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Part I

Zeta functions

Dynamical Zeta functions and closed orbits for geodesic and hyperbolic flows

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Introduction

In this article we want to give the basic definitions and properties of dynamical zeta functions, and describe a few of their applications. The emphasis is on giving the flavour of the subject rather than a detailed summary.

To fix ideas, let us assume that V is a compact surface with some appropriate Riemannian metric $< \cdot, \cdot >_{TV}$, say. We shall always assume that V has negative curvature at every point on V (although we will not necessarily assume that it has constant negative curvature). In studying geometric properties of manifolds it is sometimes convenient to study the associated geodesic flow. Fortunately, geodesic flows for negatively curved surfaces are important examples of a broader class of flows, namely hyperbolic flows, which are amenable to quite powerful techniques in dynamical systems which have evolved over the last 40 years (from the work of Anosov, Sinai, Ratner, Smale, Bowen, Ruelle, and many others). In particular, it is often (but not always) convenient to introduce simple symbolic models for these flows. The basic hope is that, despite the sacrifice of some of the geometry, we can benefit from being able to apply fairly directly ideas from ergodic theory and what is often colloquially called "Thermodynamic Formalism". Somewhat surprisingly, this method is successful for various classes of problems, including:

(a) Geometric problems (e.g., counting closed geodesics, or equivalently closed orbits for the geodesic flow);
(b) Statistical Properties (e.g., determining rates of mixing for flows); and
(c) Distributional properties (e.g., linear actions associated to the horocycle foliation).

Of course, anyone familiar with the Selberg zeta function for surfaces of constant negative curvature will recognise many of the ideas in (a), for example. The main difference is that instead of using the Selberg trace formula, say, we use transfer operators to study the zeta function. What we lose in elegance (and error terms!) we hope to make up for in the generality of the setting.

In this overview we want to recall a number of the key themes and outline some recent and ongoing developments. The choice of topics reflects the author's idiosyncratic tastes. The results are organised so as to give the illusion of coherence, but are in fact a mixture of older and more recent material. For different accounts and perspectives, the reader is referred to [7], [62]. In particular, nowadays non-symbolic methods are catching up in terms of efficiency in the above areas.

Finally, I would like to express my gratitude to the organisers of the Les Houches School for their invitation to participate.

1 Symbolic dynamics and zeta functions

The familiar geodesic flow for V is a flow ϕ_t $(t \in \mathbb{R})$ defined on the (three dimensional) unit tangent bundle $T_1V = \{(x,v) \in TV : ||v||_{TV} = 1\}$, i.e., those tangent vectors to V having length one with respect to the ambient Riemannian norm. The flow acts in the standard way by moving one tangent vector $v \in T_1V$ to another $v' =: \phi_t x$, using parallel transport [5]. [1] It is the hypothesis of negative curvature ensures that this geodesic flow is a *hyperbolic flow*, i.e., one for which directions transverse to the flow direction (in a natural sense) are either expanding or contracting. [2]

1.1 Sections

The modern use of symbolic dynamics to model hyperbolic systems probably dates back to the work of Adler and Weiss [2], who showed that the famous Arnold CAT map could be modelled by a shift map on the space of sequences from a finite alphabet of symbols. This lead to Sinai's seminal work introducing Markov partitions for more general hyperbolic maps and then Ratner and Bowen's extension to hyperbolic flows [10], [56]. Historically, the

[1] More precisely, given any $(x,v) \in M$ we let $\gamma_{(x,v)} : \mathbb{R} \to M$ be the unit speed geodesic with $\gamma_{(x,v)}(0) = x$ and $\dot\gamma_{(x,v)}(0) = v$. We define the *geodesic flow* $\phi_t : M \to M$ by $\phi_t(x,v) = (\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t))$.

[2] For completeness we recall the formal definition, although we won't need it in the sequel. Let M be any C^∞ compact manifold then we call a C^1 flow $\phi_t : M \to M$ *hyperbolic* (or Anosov) if:

(a) the tangent bundle TM has a continuous splitting $T_\Lambda M = E^0 \oplus E^u \oplus E^s$ into $D\phi_t$-invariant sub-bundles E^0 is the one-dimensional bundle tangent to the flow; and there exist $C, \lambda > 0$ such that $||D\phi_t|E^s|| \leq Ce^{-\lambda t}$ for $t \geq 0$ and $||D\phi_{-t}|E^u|| \leq Ce^{-\lambda t}$ for $t \geq 0$;
(b) $\phi_t : M \to M$ is transitive (i.e., there exists a dense orbit); and
(c) the periodic orbits are dense in M.

(More generally, if there is a closed ϕ-invariant set Λ with the above properties then $\phi_t : \Lambda \to \Lambda$ is called a hyperbolic flow.)

use of sequences to model geodesic flows goes back even further to the work of Morse and Hedlund [25] who coded geodesics in terms of generators for the fundamental group.

Step 1 (Discrete maps from flows)

At its most general (and probably least canonical) the coding of orbits for hyperbolic flows $\phi_t : M \to M$ on any compact manifold M starts with a finite number of codimension one sections $T_1, \ldots, T_k$ to the flow. Let $X = \cup_i T_i$ denote the union of the sections. We can consider the discrete Poincaré return map $T : X \to X$, i.e, the map which takes a point x on a section to the point $T(x)$ where its ϕ-orbit next intersects a section. Of course, we need to assume that the sections are chosen so that

(i) every orbit hits the union of the sections infinitely often.

We would also like to consider the map $r : \cup_i T_i \to \mathbb{R}^+$ which gives the time it takes for $x \in X$ to flow to $T(x) \in X$, i.e., $\phi_{r(x)}(x) = T(x)$.

