

Dynamical Zeta functions

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1 Introduction

1.1 Zeta functions as complex functions

We will want to define zeta functions which, in different settings, are functions of a single complex variable defined in terms of a countable collection of suitable real numbers. One might think of them as being a device to keep track of these values, but more often they are a useful tool for extracting information about these values.

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There are two basic questions which apply equally well to all zeta functions (or, more generally, any complex function).

Question 1.1. *Where are these functions defined? How far can we extend them as analytic or meromorphic functions?*

Once this is established one can next ask:

Question 1.2. *Where are the zeros and poles of these functions? What values do they take at specific points in their domains?*

These questions are particularly important for complex functions arising both in number theory, geometry and ergodic theory.

1.2 Different types of zeta functions

In general terms we will discuss four different types of zeta function in four different settings:

1. Number Theory and the Riemann zeta function;
2. Graph Theory and the Ihara zeta function;
3. Geometry and the Selberg zeta function;
4. Dynamical Systems and the Ruelle zeta functions.

What they all have in common is that they are complex functions defined in terms of a countable collections on numbers (which in these four examples are: prime numbers, lengths of closed paths, lengths of closed geodesics, periods of closed orbits).

2 Zeta functions in Number Theory

The most famous setting for zeta functions is in number theory, and we begin with the most famous.

2.1 Riemann zeta function

We recall the well known definition:

Definition 2.1. *The Riemann zeta function is defined for $\operatorname{Re}(s) > 1$ by*

$$\zeta(s) = \prod_{p=\text{prime}} (1 - p^{-s})^{-1}$$

Of course, by multiplying out the terms in the definition one gets an even better known definition:

Lemma 2.2. *We can also write*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\operatorname{Re}(s) > 1$

The Riemann zeta function was actually studied earlier in 1737 by Euler (at least when s was real). However, in 1859 Riemann was elected member of the Berlin Academy of Sciences and had to report on his research, and in a departure from his previous (or subsequent) research he sent in a report “On the number of primes less than a given magnitude”. In particular, he established the following basic properties of his zeta function:

Theorem 2.3 (Riemann, 1859). *The zeta function $\zeta(s)$ converges to a non-zero analytic function for $\operatorname{Re}(s) > 1$. Moreover,*

1. $\zeta(s)$ has a single (simple) pole at $s = 1$;
2. $\zeta(s)$ extends to all complex numbers $s \in \mathbb{C}$ as a meromorphic function;
3. There is a functional equation relating $\zeta(s)$ and $\zeta(1-s)$

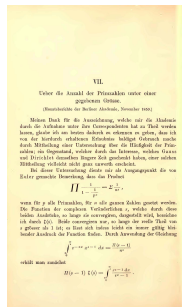


Figure 1: Riemann (1826-1866); His 1859 report; Euler (1707-1783)

One of the main subsequent applications of the zeta function was as a means to prove the prime number theorem (which was eventually proved by Hadamard and de la Vallée Poussin in 1896) and says:

Theorem 2.4 (Prime Number Theorem).

$$\#\{\text{prime } p \leq x\} \sim \frac{x}{\log x} \text{ as } x \rightarrow +\infty.$$

However, there remain a number of interesting problems on the Riemann zeta function which have important applications to counting primes. There are always trivial zeros at the negative even integers (the trivial zeros). The still unproved Riemann hypothesis says the following:

Conjecture 2.5 (Riemann Hypothesis). *The non-trivial zeros of $\zeta(s)$ are on the line $\operatorname{Re}(s) = \frac{1}{2}$.*

More precisely, $\Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)$ satisfies the functional equation $\Lambda(s) = \Lambda(1-s)$
 The proof required additionally showing that $\zeta(s)$ has no zeros on the line $\operatorname{Re}(s) = 1$



Figure 2: Hadamard (1863-1963); de la Vallée Poussin (1869-1962); Hardy (1877-1947)

This was formulated by Riemann in 1859 and was restated Hilbert's 8th problem and one of the Clay Institute million dollar problems. If it was valid then it would lead to significant improvements in the Prime Number Theorem.

Corollary 2.6 (Of Riemann Hypothesis). *If the Riemann Hypothesis holds then for any $\epsilon > 0$ we have that:*

$$\#\{\text{prime } p \leq x\} = \int_2^x \frac{du}{\log u} + O(x^{\frac{1}{2}+\epsilon}) \text{ as } x \rightarrow +\infty.$$

The principle term on the right hand side of the above expression is often called the *logarithmic integral* and is asymptotic to $x/\log x$ as $x \rightarrow +\infty$, as required by Theorem 2.4.

It has been experimentally verified for a very large number of zeros. An early result was the following result of Hardy

Theorem 2.7 (Hardy, 1914). *There are infinitely many zeros lie on the line $\text{Re}(s) = \frac{1}{2}$.*

In 1941, Selberg improved this to show that at least a (small) positive proportion of zeros lie on the line.

Remark 2.8. Hilbert and Polya proposed the idea of trying to understand the location of the zeros of the Riemann zeta functions in terms of eigenvalues of some (as of yet) undiscovered self-adjoint operator whose necessarily real eigenvalues are related to the zeros. This idea has yet to reach fruition for the Riemann zeta function but the approach works particularly well for the Selberg Zeta function. Interestingly, Selberg showed that the Riemann zeta function wasn't needed in counting primes.

In fact, Riemann Conjecture topped Hardy's famous wish list from the 1920s:

1. Prove the Riemann Hypothesis.
2. Make 211 not out in the fourth innings of the last test match at the Oval.
3. Find an argument for the nonexistence of God which shall convince the general public.
4. Be the first man at the top of Mount Everest.
5. Be proclaimed the first president of the U.S.S.R., Great Britain and Germany.
6. Murder Mussolini.

