VAN DER WAERDEN’S THEOREM
ON ARITHMETIC PROGRESSIONS

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Abstract. This is a short exposition of the dynamical approach to the proof of van der Waerden’s theorem on arithmetic progressions.

0. Introduction

Let \( \mathbb{Z} = \bigcup_{i=1}^{N} C_i \) be a finite partition of the integers (i.e., \( C_i \cap C_j = \emptyset \) for \( i \neq j \)). Of course, at least one of sets must have infinitely many integers, but the following theorem shows there is a far stronger result.

Theorem (van der Waerden, 1927 [6]). There exists \( 1 \leq i \leq N \) such that \( C_i \) contains arithmetic progressions of arbitrary length, (i.e., \( \forall k \geq 1, \ c \in \mathbb{Z}, \text{ and } d \in \mathbb{N} \text{ such that } c + jd \in C_i \text{ for } 0 \leq j \leq k - 1 \))

Here \( k \) is called the length of the arithmetic progression.

Examples.

(1) We could let \( C_1 \) be the odd numbers and let \( C_2 \) be the even numbers, then both sets have arithmetic progressions of all lengths;

(2) We could let \( C_1 \) be \( \pm \)prime numbers and \( C_2 \) the compliment.

The original proof of van der Waerden is combinatorial, and was one of the “Three pearls of number theory” in Khintchine’s famous book [5]. There is a dynamical approach to this theorem due to Furstenberg and Weiss [4]. We can associate to the partition a sequence \( x = (x_n)_{n \in \mathbb{Z}} \in \Sigma = \{1, \ldots, N\}^\mathbb{Z} \) defined by

\[ x_n = i \text{ if } n \in C_i. \]

If \( \Sigma \) has the usual (Tychanoff product) topology, we let \( Y = \bigcup_{n \in \mathbb{Z}} \sigma^n x \) be the closure of the orbit of \( x \). We define the continuous shift map \( \sigma : Y \to Y \), by

\[ \sigma((w_n)_{n \in \mathbb{Z}}) = (w_{n+1})_{n \in \mathbb{Z}}. \]

A basic key observation is the following:

Lemma 1 (Dynamical approach to arithmetic progressions). Assume that for some \( 1 \leq i \leq N \) and \( [i] = \{w = (w_n)_{n \in \mathbb{Z}} : w_0 = i\} \) we have that

\[ Y \cap [i] \cap \sigma^{-d}[i] \cap \sigma^{-2d}[i] \cap \cdots \cap \sigma^{-(k-1)d}[i] \neq \emptyset, \]

for some \( d \geq 1 \) and \( k \geq 1 \) \( C_i \) contains an arithmetic progression of length \( k \).

Proof. Since the intersection is an open set in \( Y \), it follows that it contains \( \sigma^n x \), for some \( n \in \mathbb{Z} \). However, this means \( x_{n+jd} = i \) for \( 0 \leq j \leq d - 1 \). \( \Box \)
Minimal subsets

It helps to consider the restriction of \( \sigma \) to a smaller set. We say that a \( \sigma \)-invariant subset \( X \subset Y \) is \emph{minimal} if the restriction \( \sigma : X \to X \) is minimal (i.e., for every \( w \in X \) we have \( X = \cup_{n \in \mathbb{Z}} \sigma^n w \)).

**Lemma 2 (Existence of Minimal sets).**

1. There exists a minimal subset \( X \subset Y \).
2. For any open set \( V \subset X \) there exists \( M > 0 \) such that \( X = \cup_{|n| \leq M} \sigma^{-n} V \)

**Proof.** Following [7], we can choose an enumeration \( \{U_k\}_{k=1}^\infty \) of all the (non-empty) cylinder sets \( \{\alpha_{-L}, \ldots, \alpha_L\} := \{w \in X : w_{-L} = \alpha_{-L}, \ldots, w_L = \alpha_L\} \), where \( \alpha_{-L}, \ldots, \alpha_L \in \{1, \ldots, N\} \) and \( L \geq 0 \). Let \( X_1 = Y \). We then proceed inductively, given \( X_k \) we define

\[
X_{k+1} = \begin{cases} 
X_k & \text{if } X \subset \cup_{n \in \mathbb{Z}} \sigma^{-n} U_k \\
X_k - (\cup_{n \in \mathbb{Z}} \sigma^{-n} U_k) & \text{otherwise}
\end{cases}
\]

We then define \( X = \cap_{k=1}^\infty X_k \) which is closed, invariant and (by construction) a minimal set. For the second part, since \( X \) is minimal, \( X = \cup_{n \in \mathbb{Z}} \sigma^{-n} V \) and the result then follows by compactness. \( \square \)

**Proof of van der Waerden’s Theorem**

We proceed following [2], [1] and [3]. It suffices to restrict our attention to a minimal subset \( X \subset \Sigma \) and to prove the following:

**Proposition 1 (Multiple Recurrence).** Let \( V \subset X \) be an open set. \( \forall k_0 \), \( \exists k \geq k_0 \) and \( d \geq 1 \) such that

\[
V \cap \sigma^{-d} V \cap \sigma^{-2d} V \cap \cdots \cap \sigma^{-(k-1)d} V \neq \emptyset. \quad (P(k))
\]

**Proof of Theorem (assuming Proposition 1).** This follows from Lemma 1 and Proposition 1 (with the particular choice \( V = [i] \), chosen so that \([i] \cap X_0 \neq \emptyset \)). \( \square \)

The importance of minimality of \( X \) is that it allows us (by part (2) of Lemma 2) to write \( X = \cup_{|n| \leq M} \sigma^{-n} V \), say.