Key idea (modulo a slight fudge) There is a natural correspondence between periodic discrete orbits $T^n x = x$ and continuous periodic orbits τ of period $\lambda = \lambda(\tau) > 0$ (i.e., the smallest value such that $\phi_\lambda(x_\tau) = x_\tau$ for all $x_\tau \in \tau$), where

$$\lambda = r(x) + r(Tx) + \ldots + r(T^{n-1}x).$$

Like many simple ideas, it is not quite true. There is an additional technical complication because of the closed orbits which pass through the boundaries of sections. However, this is not the typical case and an extra level of technical analysis sorts out this problem [10].

Step 2 (Sequence spaces from the Poincaré map)

The essential idea in symbolic dynamics is that a typical orbit $\{\phi_t(x) : -\infty < t < \infty\}$ will traverse these sections infinitely often (both in forward time and backward time) giving rise to a bi-infinite sequence $(x_n)_{n=-\infty}^{\infty}$ of labels of the sections it traverses [10], [56].

(ii) The sections are chosen to have a Markov property (i.e., essentially that the space Σ of all possible sequences $(x_n)_{n=-\infty}^{\infty}$ is given by a nearest neighbour condition: there exists a $k \times k$ matrix A with entries either 0 or 1 such that the sequence occurs if and only if $A(x_n, x_{n+1}) = 1$).

Alternatively, we can retain a little of the regularity of the functions as follows.

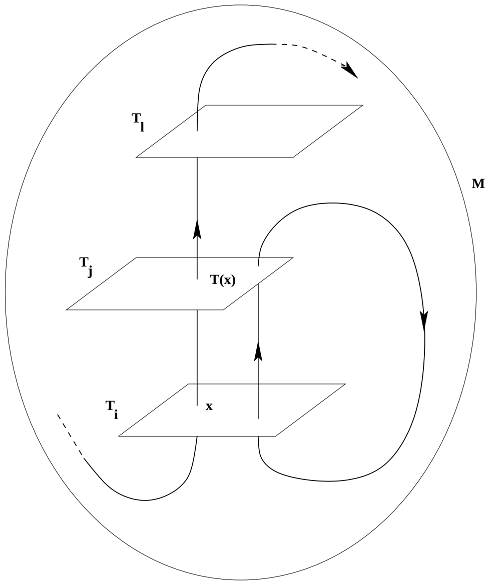

Fig. 1. Figure 1. Transverse (Markov) sections for a hyperbolic flow code a typical orbit and a closed orbit

Step 2′ (Expanding maps from the Poincaré map)

Instead of a reducing orbits to sequences, we can replace the invertible Poincaré map by an expanding map (on a smaller space). The basic idea is to remove the contracting direction by identifying the sections X along the stable directions. We can then replace the union of two dimensional sections X by a union of one dimensional intervals Y. The Poincaré map $T : X \to X$ then quotients down to an expanding map $S : Y \to Y$ [59]. Of course, we lose track of the "pasts" of orbits, but for most purposes this is not a real problem.

1.2 An alternative approach for constant curvature: The Modular surface and compact surfaces

We mentioned that for geodesic flows on surfaces of constant negative curvature there is an alternative method of Hedlund and Hopf to code geodesics. This method was further developed by Adler and Flatto [1] and Series [69]. Again it leads to a C^ω expanding Markov map $T : Y \to Y$. In this case, Y corresponds to the boundary of the universal cover

$$\mathbb{D}^2 = \{z = x + iy \in \mathbb{C} : |z| < 1\}$$

of the surface,i.e., the unit circle. This is divided into a finite number of arcs (actually determined by the sides of a fundamental domain for the surface).

The corresponding metric on $\mathbb{D}^2$ is $ds^2 = (dx^2 + dy^2)/(1 - x^2 - y^2)^2$. The side pairs of the fundamental domain correspond to linear fractional transformations which preserve $\mathbb{D}^2$. On the boundary they give rise to expanding interval maps. A geodesic on $\mathbb{D}^2$ is uniquely determined by its two end points on the unit circle. We can associate a function $r : Y \to \mathbb{R}$ by $r(x) = \log |T'(x)|$, then we have the ingredients of the symbolic model.

Example: Modular surface We can consider the geodesic flow on the modular surface. In this case the surface is non-compact, and the difference is that the linear fractional transformation $T : Y \to Y$ is on an infinite number of intervals. However, in this case the transformation T is the well known continued fractional transformation on $[0, 1]$, i.e., $T : [0, 1] \to [0, 1]$ by $Tx = \frac{1}{x}$ (mod 1). The corresponding function $r : I \to \mathbb{R}$ is $r(x) = -2 \log x$, as is easily checked.

In this case the associated transfer operator is very easy to describe. We look at the Banach space B of analytic functions (with a continuous extension to the boundary) on a disk $\{z \in \mathbb{C} : |z - \frac{1}{2}| < \frac{3}{2}\}$. The transfer operator is given by $\mathcal{L}_s h(x) = \sum_{n=1}^{\infty} h\left(\frac{1}{x+n}\right) \frac{1}{(x+n)^{2s}}$ and the determinant $\det(I - \mathcal{L}_s)$ is analytic for $Re(s) > 1$. Using an approach of Ruelle [58], [59], Mayer [40], [41] showed that

(i) $\det(I - \mathcal{L}_s)$ has analytic extension to $\mathbb{C}$; and
(ii) $\zeta(s) = \det(I - \mathcal{L}_{s+1})/\det(I - \mathcal{L}_s)$

For the Modular surface (and related surfaces) this special model leads to very elegant connections with functional equations, the Riemann zeta function and Modular functions cf. [38].