Remark 2.9. Recently it has been noticed that (assuming the Riemann Hypothesis) the spacings of the imaginary parts of the non-trivial zeros of $\zeta(s)$ behave like the eigenvalues of random Hermitian matrices. Katz and Sarnak call this the Montgomery-Odlyzko law (another name is GUE for Gaussian Unitary Ensemble) although the law does not appear to have found a proof.

2.2 Other types of complex functions in Number Theory

2.2.1 L -functions

One generalisation of the Riemann zeta function is the L -function associated to a character $\chi : Z \rightarrow \mathbb{C}$ of the form $\chi(n) = e^{2\pi i n m}$, say. In particular, we can define

$$L(s, \chi) = \prod_p (1 - \chi(n)p^{-s})^{-1}$$

which converges for $\operatorname{Re}(s) > 1$. When χ is trivial then this reduces to the Riemann zeta function.

The study of L -functions in place of the Riemann zeta function leads to a generalisation of the Prime Number Theorem:

Theorem 2.10 (Dirichlet, Vallée-Poussin). *Let $1 \leq m \leq n$ be coprime to n then*

$$\#\{\text{primes } p \leq x \text{ and } m = p \pmod{n}\} \sim \frac{1}{\phi(m)} \frac{x}{\log x} \text{ as } x \rightarrow +\infty$$

where $\phi(m) = \#\{1 \leq a \leq m : (a, m) = 1\}$.

Example 2.11. *For example, if we let $m = 3$ then $\phi(3) = 2$. Thus “half the primes lie in $3\mathbb{N} + 1$ and half the primes lie in $3\mathbb{N} + 2$ ”.*

2.2.2 Dedekind Zeta functions

In number theory, the Dedekind zeta function is associated to an algebraic number field K , such as $\mathbb{Q}(\sqrt{2})$, for example. This zeta function is an infinite product over prime ideals p in O_K , the ring of algebraic integers of K . The terms in the product are $\zeta_K(s) = \prod_{p \subset O_K} (1 - (N_{K/Q}^{-s}))^{-1}$, where $N(p) = \#(O_K/p)$ for each prime ideal p . The results on the Riemann zeta function can be extended to this case and can be used to prove the analogous Prime Ideal Theorem. When $K = \mathbb{Q}$ this reduces to the Riemann zeta function.

2.2.3 Finite Field Zeta functions

In 1949, Andre Weil proposed some fundamental conjectures related to the number of solutions to a system of polynomial equations over finite fields. Consider $y^2 = x^3 - 1 \pmod{7}$, say. There are 4 solutions in F_7 ; 47 solutions in F_{7^2} ; 364 solutions in F_{7^3} ; etc. More generally, we can consider $q = p^r$ there are also zeta functions for projective algebraic varieties X over a finite field F_q defined by

$$\zeta(s) = \exp \left(\sum_{m=1}^{\infty} \frac{N_m}{m} q^{-sm} \right).$$

where N_m is the number of points of X defined over the degree m -extension F_{q^m} of F_q . This is a rational function of q^{-s} , as was proved by Dwork in 1960. Deligne proved the analogue of the Riemann Hypothesis in 1974.

3 Zeta functions for graphs

We can consider two types of finite graphs: Directed graphs and undirected graphs. If the graphs were road maps of a town then directed graph would be where all of the streets were one way, and the undirected graph would be where traffic could go either way down each street.

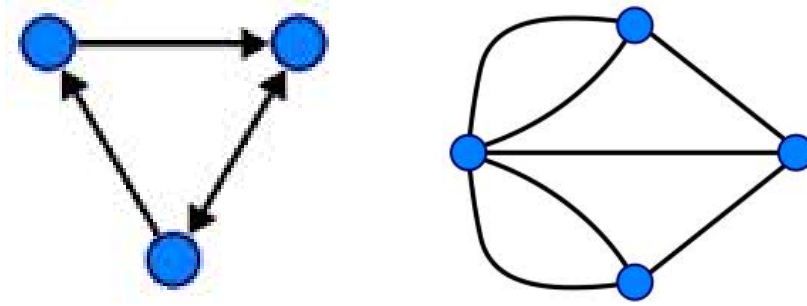


Figure 3: (i) A directed graph; and (ii) an un directed graph

We treat first the zeta functions of finite directed graphs and then the case of undirected graphs.

3.1 Directed graphs: Bowen-Lanford zeta functions

3.1.1 The zeta function for directed graphs

We denote a directed graph by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} denotes the set vertices and \mathcal{E} denotes the set of directed edges. Given an edge $e \in \mathcal{E}$ we will assume that it has a specific orientation.

We will assume for simplicity that there is an most one edge e between any two pairs of vertices v, v' , say. In particular, this allows the following simplification:

- We can uniquely identify pairs of neighbouring vertices (v, v') with ' unique directed edge $e \in \mathcal{E}$ joining them;
- Given a directed edge $e \in \mathcal{E}$ we can associate the vertices $v = e(0)$ and $v' = e(1)$ in \mathcal{V} (i.e., the starting and finishing vertices of the edge e).

We are interested in *closed paths* in \mathcal{G} which follow the orientation of the edges. In particular, under the above simplifying assumption we can either represent this as either:

1. a sequence of edges (e_1, \dots, e_n) with $e_k(1) = e_{k+1}(0) \in \mathcal{V}$, for $k = 1, \dots, n-1$, and $e_n(1) = e_1(0) \in \mathcal{V}$ or, equivalently,
2. a sequence of vertices (v_1, \dots, v_n) with $(v_k, v_{k+1}) \in \mathcal{E}$, for $k = 1, \dots, n-1$, and $(v_n, v_1) \in \mathcal{E}$.

