ is real, it follows that

$$(Tx - \alpha x, \beta x) = (\beta x, Tx - \alpha x),$$

and so

$$\|(T - \lambda I)x\|^2 = \|Tx - \alpha x\|^2 + \beta^2\|x\|^2 \geq \beta^2\|x\|^2,$$

i.e. $T - \lambda I$ is bounded below as claimed.

(iii) If $\lambda \in \mathbb{C}$ and $T - \lambda I$ has a densely defined inverse then $(T - \lambda I)^{-1} \in B(H, H)$, i.e. $\lambda \in R(T)$. So $\sigma_c(T) \subseteq \mathbb{R}$.

(iv) If the range of $T - \lambda I$ is not dense in $H$ then by Question 3 we know that $\lambda$ is an eigenvalue of $T^*$. If $T$ is self-adjoint, we must have $\lambda$ an eigenvalue of $T$. But all eigenvalues of $T$ are real, so in fact $\lambda$ is an eigenvalue of $T$. So $\lambda \in \sigma_p(T)$ rather than $\sigma_r(T)$, which implies that $\sigma_r(T)$ is empty.

15.(i) If $y \in \ell^q$ then

$$|l(y)| = \left| \sum_{j=1}^{\infty} x_j y_j \right| \leq \|x\|_p \|y\|_q$$

using Hölder’s inequality. So $\|l\|_p^* \leq \|y\|_q$ as claimed.

(ii) Any $x = (x_1, x_2, x_3, \ldots) \in \ell^p$ can be written as

$$x = \sum_{j=1}^{\infty} x_j \xi_j,$$

for the elements $\xi_j$ defined in the question. Since $l$ is bounded it is
continuous, so

\[ l(x) = l \left( \lim_{n \to \infty} \sum_{j=1}^{n} x_j \varepsilon_j \right) \]

\[ = \lim_{n \to \infty} l \left( \sum_{j=1}^{n} x_j \varepsilon_j \right) \]

\[ = \lim_{n \to \infty} \sum_{j=1}^{n} x_j l(\varepsilon_j) \]

\[ = \sum_{j=1}^{\infty} x_j l(\varepsilon_j) \]

\[ = \sum_{j=1}^{\infty} x_j y_j, \]

if we define \( y_j = l(\varepsilon_j) \). [Should you come across something similar in an exam you should at least mention continuity of \( l \) as the justification for taking it inside the infinite sum.]

(iii) We have

\[ \|x^{(n)}\|_p^p = \sum_{j=1}^{n} |y_j|^q = \sum_{j=1}^{n} |y_j|^p q - 1 = \sum_{j=1}^{n} |y_j|^q, \]

and so, as claimed,

\[ \|x^{(n)}\|_p = \left( \sum_{j=1}^{n} |y_j|^q \right)^{1/p}. \]

Also

\[ l(x^{(n)}) = \sum_{j=1}^{n} \frac{|y_j|^q}{y_j} y_j = \sum_{j=1}^{n} |y_j|^q. \]

Since \( |l(x^{(n)})| \leq \|l\|_p \|x^{(n)}\|_p \), we have

\[ \sum_{j=1}^{n} |y_j|^q \leq \|l\|_p \left( \sum_{j=1}^{n} |y_j|^q \right)^{1/p}, \]
from which it follows, since $1 - 1/p = 1/q$, that

$$\left( \sum_{j=1}^{n} |y_j|^q \right)^{1/q} \leq \|l\|_{p^*}.$$ 

Since this holds for every $n$, we must have $y \in \ell^q$ with $\|y\|_q \leq \|l\|_{p^*}$. 