2 Zeta functions, symbolic dynamics and determinants

Let us denote by τ closed orbits for ϕ and let us write $\lambda(\tau) > 0$ for the period, (i.e., given $x \in \tau$ we have $\phi_{\lambda(\tau)}(x) = x$). We shall call τ a primitive closed orbit if $\lambda(\tau)$ is the smallest such value. Let us assume for simplicity a fact which is patently not true (but which has the virtue that it makes a complicated argument into a simple one!) that $r(x) = r(x_0, x_1)$ depends on only two terms in the sequence $x = (x_n)_{n=-\infty}^{\infty} \in \Sigma$. We can then associate to A a weighted $k \times k$ matrix $M_s(i, j) = A(i, j)e^{-sr(i,j)}$, i.e., the entries 1 in A are replaced by values of the exponential of $-sr$ (with $s \in \mathbb{C}$) [45], [44]:

$$\begin{aligned}
\zeta(s) &= \prod_{\tau=\substack{\text{prime}\\ \text{orbits}}} \left(1 - e^{-s\lambda(\tau)}\right)^{-1} = \exp\left(\sum_{\tau=\substack{\text{prime}\\ \text{orbits}}} \sum_{m=1}^{\infty} \frac{(e^{-s\lambda(\tau)})^m}{m} \right) \\
&= \exp\left(\sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \sum_{\substack{\text{prime orbits}\\ \{x,\dots,\sigma^{p-1}x\}}} \frac{e^{-sm[r(x_0,x_1)+r(x_1,x_2)+\cdots+r(x_{p-1},x_0)]}}{m} \right) \\
&= \exp\left(\sum_{n=1}^{\infty} \sum_{\sigma^n x = x} \frac{e^{-sn[r(x_0,x_1)+r(x_1,x_2)+\cdots+r(x_{p-1},x_0)]}}{n} \right) \qquad (2.1) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{\operatorname{trace}(M_s^n)}{n} \right) = \frac{1}{\det(I - M_s)}.
\end{aligned}$$

In particular, in this model case we see that $\zeta(s)$ has a (non-zero) meromorphic extension to the entire complex plane. Moreover, the poles are characterised as those values s for which the matrix M_s has 1 as an eigenvalue.

More generally, the function r will be more complicated, but still retains sufficient regularity that the spirit of the above simple argument applies. In the more general setting, the matrix is replaced by a bounded linear operator (the Ruelle transfer operator). [3] The spectrum of this operator is quasi-compact (i.e., aside from isolated eigenvalues of finite multiplicity, the remaining essential spectrum is in a "small" disk). The corresponding result is then in general [58], [44], [49]:

Theorem 2.1 *The zeta function $\zeta(s)$ converges on a half-plane $Re(s) > h$. The zeta function $\zeta(s)$ has a non-zero meromorphic extension to a larger half-plane $Re(s) > h - \epsilon$, for some $\epsilon > 0$.*

There is a simple pole at $s = h$ and, for geodesic flows, there are no other poles on the line $Re(s) = h$.

In the special case of hyperbolic flows with analytic horocycle foliations it is possible to show much more. This includes, for example, constant curvature geodesic flows. This gives much stronger results [59]:

Theorem 2.2 *The zeta function $\zeta(s)$ has a non-zero meromorphic extension to $\mathbb{C}$.*

The proof of this result is similar in spirit, except that from the hypothesis on the foliations the expanding map on the sections (in Case $2'$ before) is also C^ω. The transfer operator on analytic functions is trace class and so the determinant makes perfect sense.

If the foliations are not analytic (which is the case for variable curvature surfaces) then slightly less is known [32], [62] and [21].

[3] The transfer operator in the context of the Modular surface is the operator $\mathcal{L}_s$ described in the last section

3 Counting orbits

We want to mimic the use of the Riemann zeta function in prime number theory, except we want to count closed orbits instead of prime numbers. The aim is to describe that asymptotic behaviour of the number of prime numbers less than x, i.e.,

$$\pi(x) = \#\{p \leq x : p \text{ is a prime } \} \text{ for } x > 0.$$

Notation: We write $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to +\infty$.

In 1896, Hadamard and de la Vallée Poussin independently showed the asymptotic estimate $\pi(x) \sim \frac{x}{\log x}$, as $x \to +\infty$, i.e., the *prime number theorem* [19]. The basic properties of $\pi(x)$ come from the *Riemann zeta function* defined by

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=prime} \left(1 - p^{-s}\right)^{-1}.$$

This converges to an analytic non-zero function on the domain $Re(s) > 1$. Moreover, $\zeta_R(s)$ has the following important properties [19]:

(1) $\zeta_R(s)$ has an analytic non-zero extension to a neighbourhood of $Re(s) \geq 1$, *except* for a simple pole at $s = 1$;
(2) $\zeta_R(s)$ has a meromorphic extension to all of $\mathbb{C}$; and $\zeta_R(s)$ and $\zeta_R(1-s)$ are related by a functional equation.

Property (1) has a direct analogue for most hyperbolic flows (including geodesic flows). We say that a hyperbolic flow is weak mixing if the set of lengths of closed orbits $\{\lambda(\tau) : \tau = \text{ closed orbit}\}$ isn't contained in $a\mathbb{N}$, for some $a > 0$. In particular, any geodesic flow is weak mixing. [4] The following is the analogue of property (1) for the Riemann zeta function.

Theorem 3.1 *Let ϕ be a weak mixing hyperbolic flow. There exists $h > 0$ such that $\zeta(s)$ has an analytic non-zero extension to a neighbourhood of $Re(s) > h$, except for a simple pole at $s = h$.*

The value h is the topological entropy of the geodesic flow.