A cyclic permutations of such words will represent the same closed path \mathcal{G} . We let n denote the length of the closed path. We will let τ denote a *prime* closed path and we denote the length of the path by $n = |\tau|$.

Definition 3.1. We can then define the Bowen-Lanford zeta function by

$$\zeta(z) = \prod_{\tau} (1 - z^{|\tau|})^{-1}$$

which converges provided $|z|$ is sufficiently small.

We can also multiply out the terms in this zeta function $\zeta(z)$ in much the same way as we did for the Riemann zeta function (cf. Lemma 2.1)

Lemma 3.2. Let $N(n)$ denote the number of all possible strings (v_1, \dots, v_n) representing a closed path in \mathcal{G} of length n . Then

$$\zeta(z) = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n} N(n) \right)$$

Proof. This is just a simple expansion of $\log(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$:

$$\begin{aligned} \log \zeta(z) &= \sum_{\tau} \log(1 - z^{|\tau|}) = \sum_{\tau} \left(-\sum_{k=1}^{\infty} \frac{z^{k|\tau|}}{k} \right) \\ &= -\sum_{k=1}^{\infty} \left(\sum_{\tau} |\tau| \left(\frac{z^{k|\tau|}}{k|\tau|} \right) \right) = -\sum_{n=1}^{\infty} \frac{z^n}{n} N(n) \end{aligned}$$

provided $|z|$ is sufficiently small to give convergence. □

We would like to show that $\zeta(z)$ has a meromorphic extension in \mathbb{C} to the entire complex plane. The most convenient way to do this is to introduce a matrix:

Definition 3.3. We can associate to the graph \mathcal{G} a $|\mathcal{V}| \times |\mathcal{V}|$ transition matrix A with

$$A(v, v') = \begin{cases} 1 & \text{if } e = (v, v') \in \mathcal{E} \\ 0 & \text{if } e = (v, v') \notin \mathcal{E} \end{cases}$$

In particular, we immediately have the following simple identity (which is equally easy to prove).

Lemma 3.4. For each $n \geq 1$, $N(n) = \text{trace}(A^n)$.

A prime closed path is simply one that it is not merely a shorter path traversed more than once
It would suffice that $|z| < \frac{1}{|\mathcal{E}|}$, say

In particular, they need not be prime, nor are they necessarily cyclically reduced

It is convenient to further assume that the graph \mathcal{G} has the property that the transition matrix is aperiodic (i.e., there exists $N > 0$ such that $A^N > 0$).

Moreover, having introduced a (non-negative) matrix and having assumed it is aperiodic it is convenient to use the classical Perron-Frobenius theorem for aperiodic matrices which describes their eigenvalues.

Lemma 3.5 (Perron-Frobenius Theorem, 1908). *Assume that the matrix A is aperiodic then:*

1. A has a simple positive eigenvalue $\lambda_1 > 0$;
2. all the other eigenvalues λ_i ($i = 2, \dots, k$) satisfy $\max_{2 \leq i \leq k} |\lambda_i| < \lambda_1$.



Figure 4: Perron (1880-1975) and Frobenius (1849 - 1917)

Since A has natural number entries and is aperiodic it is easy to see that $\lambda_1 > 0$. Lemma 3.5 leads easily to the following basic result on zeta functions for directed graphs.

Theorem 3.6 (Bowen-Lanford, 1968). *The zeta function $\zeta(z)$ is non-zero and analytic for $|z| < 1/\lambda_1$. Moreover,*

1. The zeta function $\zeta(z)$ has a simple pole at $z = 1/\lambda_1$;
2. The zeta function $\zeta(z)$ has a meromorphic extension to \mathbb{C} of the form

$$\zeta(z) = 1/\det(I - zA).$$

In particular, it is the reciprocal of a polynomial.

Proof. By Lemmas 3.4 and 3.5 we see that the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n} N(n) = \sum_{n=1}^{\infty} \frac{z^n}{n} \text{trace}(A^n) = O\left(\sum_{n=1}^{\infty} \frac{|z|^n}{n} \lambda_1^n\right)$$

This is equivalent to saying that one can get from any vertex to any other vertex along a path of precisely length N



Figure 5: Bowen (1947-1978) and Lanford (1940 -)

converges for $|z| < 1/\lambda_1$ and thus $\zeta(z)$ is non-zero and analytic on this domain. Moreover, since we can write

$$N(n) = \frac{\text{trace}(A^n)}{n} = \frac{1}{n} \left(\lambda_1^n + \sum_{i=1}^k \lambda_i^n \right)$$

and thus

$$\begin{aligned} \zeta(z) &= \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \left(\lambda_1^n + \sum_{i=1}^k \lambda_i^n \right) \right) \\ &= \exp(-\log(1 - \lambda_1 z)) \prod_{i=1}^k \exp(-\log(1 - \lambda_i z)) \\ &= (1 - \lambda_1 z)^{-1} \prod_{i=1}^k (1 - \lambda_i z)^{-1} = 1/\det(I - zA), \end{aligned}$$

providing $|z|$ is sufficiently small, from which part (3) follows. Finally, part (2) follows from this explicit expression. \square

3.1.2 The prime graph theorem (for directed graphs)

We can use the Bowen-Lanford zeta function to prove a simple analogue of the Prime Number Theorem for paths in the directed graph. Fortunately, the proof is even easier than for prime numbers.