The proof of Proposition 1 is now by induction on \( k \). When \( k = 1 \) the result \( P(1) \) is trivial. Assume we know that \( P(k-1) \) holds, then we shall use the following lemma to extend this result to \( P(k) \).

**Lemma 3.** For each \( l \geq 1 \) we can choose points and open sets \( x_j \in \sigma^{-n_j} V \) \((-M \leq n_j \leq M \) \( j = 0, \ldots, l \) and natural numbers \( N_1 < N_2 < \cdots < N_l \) such that \( \forall 0 \leq r \leq s \leq l \)

\[
\sigma^{j(N_r-N_s)} x_r \in \sigma^{-n_l} V, \quad (Q(l))
\]

for \( j = 0, \ldots, l \).

**Proof.** This is proved by induction on \( l \) (within the induction on \( k \), for which we are currently assuming \( P(k-1) \) holds). When \( l = 0 \) it is trivial that \( Q(0) \). Assume we know \( Q(l-1) \) holds. Let us choose a small neighbourhood \( V_0 \ni x_{l-1} \). By \( P(k-1) \) we can choose \( d \) such that there exists \( y \in V_0 \cap \sigma^{-d} V_0 \cap \sigma^{-2d} V_0 \cap \cdots \cap \sigma^{-(k-2)d} V_0 \). We set \( x_l := \sigma^{-d} y \) and \( N_1 := N_{l-1} + d \). Moreover, we can choose some \( \sigma^{-n_l} V_0 \ni x_l \) with \(|n_l| \leq M \) (by Lemma 2 (2)). In particular, for each \( 0 \leq r < l \):

\[
\sigma^{j(N_r-N_s)} x_l = \sigma^{j(N_{l-1}-N_r)} \sigma^{(j-1)(d)} y \in \sigma^{j(N_{l-1}-N_r)} (V),
\]
for \( j = 0, \cdots, k - 1 \) Moreover, providing \( V_0 \) is sufficiently small \( Q(l) \) follows from \( Q(l - 1) \), the additional results for \( x_l \) coming from those for \( x_{l-1} \) and by continuity of \( \sigma^{j(N_l - N_r)} \).

To prove \( P(k) \) holds it suffices to apply the Lemma 3 where \( l = 2M + 1 \). By the pigeonhole principle, we can choose \( 0 \leq r < s \leq 2M + 1 \) such that \( n_r = n_s \in \{-M, \cdots, M\} \). Then setting \( x = x_r \) and now setting \( d = N_s - N_r \) gives a point in the intersection for \( P(k) \). This completes the inductive step, and thus the proof of Proposition 2.

Some final comments

Observe that the proof of Proposition 1 holds for any homeomorphism of a compact topological space (without any assumption of a metric).

D. Birkhoff showed that for a homeomorphism \( T : X \to X \) of a compact metric space there exists \( x \in X \) and a sequence \( n_l \to \infty \) such that \( d(T^{n_l}x, x) \to 0 \). The following is a corollary of Proposition 1:

**Proposition 2 (Multiple Birkhoff recurrence).** Let \( T : X \to X \) be a homeomorphism of a compact metric space. For each \( k \geq 1 \) there exists \( x \in X \) and a a sequence \( n_l \to \infty \) such that \( \max_{0 \leq i \leq k - 1} \{d(T^{n_l}x, x)\} \to 0 \).

**Proof.** We can let \( V_l = B(y_l, 2^{-l}) \), for \( l \geq 0 \). By Proposition 1, there exists \( x_l \in X \) with \( \max_{0 \leq i \leq k - 1} \{d(T^{n_l}x_l, x_l)\} \leq 2^{-l} \). Letting \( x \) be an accumulation point of \( \{x_l\} \), the result follows.

Another application of Proposition 1 is to a higher dimensional analogue of van der Waerden’s theorem for \( \mathbb{Z}^D \), with \( D \geq 1 \).

**Theorem (Higher Dimensional van der Waerden’s Theorem).** Let \( \mathbb{Z}^D = \bigcup_{i=1}^N C_i \) be a finite partition and let \( F \subset \mathbb{Z}^D \) be a finite set. Then: \( \forall k_0, \exists k \geq k_0, c \in \mathbb{Z}^d \) and \( d \in \mathbb{Z} \) such that \( c + jd \in C_i \) for \( 0 \leq j \leq k - 1 \).

**References**