Unfortunately, property (2) doesn't always have a direct analogue for general hyperbolic flows (although it does for constant curvature geodesic flows). [5] However, since the proof of the prime number theorem only required property (1) for the Riemann zeta function, we expect that something similar will hold for closed orbits of hyperbolic flows. We can denote

$$\pi(T) = \text{Card}\{\tau : \lambda(\tau) \leq T\}, \text{ for } T > 0.$$

[4] Although height one suspended flows over hyperbolic diffeomorphisms aren't!
[5] Indeed there are examples (due to Gallavotti) of zeta functions which have logarithmic singularities.

The following result is the analogue of the prime number theorem for closed orbits.

Corollary 3.2 *Let $\phi_t : M \to M$ be a weak mixing hyperbolic flow then*

$$\pi(T) \sim \frac{e^{hT}}{hT}, \text{ as } T \to +\infty. \tag{3.1}$$

This was proved, although the details were not published at the time, by Margulis in 1969. This proof did not use zeta functions, but properties of transverse measures for the horocycle foliation. (The proof was reconstructed by Toll in his unpublished Ph.D. thesis from the University of Maryland in 1984.) An alternative proof using zeta functions was given by Parry and Pollicott in [45]. Prior to this Sinai had shown in 1966 that $\lim_{T\to+\infty} \frac{1}{T} \log \pi(T) = h$. For the special case of geodesic flows on surfaces of constant curvature $\kappa = -1$, Huber showed in 1959, using the Selberg trace formula, that $\pi(T) = \mathrm{li}(e^{hT}) + O\left(e^{cT}\right)$ where $\mathrm{li}(x) = \int_2^x \frac{du}{\log u}$ and $c < h$ is actually related to the first non-zero eigenvalue of the Laplacian on the surface.

There are related results for counting geodesic arcs between two given points in place of closed geodesics [54].

3.1 Riemann hypothesis and error terms for primes

The (still unproved) Riemann hypothesis states that: *Riemann hypothesis* $\zeta(s)$ has all of its (non-trivial) zeros on the line $Re(s) = 1/2$.

We recall the following:

Notation: We write $f(T) = g(T) + O(h(T))$ if there exists $C > 0$ such that $|f(T) - g(T)| \le C|h(T)|$.

The Riemann hypothesis would imply that for any $\epsilon > 0$ we can estimate $\pi(x) = \mathrm{li}(x) + O(x^{1/2+\epsilon})$. To date, only smaller non-uniform estimates on the zero free region are known which lead to weaker error terms [19].

3.2 Error terms for closed orbits

It turns out that it is more convenient to replace the principal asymptotic term by $\mathrm{li}(e^{hT}) \sim \frac{e^{hT}}{hT}$, as $T \to +\infty$.

The following result shows that for variable curvature geodesic flows we get exponential error terms (cf. [16] [53]).

Theorem 3.3 *Let $\phi_t : M \to M$ be the geodesic flow for a compact surface with negative curvature. There exists $0 < c < h$, where h again denotes the topological entropy, such that*

$$\pi(T) = \mathrm{li}(e^{hT}) + O\left(e^{cT}\right), \text{ as } T \to +\infty \tag{3.2}$$

Unfortunately, in contrast to the constant curvature case, there is little insight into the value of $c > 0$. The estimate (3.2) extends to counting closed geodesics on compact manifolds of arbitrary dimension providing that the sectional curvature is pinched $-4 \leq \kappa \leq -1$. The following result on $\zeta(s)$ is the main ingredient in the proof of Theorem 3.3.

Proposition 3.4 *For a geodesic flow there exists $c < h$ such that $\zeta(s)$ is analytic in the half-plane $Re(s) > c$, except for a simple pole at $s = h$. Moreover, there exists $0 < \alpha < 1$ such that $\zeta'(h+it)/\zeta(h+it) = O(|t|^\alpha)$, as $|t| \to +\infty$.*

This can be viewed as an analogue of the classical Riemann Hypothesis for the zeta function for prime numbers. It is well-known for the case of constant negative curvature (using the approach of the Selberg trace formula).

At the level of more general (weak mixing) hyperbolic flows no such result can hold. Indeed, there are very simple examples with poles $\sigma_n + it_n$ for $\zeta(s)$ such that $\sigma_n \nearrow h$ (and $t_n \nearrow \infty$) [47].

3.3 Spatial distribution of closed orbits

Given a geodesic flow $\phi_t : M \to M$, a classical result of Bowen [13] shows that the closed orbits τ are evenly distributed (according to the measure of maximal entropy μ). Consider a Hölder continuous function $g : \Lambda \to \mathbb{R}$, then we can weight a given closed orbit τ by $\lambda_g(\tau) = \int_0^{\lambda(\tau)} g(\phi_t x_\tau)dt$ (for $x_\tau \in \tau$). The following result was originally proved by Bowen [13], with a subsequent proof by Parry [43] using suitably weighted zeta functions.

Theorem 3.5 *Given a geodesic flow $\phi : M \to M$ there exists a probability measure μ such that*

$$\sum_{\lambda(\tau)\leq T} \lambda_g(\tau) / \sum_{\lambda(\tau)\leq T} \lambda(\tau) \to \int g d\mu \text{ as } T \to +\infty.$$

In the case of constant curvature surfaces the measure of maximal entropy is the Liouville measure (i.e., the natural normalised volume).

There are also Central Limit Theorems [57] and Large Deviation Theorems [31] for closed geodesics. In particular, the latter can be viewed as generalisations of Theorem 3.5. More generally, the following result of Kifer is valid for any hyperbolic flow and so, in particular, for the geodesic flow $\phi_t : SV \to SV$. Let μ_τ be the natural invariant measure supported on a closed orbit τ.