Theorem 3.7 (Prime Graph Theorem for directed graphs).

$$\text{Card}\{\tau : |\tau| = n\} \sim \frac{\lambda_1^n}{n} \text{ as } n \rightarrow +\infty$$

Proof. In fact it suffices to show that $N(n) \sim \lambda_1^n$ as $n \rightarrow +\infty$ since we can write

$$\text{Card}\{\tau : |\tau| = n\} = \frac{N(n)}{n} + O(\lambda^{n/2})$$

since:

And although we implicitly use the zeta function we don't need it explicitly

1. we need to divide by n to allow for cyclic permutations to get from sequences of vertices to closed paths τ ; and
2. we need to throw away non-prime orbits, which number at most $N\left(\frac{n}{2}\right) + N\left(\frac{n}{3}\right) + \cdots = O\left(\lambda_1^{n/2}\right)$

However, using Lemma 3.4 again we see that

$$N(n) = \frac{\text{trace}(A^n)}{n} = \frac{\lambda_1^n}{n} + \sum_{i=2}^k \frac{\lambda_i^n}{n} = \frac{\lambda_1^n}{n} + O(\lambda_1^n)$$

□

3.1.3 A dynamical viewpoint

In these notes it will later prove convenient to take a more dynamical viewpoint. To preview this idea, let us denote by X_A the space of all infinite paths in the graph \mathcal{G} . These can be labelled by the sequence of vertices they pass through and so can be identified with:

$$X_A = \{(v_n)_{n \in \mathbb{Z}} \in \prod_{n \in \mathbb{Z}} \mathcal{V} : A(v_n, v_{n+1}) = 1, n \in \mathbb{Z}\}.$$

This is a compact metrizable space (with the Tychonoff product topology). We can then associate a homeomorphism $\sigma : X_A \rightarrow X_A$ by $\sigma(v_n) = (v_{n+1})$ by shifting paths by one place. We then see that there is a natural bijection between closed paths and the set of periodic points $\sigma^n(v_k) = (v_{k+1})$ for the map σ .

We now move onto the slightly more geometric notion of an undirected graph.

3.2 Undirected graphs: Ihara zeta function

3.2.1 The zeta function for undirected graphs

Now let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite connected non-trivial *non-directed* graph (i.e., the “usual” type of graph where we don’t associate orientations to the edges).

In the same spirit as before, we want to consider *prime* closed paths τ in \mathcal{G} . and associate to these an appropriate zeta function. However, if we consider closed paths in the context of undirected graphs then it is natural to make an additional assumption.

We adopt the convention that there is no backtracking (i.e., we don’t allow paths to take the same edge in one direction and then immediately afterwards take the same edge back).

In this context we again let τ denote a closed path which passes through $|\tau|$ edges.

By analogy with the definition of the Bowen-Lanford zeta function (cf. Definition 3.1) we define a zeta function in the present context as follows:

This is simply a sub shift of finite type

In light of this dynamical interpretation, we see that the Bowen-Lanford zeta function is actually an example of the Artin-Mazur zeta function defined in terms of periodic points for transformations

Perhaps foolishly, I will still use the same notation \mathcal{E} for the now undirected edge set

Definition 3.8. We define the Ihara zeta function $\zeta(z)$ for undirected graphs by

$$\zeta(z) = \prod_{\tau} (1 - z^{|\tau|})^{-1}$$

We want to understand the properties of $\zeta(z)$. Let us assume for the simplicity of subsequent statements that:

1. the graph \mathcal{G} has valency $q + 1$ with $q \geq 2$ (i.e., every vertex has $q + 1$ edges attached);
2. there is at most one edge between any two vertices;
3. there are no edges starting and finishing at the same vertex.

We can associate a type of adjacency matrix B to this undirected graph as follows:

Definition 3.9. More generally, without the simplifying assumptions 2 and 3 the same statements will hold providing we associate instead the $|\mathcal{V}| \times |\mathcal{V}|$ matrix B by

$$B(v, v') = \begin{cases} 1 & \text{if an edge joins } v \text{ and } v' \\ 0 & \text{otherwise} \end{cases}$$

The usefulness of the matrix B in this case is shown by the following classic result:

Theorem 3.10 (Ihara). Let \mathcal{G} be a finite connected graph of valency $q + 1$ then

$$\zeta(z)^{-1} = (1 - z^2)^{r-1} \det(I - zB + qz^2I)$$

where $r = |\mathcal{E}| - |\mathcal{V}| + 1$.

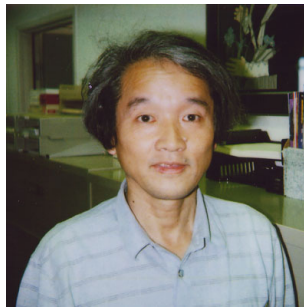


Figure 6: Ihara (1938 -) and Bass (1932 -)

We outline later a short proof of this result (due to Bass) which morally depends on the Bowen-Lanford zeta function.

Or more generally, $B(v, v') = \begin{cases} \text{Number of edges from } v \text{ to } v' & \text{if } v \neq v' \\ 2 \times \text{number of loops at } v & \text{if } v = v' \end{cases}$

Example 3.11. Consider the tetrahedron graph K_4 , with four vertices and six edges. The zeta function

$$1/\zeta(z) = (1 - z^2)^2(1 - z)(1 - 2z)(1 + z + 2z^2)^3$$

has poles at $-1, \frac{1}{2}, 1, (-1 \pm \sqrt{-7})/4$.

