Making Chaos at Home

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These are instructions on how to construct a simple mechanical system which time evolution exhibits chaotic behaviour. In particular, it approximates an ideal system which corresponds to an Anosov Hamiltonian flow.

The basic material

Most of the following household items are easily obtained from any supermarket.

- 24 penny coins
- A roll of duct tape (2 inch width)
- 2 plastic drinking straws (12 inch)
- 2 small paperclips (1 inch length)
- 2 plastic bottles (1.5 litre beverage bottles, full)
- 3 lengths of string (longer than 48 inch)
- A clothes horse (2 fold airer)
- A strip of wood (approximately 2 × 1 × 32 inch)
- A pair of scissors
- 32 penny coins
- A roll of duct tape (2 inch width)

The construction

We divide the construction up into 6 easy steps.

Step 1: The masses

Divide the coins into 3 stacks of 8 pennies and wrap each in duct tape. This gives 3 cylindrical masses.

Step 2: Constructing the two shorter rods

Cut one of the straws into two pieces of lengths of approximately 12 inch and 7 inch, respectively. For the shorter half, take a paperclip and insert the inner part into one end. Secure with duct tape around the outer part of the paper clip.

Step 3: The pivots

Cut a strip of duct tape of length approximately 5 inch. Press the drawing pin through the centre of the duct tape. Stick the duct tape over the end of a bottle so that the point of the pin points vertically. Repeat with for the second bottle. Place the two bottles approximately 12 inch apart.

Step 4: The shorter rods on the pivots

Place the strip of wood across the top of the clothes horse so that it is vertically above the tops of both the bottles, at a height of at least 36 inch. The one end of a piece of string (of approximately 48 inch) to the wood directly above one bottle. Allow the string to hang down 4 inch below the drawing of the bottles, at a height of at least 36 inch. Tie one end of a piece of string (of approximately 48 inch) to the wood directly above one bottle. Place the two bottles approximately 4 inch above the end of the rod (by tying the paperclip to the rod using a short piece of string). Attach the other end to the remaining free end of a short rod approximately 4 inch above the paperclip (by again tying the paperclip to the end using a short piece of string).

Step 5: Constructing the two longer rods

Attach one end of the remaining long rod to the central string approximately 1 inch above the end of the rod (by tying the paperclip to the rod using a short piece of string). Attach the other end to the remaining free end of a short rod approximately 4 inch above the paperclip (by again tying the paperclip to the end using a short piece of string).

Step 6: The joints

Suspend the final length of string from the piece of wood, at midpoint of the previous two, and attach to the paperclip on the end where the mass is of one of the longer rods. Arrange the length to be 1 inch shorter than the other two strings. Attach the other end of the same longer rod to one of the existing strings ½ inch above the other coins (by tying the paperclip to the taut using a short piece of string). Attach one end of the remaining long rod to the central string approximately ¼ inch above the end of the rod (by tying the paperclip to the rod using a short piece of string) and attach the other end to the remaining free end of a short rod approximately ¼ above the paperclip (by again tying the paperclip to the end using a short piece of string).

The mathematical theory

Consider the pentagonal linkage consisting of:

1. two short rods of length ε > 0 connected to fixed pivots at (0, 0) and (1, 0);
2. two long rods of length 1/2 + bε, for some fixed value of 0 < b ≤ 1, attached to joints at the other ends of the short rods;
3. the long rods are joined at a common movable pivot (u, v), and finally
4. Assume the 3 joints have suitable masses.

This is illustrated in the following figure.

Figure 3: The pentagonal linkage

The topology of the configuration space is easily understood.

1. If 0 < b < 1 then the configuration space $M$ is a surface of genus 2; and
2. If b = 1 then the configuration space $M$ is the union of two tori $T^2$ touching at a single point.

For the short rods make angles $\theta_1, \theta_2$ to the horizontal axis then we can consider the embedding into the four dimensional space $\mathbb{R}^2 \times \mathbb{S}^2$ given by $(\theta_1, \theta_2) \mapsto (R_2, R_2 \cos \theta_1, \sin \theta_2, 0)$.

For this example the configuration space is a surface of genus 2. There is a simple expression for the curvature of the embedded surface $M$ when $\epsilon$ is small. Fix $b = 1$. The asymptotic formula for the curvature $\kappa(\theta_1, \theta_2)$ as $\epsilon \to 0$ is:

$$\frac{(\cos \theta_1 - \cos \theta_2) - \cos \theta_1 \cos \theta_2 (\cos \theta_1 - \cos \theta_2)}{b(2 \cos \theta_1 - \cos \theta_2)^2} + O(\epsilon^{2/3}).$$

Figure 4: A contour plot of the asymptotic curvature $\lim_{\epsilon \to 0} \kappa(\theta_1, \theta_2)/\epsilon$ when $b = 1$. The configuration space $M$ consists of two such tori joined together; and (b) A 3-dimensional plot of the curvature. Observe that there is an "island" of positive curvature.

We come to our final claim: Provided we choose:

1. $0 < b < 1$ sufficiently close to 1; and
2. $\epsilon > 0$ sufficiently small

the associated geodesic flow for $M$ is Anosov.

More precisely, recall that the Riccati equation takes the form $x' + x^2 + u(t) = 0$. It suffices to establish the cone condition: if $u(t) \geq 0$ then there exists $x > 0$ and $b > 0$ such that $u(t) \geq x$ for $t \geq b$. We can consider geodesics starting from the curve corresponding to the boundary $C$ of the disk of positive curvature.

Figure 5: (a) The disks represent copies of the "island" of positive curvature and the lines geodesics; and (b) The superposition of the plots of the solutions to the Ricatti equation (in the case $b = 1$) for 80 points on $C$, each with 20 inwardly directed vectors.