Proposition 3.6 *Let $\mathcal{U}$ be an open neighbourhood of the measure of maximal entropy μ in the space $\mathcal{M}$ of all ϕ-invariant probability measures on M. Then*

$$\frac{1}{\pi(T)} \#\{\tau : \lambda(\tau) \leq T \text{ and } \mu_\tau/\lambda(\tau) \notin \mathcal{U}\} = O(e^{-\delta T}),$$

as $T \to +\infty$, where $\delta = \inf_{\nu \in \mathcal{M}-\mathcal{U}}\{h - h(\nu)\} > 0$.

3.4 Homological distribution of closed orbits

By way of motivation, recall the asymptotic behaviour of the number $B(x)$ of integers less than x which can be written as a square or as the sum of two squares, i.e., $B(x) = \#\{1 \leq n \leq x : n = u_1^2 + u_2^2, u_1, u_2 \in \mathbb{Z}\}$ for $x > 0$. Landau [35] showed that $B(x) \sim Kx/(\log x)^{1/2}$, for some $K > 0$, and the same result appears in Ramanujan's famous letter to Hardy in 1913 [8]. The full asymptotic expansion for $B(x)$ has the simple form

$$B(x) = \frac{Kx}{(\log x)^{1/2}} \left(1 + \sum_{n=1}^{N} \frac{\alpha_n}{(\log T)^n} + O\left(\frac{1}{(\log x)^N} \right) \right)$$

for any $N \geq 1$. [23]. The proof of the above asymptotic expansion involves studying the complex function

$$s \mapsto \frac{1}{1 - 2^{-s}} \prod_q \frac{1}{1 - q^{-s}} \prod_r \frac{1}{1 - r^{-2s}},$$

where q runs through all primes equal to 1 (mod 4) and r runs through all primes equal to 3 (mod 4). Of course, this differs from the Riemann zeta function only in the factor of 2 in the last exponent, but the result is a singularity of the form $(s-1)^{-1/2}$, rather than a simple pole, which leads to a different asymptotic behaviour.

As usual, we let V denote a compact surface of negative curvature. Let $\alpha \in H_1(V, \mathbb{Z})$ be a fixed element in the first homology. Given a closed geodesic γ we denote by $[\gamma]$ the homology class associated to a closed geodesic V. Let $\pi(T, \alpha)$ be the number of closed geodesics in the homology class α of length at most T, i.e.,

$$\pi(T, \alpha) = \#\{\gamma : l(\gamma) \leq T, \quad [\gamma] = \alpha\}.$$

The following formula was proved independently by Anantharaman [4] and Pollicott and Sharp [55].

Theorem 3.7 *Let $b = \dim(H_1(V, \mathbb{R}))$ be the first Betti number for V. There exist $C_0, C_1, C_2, \ldots$ (with $C_0 > 0$) such that*

$$\pi(T, \alpha) = \frac{e^{hT}}{T^{b/2}} \left(\sum_{n=0}^{N} \frac{C_n}{T^n} + O\left(\frac{1}{T^N} \right) \right) \quad \textit{as } T \to +\infty,$$

for any $N > 0$.

The similarity with Landau's result comes from a $(s-1)^{-1/2}$ singularity also appearing in the domain of the corresponding zeta function for $\pi(T, \alpha)$.

For surfaces of constant curvature $\kappa = -1$ this was originally proved by Phillips and Sarnak [46]. Katsuda and Sunada [30], Lalley [33] and Pollicott [50] then each independently showed that for more general surfaces of variable curvature the basic asymptotic formula $\pi(T,\alpha) \sim \frac{e^{hT}}{T^{b/2}}$, as $T \to +\infty$, still holds.

Finally, we should remark that there are interesting results on special values of the closely related homological L-functions cf. [20], [22].

3.5 Intersections of closed orbits

There are a number of results describing the average number of times a typical closed geodesic intersects itself [48] and [34] [6]. However, we shall describe a more topical result conjectured by Sieber and Richter[71].

Given $0 \le \theta_1 < \theta_2 \le 2\pi$, let $i_{\theta_1,\theta_2}(\gamma)$ denote the number of self-intersections of the closed geodesic γ such that the absolute value of the angle of intersection lies in the interval $[\theta_1, \theta_2]$.

Theorem 3.8 *Given $0 \le \theta_1 < \theta_2 \le 2\pi$, there exists $I = I(\theta_1, \theta_2)$ and $c < h$ such that, for any $\epsilon > 0$,*

$$\#\left\{\gamma : l(\gamma) \le T, \frac{i_{\theta_1\theta_2}(\gamma)}{l(\gamma)^2} \in (I-\epsilon, I+\epsilon)\right\} = \mathrm{li}(e^{hT}) + O(e^{cT}).$$

We shall outline the idea of the proof, due to Sharp and the author. Let $\mathcal{F}$ denote the foliation of SV by orbits of the geodesic flow ϕ. Given any ϕ-invariant finite measure μ (not necessarily normalised to be a probability measure) we can consider the associated transverse measure $\tilde{\mu}$ for $\mathcal{F}$. The set of such transverse measures $\mathcal{C}$ is usually called the *space of currents*. Let $E = SV \oplus SV - \Delta$ be the Whitney sum of the bundle SV with itself, minus the diagonal $\Delta = \{(x, v, v) : x \in V, v \in S_xV\}$. Let $p : E \to V$ denote the canonical projection. In particular, points of the four dimensional vector bundle E (with two dimensional fibres) consist of triples $\{(x, v, w) : x \in V \text{ and } v, w \in S_xV\}$. Let $p_1 : E \to SV$ be the projection defined by $p_1(x, v, w) = v$ and let $p_2 : E \to SV$ be defined by $p_2(x, v, w) = w$. Following closely Bonahon's construction [9], we consider the two transverse foliations (with one dimensional leaves) of E given by $\mathcal{F}_1 = p_1^{-1}(\mathcal{F})$ and $\mathcal{F}_2 = p_2^{-1}(\mathcal{F})$. Given $0 \le \theta_1 < \theta_2 \le \pi$, we define the *angular intersection bundle* $E_{\theta_1,\theta_2} \subset E$ by $E_{\theta_1,\theta_2} = \{(x, v, w) \in E : \angle vw \in [\theta_1, \theta_2]\}$, where $0 \le \angle vw \le \pi$ denotes the angle between the two vectors. This is a closed sub-bundle of E.