Some useful consequences on the location of poles of the Ihara zeta function of an undirected graph with valency $q + 1$ are the following:

Corollary 3.12. The zeta function $\zeta(z)$ is non-zero and analytic for $|z| < \frac{1}{q}$. Moreover,

1. $\zeta(z)$ has a simple pole at $z = \frac{1}{q}$; and
2. the poles of the zeta function $\zeta(z)$ lie on the union of the circle $\{z \in \mathbb{C} : |z| = 1/\sqrt{q}\}$ and the intervals $[1/q, 1] \cup [-1, -1/q]$

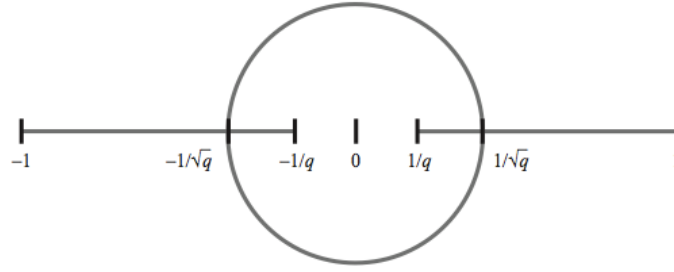


Figure 7: The poles of the Ihara zeta function $\zeta(z)$ are restricted to the region illustrated

There is an interesting extension of this to the case of zeta functions for which the valency is not constant. Assume that all the vertices have degree at most $q_1 + 1$ and at least $q_2 + 1$

Theorem 3.13 (Kotani-Sunada). Under the above hypotheses:

1. Every pole of $\zeta(z)$ lies in the annulus $\frac{1}{q_1} \leq |z| \leq 1$
2. Every pole on the real line satisfies $\frac{1}{\sqrt{q_1}} \leq |u| \leq \frac{1}{\sqrt{q_2}}$

Remark 3.14. An important class of graphs we don't have time to discuss are Ramanujan graphs, for which the eigenvalues either have modulus $q + 1$ or modulus at most $2\sqrt{q}$. This might perhaps be viewed as the analogue of the Riemann Hypothesis were we looking for one at present.

3.2.2 The prime graph theorem (for undirected graphs)

We can use the Ihara zeta function to prove a simple analogue of the Prime Number Theorem for paths in the undirected graph.

To simplify the statement, let us further assume that the lengths of closed orbits have greatest common divisor 1 (i.e., they are not all natural number multiples of some $a \geq 2$).

Theorem 3.15 (Prime Graph Theorem for undirected graphs).

$$\text{Card}\{\tau : |\tau| = n\} \sim \frac{\lambda_1^n}{n} \text{ as } n \rightarrow +\infty$$

Proof. As in Lemma 3.2, let $N(n)$ denote the number of all possible strings (v_1, \dots, v_n) representing a path in the now undirected path \mathcal{G} of length n . By analogy with Lemma 3.2 we can write the Ihara zeta function in the form:

$$\zeta(z) = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n} N(n) \right).$$

Let us now consider the logarithmic derivative

$$\frac{d}{dz} \log \zeta(z) = \sum_{n=1}^{\infty} N(n) z^{n-1}.$$

The properties that we need to extract from Theorem 3.10 are $\zeta(z)$ has a simple pole at $1/q$ and there are no other poles (or zeros) in a disk $|z| < (1+\epsilon)/q$. Writing $\zeta(z) = (z - 1/q)\phi(z)$, where $\phi(z)$ is analytic and non-zero in the disk $|z| < (1+\epsilon)/q$, then we can write

$$\frac{d}{dz} \log \zeta(z) = \frac{1}{z - 1/q} + \psi(z)$$

where $\psi(z) = \frac{\phi'(z)}{\phi(z)}$ is analytic in the disk $|z| < (1+\epsilon)/q$. Expanding $\frac{1}{z-1/q} = \sum_{n=0}^{\infty} q^{n+1} z^n$ and comparing coefficients in the power series for the two expressions for $\frac{d}{dz} \log \zeta(z)$ shows that $N(n) = q^n(1 + O((1+\epsilon)^{-n}))$. Finally, arguing as in the proof of Theorem 5.4 for directed graphs this is equivalent to the asymptotic in the statement. \square

3.3 The laplacian for undirected graphs

3.3.1 The laplacian

Instead of the adjacency matrix B we could also consider the laplacian on graphs. In particular, if we consider functions $l^2(\mathcal{V})$ on the vertex set \mathcal{V} , then we can define the Laplacian to be the linear operator $\Delta : l^2(\mathcal{V}) \rightarrow l^2(\mathcal{V})$ defined by

$$\Delta w(v) = w(v) - \frac{1}{(q+1)} \sum_{(v,v') \in \mathcal{E}} w(v').$$

Of course, since the space is finite dimensional this linear operator can also be represented by a matrix. In this case, it takes the form $I - \frac{1}{(q+1)}B$, where B is the associated adjacency matrix.

We are recycling the notation in the hope that it isn't too confusing, although we are now labelling undirected graphs

This uses the simplifying assumption to eliminate the possibility that $z = -\frac{1}{q}$ is a pole

The operator is self adjoint and thus the spectrum lies in the interval $[-1, 1]$. As for any self-adjoint operator, one can write $\Delta = \int_{-1}^1 \lambda dE(\lambda)$, where E is a spectral measure taking values in projections on $l^2(\mathcal{V})$. We can consider the spectral measure

$$\mu = \frac{1}{|\text{spec}(\Delta)|} \sum_{\lambda \in \text{spec}(\Delta)} \delta_\lambda$$

supported on the finite set of eigenvalues in the spectrum of Δ . We can then write

$$\log \zeta_{\mathcal{G}}(z) = - \left(\frac{d-2}{2} \right) \log(1-z^2) - \int_{-1}^1 \log(1-zx + (d-1)z^2) d\mu(x)$$

where μ is the spectral measure associated to \mathcal{G} .

3.3.2 Zeta functions for infinite graphs

As an application, we can consider zeta functions of infinite graphs following, for example, Grigorchuk and Zuk and thinking of them as taking limits of finite graphs.

Let \mathcal{G}_n be a finite family of graphs and let μ_n be the associated measures. In this context, there are two natural ways to approach this, which can be shown to be equivalent using the following result.