Given currents $\tilde{\mu}, \tilde{\mu}' \in \mathcal{C}$, we can take the lifts $\widehat{\mu}_1 := p_1^{-1}\tilde{\mu}$ and $\widehat{\mu}_2' := p_2^{-1}\tilde{\mu}'$, which are transverse measures to the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ for E, respectively. Bonahon defined the *intersection form* $i : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$ to be the total mass of the E with respect to the product measure $\widehat{\mu}_1 \times \widehat{\mu}_2'$, i.e., $i(\tilde{\mu}, \tilde{\mu}') = (\widehat{\mu}_1 \times \widehat{\mu}_2')(E)$ [9]. By analogy, we can define an *angular intersection form* $i_{\theta_1,\theta_2} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}^+$ to be the total mass of the E_{θ_1,θ_2} with respect to the product measure $\widehat{\mu}_1 \times \widehat{\mu}_2'$, i.e., $i_{\theta_1,\theta_2}(\tilde{\mu}, \tilde{\mu}') = (\widehat{\mu}_1 \times \widehat{\mu}_2')(E_{\theta_1,\theta_2})$.

[6] Which also corrects an error in the asymptotic expression in [48]

In the present context, the large deviation result Proposition 3.6 gives the following estimates.

Lemma 3.9 *Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\frac{1}{\pi(T)} \#\{\gamma : l(\gamma) \le T \text{ and } |l(\gamma)^{-2}(\widehat{\mu}_{\gamma,1} \times \widehat{\mu}_{\gamma,2})(E_{\theta_1,\theta_2}) - (\widehat{m}_1 \times \widehat{m}_2)(E_{\theta_1,\theta_2})| \ge \epsilon\}$$
$$= O(e^{-\delta T}), \text{ as } T \to +\infty.$$

In particular, we can set $I(\theta_1, \theta_2) := i_{\theta_1,\theta_2}(\tilde{\mu}, \tilde{\mu})$, where μ is the measure of maximal entropy. We deduce that, except for an exceptional set with cardinality of order $O(e^{(h-\delta)T})$, the set of closed geodesics of length at most T satisfy $|l(\gamma)^{-2} i_{\theta_1,\theta_2}(\gamma) - I(\theta_1, \theta_2)| < \epsilon$. Theorem 3.8 then follows easily by applying the asymptotic counting results described in §3.2.

3.6 Decay of Correlations (a compliment to counting orbits)

A closely related problem to that of counting closed orbits is that of decay of correlations. Let $\phi_t : M \to M$ be a weak-mixing hyperbolic flow and let μ again be the measure of maximal entropy (i.e., the measure in Theorem 3.5). The flow ϕ is strong mixing, i.e.,

$$\rho_{F,G}(t) := \int F \circ \phi_t G d\mu - \int F d\mu \int G d\mu \to 0, \text{ for all } F, G \in L^2(X, \mu).$$

(i.e., the "correlation of the flow tends to zero".)

Dolgopyat proved the following result on exponential decay of correlations in the case of geodesic flows on compact negatively curved surfaces [17].

Theorem 3.10 *Let $\phi_t : M \to M$ be the geodesic flow for a surface of variable negative curvature. There exists $\epsilon > 0$ such that for any smooth functions $F, G : M \to \mathbb{R}$ there exists $C > 0$ with $\rho_{F,G}(t) \le Ce^{-\epsilon|t|}$.*

For constant negative curvature surfaces this result can be proved using representation theory [42], [15]. Moreover, there are very few examples of hyperbolic flows for which exponential decay of correlations is known to hold [17].

The complex function used in the study of $\rho_{F,G}(t)$ is its Fourier transform $\widehat{\rho}_{F,G}(s) = \int e^{ist} \rho_{F,G}(t) dt$.

Theorem 3.11 *Let $\phi : M \to M$ be a C^r hyperbolic flow ($r \ge 2$ or $r = \omega$). There is a neighbourhood $\mathcal{V}$ of ϕ (amongst C^r flows on M) such that:*

there exists $\epsilon > 0$ such that the associated correlation function $\widehat{\rho}^{(\psi)}(s)$ has a meromorphic extension to a strip $|Im(s)| < \epsilon$, for each $\psi \in \mathcal{V}$ [47]; and *whenever $s_i = s_i(\phi)$ is a simple pole for $\widehat{\rho}^{(\phi)}(s)$ in the strip $|Im(s)| < \epsilon$ then the map $\mathcal{V} \ni \psi \mapsto s_i(\psi)$ is C^{r-2} [51].*

Moreover, since the analysis of the Fourier transform also depends on the Ruelle transfer operator there is a direct relationship between the poles of $\hat{\rho}_{F,G}(s)$ and $\zeta_\phi(s)$ (described in [47]). More precisely, s (with $Im(s) < 0$) is a pole for $\widehat{\rho}(s)$ if and only if $h + is$ is a pole for $\zeta(s)$.

If we replace μ by the Liouville measure (or any other suitable Gibbs measure) analogous results hold, with a suitably weighted zeta function.

4 Other applications of closed geodesics

Ruelle's approach to the proof of theorem 2.2 has a number of other applications. Here we recall a couple of our favourites.

4.1 Determinants of the Laplacian

A very interesting object in the case of surfaces V of constant negative curvature $\kappa = -1$ is the (functional) *determinant of the Laplacian.* The Laplacian $\Delta : L^2(V) \to L^2(V)$ is a self-adjoint linear differential operator. Let us write the spectrum of $-\Delta$ as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow +\infty$ and consider the associated Dirichlet series

$$\eta(s) := \sum_{n=1}^{\infty} \lambda_n^{-s}.$$

This converges for $Re(s) > 1$, as is easily seen using Weyl's theorem. The function $\eta(s)$ has a meromorphic extension to $\mathbb{C}$ and we define the determinant by $\det(\Delta) := \exp(-\eta'(0))$. [7] The function $\det(\Delta)$ depends smoothly on the Riemann metric. There is considerable interest in understanding its critical points [66].

Somewhat surprisingly this quantity can be explicitly expressed in terms of the closed geodesics. The starting point is that is a direct connection between $\det(\Delta)$ and the Selberg zeta function. First we define for each $n \geq 1$ the function

$$a_n := \sum_{|\tau_1| \cdots + |\tau_r| = n} (-1)^{r+1} \frac{\lambda(\tau_1) + \cdots \lambda(\tau_k)}{(e^{\lambda(\tau_1)} - 1) \cdots (e^{\lambda(\tau_k)} - 1)},$$

where the sum is over collections of closed orbits for the geodesic flow (or, equivalently, closed geodesics) and $|\tau|$ denotes the word length of a corresponding conjugacy class in $\pi_1(V)$ with respect to a suitable choice of generators (i.e., the smallest number of generators needed to write an element in this conjugacy class). The following theorem was proved in [52]. [8]

[7] A particularly nice introduction to this subject is [66].

[8] The title of this article is good humoured reference to the title of the Ph.D. thesis of G. McShane.

Theorem 4.1 *We can write* $\det(\Delta) = C(g) \sum_{n=1}^{\infty} a_n$, *where the series is absolutely convergent (and* $|a_n| = O(\theta^{n^2})$*) and* $C(g)$ *is a constant depending only on the genus* g *of the surface* V*.*

It is also possible to use the zeta functions to describe the dependence of other dynamical invariants, such as entropy [28].

4.2 Computation

It is an interesting problem to get numerical estimates on dynamical properties for interval maps. For example, given an expanding interval maps it might be interesting to estimate the entropy (of the absolutely continuous invariant measure). The "classical" approach to this problem is the Ulam method, in which the map is essentially approximated by a piecewise linear map and the density can be estimated from the eigenvectors of the matrix.

We can now describe a somewhat different method which applies to C^ω expanding maps $T : I \to I$ on an interval I. We can define invariant (signed) measures ν_M defined by

$$\nu_M = \sum_{\substack{(n_1,\ldots,n_m)\\ n_1+\cdots+n_m \le M}} \frac{(-1)^m}{m!} \sum_{i=1}^{m} \sum_{x \in \mathrm{Fix}(n_i)} \left(\prod_{\substack{j=1\\ j\ne i}}^{m} \sum_{z \in \mathrm{Fix}(n_j)} r(z, n_j) \right) \frac{\delta_x}{|(T^{n_i})'(x) - 1|}$$

where δ_x is the Dirac measure and the first summation is over ordered m-tuples of positive integers whose sum is not greater than M, where $\mathrm{Fix}(n)$ denotes the set of fixed points of T^n, and where

$$r(x, n) = \frac{1}{n|(T^n)'(x) - 1|}.$$

The measure ν_M is supported on those periodic points of period at most M, which can easily be computed in practise. Introducing the normalisation constant

$$I_M = \sum_{\substack{(n_1,\ldots,n_m)\\ n_1+\cdots+n_m \le M}} \frac{(-1)^m}{m!} \sum_{i=1}^{m} \sum_{x \in \mathrm{Fix}(n_i)} \left(\prod_{\substack{j=1\\ j\ne i}}^{m} \sum_{z \in \mathrm{Fix}(n_j)} r(z, n_j) \right) \frac{1}{|(T^{n_i})'(x) - 1|},$$

we then define the invariant signed probability measures $\mu_M = I_M^{-1} \nu_M$. For real analytic maps we have the following [27]:

Theorem 4.2 *Let* μ *be the absolutely continuous* T*-invariant probability measure. There is a sequence of* T*-invariant signed probability measures* μ_M*, supported on the points of period at most* M*, such that for every* C^ω *function* $g : I \to \mathbb{R}$*, there exists* $0 < \theta < 1$ *and* $C > 0$ *with*

$$\left| \int g \, d\mu_M - \int g \, d\mu \right| \leq C\theta^{M(M+1)/2}.$$

In particular, with the choice $g = \log |T'|$ we have as a corollary that the "Lyapunov exponent" $\lambda_\mu = \int \log |T'| \, d\mu$ can be quickly approximated, i.e.,

$$\left| \int \log |T'| \, d\mu_M - \lambda_\mu \right| \leq C\theta^{M(M+1)/2}.$$

Many related ideas appear in the beautiful work of Cvitanovic and his coauthors.

5 Frame flows

Recently, there has been interest in extending results for hyperbolic flows to partially hyperbolic flows. That is, we allow some transverse directions to the flow that are neither expanding nor contracting. The principle example of such systems are probably the frame flow, which is an extension of the geodesic flow $\phi_t : M \to M$ on the unit tangent bundle, for a manifold V with negative sectional curvatures.