Lemma 3.16 (Serre). *The convergence of $\mu_n \rightarrow \mu_\infty$ in the weak star topology is equivalent to the formal power series*

$$\log \zeta_{\mathcal{G}_n}(z) = - \sum_{k=1}^{\infty} \frac{c_k^{(n)} z^k}{k}$$

converging termwise to a limit, i.e., $c_k = \lim_{n \rightarrow +\infty} c_k^{(n)}$ exists for each $k \geq 1$.

When convergence in either of these equivalent senses holds then we can define the zeta function for the limiting graph \mathcal{G} by:

$$\log \zeta_{\mathcal{G}}(z) := \lim_{n \rightarrow +\infty} \frac{1}{|\mathcal{V}_n|} \log \zeta_{\mathcal{G}_n}(z).$$

The uniqueness of the definition in the connected component V_0 of $\mathbb{C} - X$ follows from the uniqueness of the analytic extension. Furthermore, given the explicit form of the Ihara zeta function for finite graphs we have the following:

Theorem 3.17 (Grigorchuk-Zuk). *When the limit of $\log \zeta_{\mathcal{G}_n}(z)$ exists it takes the form*

$$\log \zeta_{\mathcal{G}_\infty}(z) := - \frac{d-2}{2} \log(1-z^2) - \int_{-1}^1 \log(1-zx + (d-1)z^2) d\mu_\infty(x)$$

where μ_∞ is the weak star limit of the associated spectral measures μ_n .

In particular, the expression for $\log \zeta_{\mathcal{G}_\infty}(z)$ converges to an analytic function on $|z| < \frac{1}{d-1}$ and has an analytic extension to V_0 . More generally, for the Cayley graph associated to an infinite group we can associate the logarithm of the zeta function $\log \zeta(z)$ by using the definition where μ is taken to be the Kesten spectral measure for the random walk.

4 Zeta functions for Diffeomorphisms

We next want to introduce the first of our dynamical zeta functions (not withstanding the dynamical interpretation of the Bowen-Lanford zeta function).

Let $f : M \rightarrow M$ be a C^∞ diffeomorphism of a compact manifold. Let us denote

$$\text{Fix}(f^n) = \{x : f^n x = x\}$$

and assume that for every $n \geq 1$ we have that $\text{Card}(\text{Fix}(f^n)) < +\infty$. This gives us some hope of making the following definition.

Definition 4.1. *We can then formally define the Artin Mazur zeta function by*

$$\zeta(z) = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \text{Fix}(f^n) \right)$$



Figure 8: Artin (1934-) and Mazur (1937-)

We begin with a class of diffeomorphisms about which we can say most, and for which the zeta functions have properties closest to those for the graphs.

4.1 Anosov diffeomorphisms

Let $f : M \rightarrow M$ be a C^∞ diffeomorphism of a compact manifold M .

Definition 4.2. *We say that f is Anosov if*

1. *there is a continuous splitting $TM = E^s \oplus E^u$ and there exist $C > 0$, $0 < \lambda < 1$ such that:*

$$(a) \|Df^n|E^s\| \leq C\lambda^n \text{ for } n \geq 0, \text{ and}$$

$$(b) \|Df^{-n}|E^u\| \leq C\lambda^n \text{ for } n \geq 0.$$

2. *$f : M \rightarrow M$ is transitive (i.e., there exists a dense orbit)*

The following result is well known for Anosov diffeomorphisms.

Lemma 4.3. *The number $N(n)$ of periodic points grows exponentially fast, i.e.,*

$$h(f) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log N(n) > 0.$$

This leads naturally to the following definition.

Definition 4.4. *The value $h(f)$ is called the topological entropy of the Anosov diffeomorphism.*

In particular, this bound on the rate of growth of the numbers $N(n)$ shows that the zeta function converges for $|z| < e^{-h(f)}$. The classical result on the extension of the Artin-Mazur zeta function for Anosov diffeomorphisms is the following:

Theorem 4.5 (Williams, Guckenheimer, Manning, 1971). *The Artin Mazur zeta function for Anosov diffeomorphisms has a meromorphic extension to \mathbb{C} . Moreover, it is a rational function, i.e., a quotient $P(z)/Q(z)$ of two polynomials $P, Q \in \mathbb{C}[z]$.*



Figure 9: Anosov (1936-), Manning (1946-) and the CAT map

The proof, which we omit, is based on modelling the diffeomorphism using the maps $\sigma : X_A \rightarrow X_A$ associated to directed graphs which arise from Markov partitions constructed for Anosov diffeomorphisms. However, for some simple examples of Anosov diffeomorphisms it is possible to compute the Artin-Mazur zeta functions directly.

Example 4.6 (The Arnol'd CAT map). *Let $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$ be an a hyperbolic linear toral automorphism associated to $A \in SL(d, \mathbb{Z})$ with no eigenvalues of modulus 1, i.e., $f(x + \mathbb{Z}^d) = Ax + \mathbb{Z}^d$. For definiteness, we can take $d = 2$ and consider $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The zeta function in this case is $\zeta(z) = \frac{(1-z)^2}{z^2-3z+1}$.*

Remark 4.7. In this particular case, where $f : \mathbb{T}^d \rightarrow \mathbb{T}^d$ preserves orientation, one can apply a useful trick and write that

$$N(n) = \sum_{k=0}^d (-1)^k \text{trace} (f_{k*}^n : H_k(\mathbb{T}^d, \mathbb{R}) \rightarrow H_k(\mathbb{T}^d, \mathbb{R}))$$

where f_{k*} is a matrix corresponding to the induced action on the k th homology group. This uses the Lefschetz fixed point theorem (where, in greater generality one sums the Lefschetz index ± 1 over fixed points, but in this case it always takes the value 1). Moreover, f_{k*} is represented by a matrix and we can deduce that $\zeta(z) = \prod_{k=0}^d \det(I - z f_{k*})^{(-1)^k}$.

4.2 Generic diffeomorphisms

The Artin Mazur zeta function was originally introduced to deal with more general diffeomorphisms. Whereas there is a very simple result for all Anosov diffeomorphisms, they form a relatively small class of amongst all diffeomorphisms. In the opposite direction, the original application of the Artin-Mazur zeta function was to show that “generically” for diffeomorphisms one expects rationality.

Given a diffeomorphism $f : M \rightarrow M$, let $N_f(n)$ denote the number of isolated fixed points for $f^n : M \rightarrow M$, for $n \geq 1$.

Theorem 4.8 (Artin-Mazur). *Let M be a compact manifold. For $k \geq 1$, there is a C^k dense subset $\mathcal{F}_k \subset C^k(M)$ such that for each $f \in \mathcal{F}_k$ we have $N_f(n) < +\infty$, for each $n \geq 1$ and, furthermore,*

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log N_f(n) < +\infty.$$

In particular, we have the following.

Corollary 4.9. *If $f \in \mathcal{F}$ then the associated zeta function $\zeta(z)$ converges on a sufficiently small disk around the origin.*

The proof of Theorem 4.8 used results of Nash on approximation of manifolds and diffeomorphisms. There is also a generalisation by Kaloshin to the effect that for each $k \geq 1$, there is a C^k dense subset $\mathcal{H}_k \subset C^k(M)$ such that $f \in \mathcal{H}_k$ has only hyperbolic periodic points.

4.3 Ruelle zeta function for Anosov diffeomorphisms

In the context of diffeomorphisms it is natural not only to count periodic points, but also to introduce weightings for the orbits that might, for example, reflect where the periodic points are on the manifold.

Let $g : M \rightarrow \mathbb{R}$ be a C^∞ function. We can consider the following generalisation of the Artin-Mazur zeta function.

Definition 4.10. *We can formally define the Ruelle zeta function for an Anosov diffeomorphism by*

$$\zeta_g(z) = \exp \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \exp \left(\sum_{k=0}^{n-1} g(T^k x) \right) \right)$$

In particular, this weights the periodic points $\{x, fx, \dots, f^{n-1}x\}$, where $f^n x = x$, by

$$\exp \left(\sum_{k=0}^{n-1} g(T^k x) \right).$$

Of course, in the particular case that $g = 0$ this weight reduces to 1 and the zeta function reduces to the classical Artin-Mazur zeta function defined earlier. Whereas for Anosov diffeomorphisms $f : M \rightarrow M$ the Artin Mazur zeta function $\zeta(z)$ converged to an analytic function on a disk of radius $e^{-h(f)}$, where $h(f)$ denotes the g, the corresponding region for the Ruelle zeta function depends on the following:

Definition 4.11. We can denote by

$$P(g) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \left(\sum_{T^n x = x} \exp \left(\sum_{i=0}^{n-1} g(T^i x) \right) \right).$$

the pressure of the function $g : M \rightarrow \mathbb{R}$.

In particular, it from Definition 4.10 that the zeta function $\zeta_g(z)$ converges to a non-zero analytic function for $|z| < e^{-P(g)}$. Furthermore, it is relatively easy to show that $e^{-P(g)}$ is a simple pole for $\zeta_g(z)$.

Unfortunately, the meromorphic extension for this zeta function is harder to establish than in the case of the Artin-Mazur zeta function. Moreover, we can no longer expect the zeta function to be a simple rational function. However, it was proved relatively recently that:

Theorem 4.12 (Liverani-Tsujii, Baladi). *If $f : M \rightarrow M$ is a C^∞ Anosov diffeomorphism then $\zeta_g(z)$ has a meromorphic extension to \mathbb{C} .*



Figure 10: Liverani (1956-), Tsujii (1964-) and Baladi (1963-)

The basic method of proof is to introduce a linear operator called a Ruelle transfer operator, which heuristically plays the role of the matrix A in the case of the simpler Bowen-Lanford zeta function for directed graphs. The difficulty is in identifying an appropriate Banach space relative to which this operator has good spectral properties, and then deriving from these the meromorphic extension.

However, the situation for Anosov flows is even more complicated. Therefore, we first want to consider the classical special case of geodesic flows on negatively curved surfaces.

5 Selberg Zeta function

Perhaps one of the easier routes into understanding dynamical zeta functions for flows is via the more geometric example of a geodesic flow on negatively curved manifolds.

5.1 Hyperbolic geometry and closed geodesics

A particularly important zeta function in both analysis and geometry is the Selberg zeta function. A comprehensive account appears in the book of Hejhal. A lighter account appears in the book of Terras.

Assume that V is a compact surface with constant curvature $\kappa = -1$. The covering space is the Poincaré disc $\mathbb{D}^2 = \{z = x + iy : |z| < 1\}$ the Poincaré metric

$$ds^2 = 4 \frac{dx^2 + dy^2}{(1 - |z|^2)^2}$$

(which again has constant curvature $\kappa = -1$). In particular, compared with the usual Euclidean distance, the distance in the Poincaré metric tends to infinity as $|z|$ tends to 1, to the extent that geodesics never actually reach the boundary.

We begin with an elementary result

Lemma 5.1. *There are a countable infinity of closed geodesics.*

Proof. There is a one-one correspondence between closed geodesics and conjugacy classes in the fundamental group $\pi_1(V)$. The group $\pi_1(V)$ is finitely generated and countable, as are its conjugacy classes. \square

We can denote by γ one of the countably many closed geodesics on V . We can then write $l(\gamma)$ for the length of γ .