5.1 Frame flows: Archimedean version

Let $St_{n+1}(V)$ be the space of (positively oriented) orthonormal $(n+1)$-frames. The frames $St_{n+1}(V)$ form a fibre bundle over M with a natural projection $\pi : St_{n+1}(V) \to M$ which simply forgets all but the first vector in the frame, i.e., $\pi(v_1, \ldots, v_{n+1}) = v_1$. The frame flow $f_t : St_{n+1}(V) \to St_{n+1}(V)$ acts on frames $(v_1, \ldots, v_{n+1}) \in St_{n+1}(V)$ by parallel transporting for time t the frame along the geodesic $\gamma_{v_1} : \mathbb{R} \to V$ satisfying $v_1 = \dot{\gamma}_{v_1}(0)$. In particular, the frame flow semi-conjugates to the geodesic flow, i.e., $\pi f_t = \phi_t \pi$ for all $t \in \mathbb{R}$.

The associated structure group acts on each fibre by rotating the frames about the first vector v_1. In particular, we can identify each fibre $\pi^{-1}(v) \subset St_{n+1}(V)$, for $v \in St_1(V)$, with the compact group $SO(n)$. We can associate to each closed orbit τ a holonomy element $[\tau] \in SO(n-1)$ (defined up to conjugacy). The following is the natural analogue of Theorem 3.5 [44].

Theorem 5.1 *Let $f : SO(n-1) \to \mathbb{R}$ be a continuous function constant on conjugacy classes. Then*

$$\frac{1}{\pi(T)} \sum_{\lambda(\tau) \leq T} f([\tau]) \to \int f d\omega, \ \text{as } T \to +\infty,$$

where ω is the Haar measure on $SO(n-1)$.

The idea of the proof is that we can model the underlying geodesic flow symbolically by a sequence space Σ, etc. But for the frame flow we additionally have an associated map $\Theta : \Sigma \to SO(n-1)$ which essentially measures the "twist" in $SO(n-1)$ along the orbits.

The distribution properties of frame flows on certain non-compact manifolds have been considered in [36]. In this context, there is a particularly interesting connection with Clifford numbers [3].

5.2 Non-Archimedean version

Let $\mathbb{Q}_p$ denote the p-adic numbers with the usual valuation $|\cdot|_p$. Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 0\}$ denote the p-adic integers. We can study a natural analogue of the frame flow and geodesic flow for $G = PSL(2, \mathbb{Q}_p)$. The rôle of the hyperbolic upper half plane $\mathbb{H}^2$ in the usual archimedean case is taken here by a regular tree X, say. We recall the basic construction.

Vertices Given any pair of vectors $v_1, v_2 \in \mathbb{Q}_p^2$ one associates a *lattice* $L = v_1\mathbb{Z}_p + v_2\mathbb{Z}_p$. We can define an equivalence relation on lattices: $L \sim L'$ if lattices L, L' are homothetically related (i.e., there exists $\alpha \in \mathbb{Q}_p$ such that $L' = \alpha L$). We take the equivalence classes $[L]$ to be the vertices of the tree X.

Edges Given two vertices (equivalence classes) $[L_1], [L_2]$ we can associate an edge $[L_1] \to [L_2]$ whenever we can find a basis $\{v_1, v_2\}$ for L_1 and $\{\pi v_1, v_2\}$ for L_2, where $\pi = \frac{1}{p}$ is called the *uniformizer.*

Lemma 5.2 *[70] X is a homogeneous tree, with every vertex having* $(p+1)$-*edges.*

There is a natural action $GL(2, \mathbb{Q}_p) \times X \to X$ on the tree given by $\gamma[v_1\mathbb{Z}_p + v_2\mathbb{Z}_p] = [(\gamma v_1)\mathbb{Z}_p + (\gamma v_2)\mathbb{Z}_p]$. The construction and action is elegantly described by Serre [70]. The frame flow is actually a discrete action defined on the quotient space $\Gamma \backslash X$ of the associated tree X by a lattice Γ and is given as multiplication by $\left(\begin{smallmatrix} 1 & p \\ 0 & \pi \end{smallmatrix}\right)$. If Γ is torsion free then there is a natural shift map on the space of paths $\sigma : \Sigma \to \Sigma$ which plays the role of the geodesic flow. Let $\mathcal{S}$ be the closed multiplicative subgroup of squares in $\mathcal{O}^\times = \{x \in \mathbb{Q}_p : |x|_p = 0\}$. There exists a Hölder continuous function $\Theta : \Sigma \to \mathcal{S}$ such that the p-adic frame flow for a lattice Γ corresponds to a simple skew product

$$\begin{aligned} &\widehat{\sigma} : \Sigma \times \mathcal{S} \to \Sigma \times \mathcal{S} \\ &\widehat{\sigma}(x, s) = (\sigma x, \Theta(x)s). \end{aligned} \tag{5.2}$$

Let Γ_n be the set of conjugacy classes of $\gamma \in \Gamma$ with $|\mathrm{tr}\gamma|_p = n$. For each conjugacy class $[\gamma] \in \Gamma_n$, denote by $\sigma([\gamma]) \in \mathcal{S}$ the common value of $p^{|\lambda_\gamma^2|_p}\lambda_\gamma^2$, where λ_γ denotes the maximal eigenvalue. The analogue of Theorem 5.1 is the following result of Ledrappier and Pollicott.

Theorem 5.3 *Eigenvalues of matrices in Γ are uniformly distributed in the sense that for any continuous function ϕ on $\mathcal{S}$, we have:*

$$\lim_{n\to\infty} \frac{1}{p^{2n}} \sum_{[\gamma]\in\Gamma_n} \phi(\sigma([\gamma])) = \int \phi(s)d\omega(s),$$

where ω is the Haar probability measure on $\mathcal{S}$.

This can be viewed as a non-archemidean version of the results in [67]. Moreover, in the particular case that Γ is an arithmetic lattice it is possible to use Deligne's solution of the Ramanujan-Petersson conjecture to get uniform exponential convergence in Theorem 5.3.

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Part II

Appendices

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