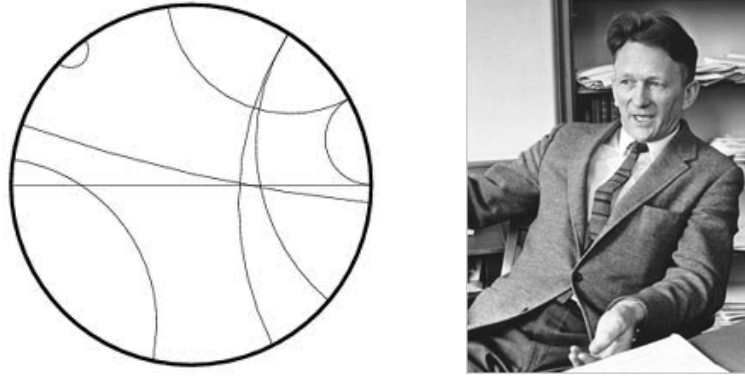


Figure 11: The Poincaré disk with some geodesics, and Selberg

5.2 The definition of the Selberg zeta function

We now come to the definition of the Selberg zeta function for closed geodesics on the compact surface V .

Definition 5.2. *The Selberg zeta function is a function of $s \in \mathbb{C}$ defined by*

$$Z(s) = \prod_{n=0}^{\infty} \prod_{\gamma} (1 - e^{-(s+n)l(\gamma)}),$$

which converges for $\operatorname{Re}(s) > 1$.

The fact that we have a zero at $s = 1$, rather than pole, is simply an artefact of the way the Selberg zeta function is defined. Rather than defy convention, we prefer to use this definition. However, when we (eventually) define the Ruelle zeta function for flows we will use a convention closer to that of the Riemann zeta function.

One explanation for the extra product over n is that it simplifies the statement on the location of the zeros for $Z(s)$. The main result on the Selberg zeta function is the following:

Theorem 5.3 (Selberg, 1956). *The zeta function $Z(s)$ converges to a non-zero analytic function for $\operatorname{Re}(s)$. Moreover,*

1. $Z(s)$ has a simple zero at $s = 1$;
2. $Z(s)$ has no further zeros on $\operatorname{Re}(s) = 1$;
3. $Z(s)$ has an analytic extension to \mathbb{C} .

The original proof used the Selberg trace formulae, which we will briefly describe later.

As an immediate application of Theorem 5.3 (in particular parts 1 and 2) one could proceed by complete analogy with the proof of the Prime Number Theorem to deduce the following:

Theorem 5.4 (Prime geodesic theorem).

$$\operatorname{Card}\{\gamma : l(\gamma) \leq T\} \sim \frac{e^T}{T} \text{ as } T \rightarrow +\infty$$

Here the lengths appear in the exponent, which is why the principle term takes the form e^T/T , compared with $x/\log T$ for the Prime Number Theorem (Theorem 2.4). Moreover, that the exponent is 1 is a consequence of the choice of the curvature being -1 . More generally, if the curvature were $-|\kappa| < 0$ then the asymptotic would be of the form $e^{\sqrt{|\kappa|}T}/(\sqrt{|\kappa|}T)$.

5.3 The Laplacian and the “Riemann Hypothesis”

The Selberg zeta function $Z(s)$ has a distinct advantage over the Riemann zeta function $\zeta(s)$ that one can show a version of the Riemann Hypothesis for $Z(s)$ and get stronger versions of the Prime geodesic theorem. This stems from a characterisation of the zeros of $Z(s)$ in terms of the spectrum of the Laplacian.

Definition 5.5. *Let $\Delta : L^2(V) \rightarrow L^2(V)$ be the Laplacian (i.e., the operator is actually defined on the $C^\infty(M)$ functions and extends to the square integrable functions in which they are dense).*

The Laplacian is defined locally, so we can more consider the Poincaré Disc (i.e., its Universal Cover with the associated lifted metric) where the corresponding Laplacian takes the explicit form

$$\Delta_{\mathbb{D}} = \frac{1}{4}(1 - |z|^2) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

where $z = x + iy \in \mathbb{D}$. The following result is standard:

Lemma 5.6. *The laplacian is a self-adjoint operator. In particular, its spectrum lies on the real line.*

We are interested in the solutions

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

for the eigenvalue equation $\Delta\phi_n + \lambda_n\phi_n$. The location of the zeros s_n for $Z(s)$ are described in terms of the eigenvalues λ_n for the Laplacian:

Theorem 5.7 (Selberg). *The zeros of the Selberg zeta function $Z(s)$ can be described by:*

1. $s = 1$ is a simple zero;
2. $s_n = \frac{1}{2} \pm i\sqrt{\frac{1}{4} - \lambda_n}$, for $n \geq 1$, are “spectral zeros”; and
3. $s = -m$, for $m = 0, 1, 2, \dots$, are “trivial zeros”.

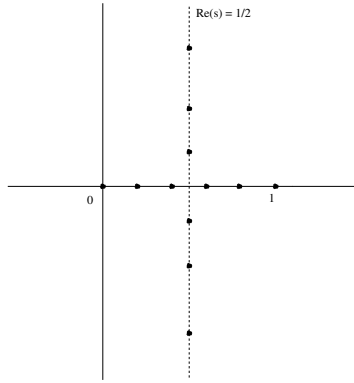


Figure 12: Zeros of the Selberg zeta function $Z(s)$

In particular, we have a variant of the “Riemann Hypothesis” for $Z(s)$.

Corollary 5.8. *$Z(s)$ has only finitely many zeros in the half-plane $\text{Re}(s) > \frac{1}{2}$ (which lie on the real axis).*

Thus we have the corresponding improvement to Prime Geodesic Theorem 5.4.

Corollary 5.9. *There exists $\epsilon > 0$ such that*

$$\text{Card}\{\gamma : l(\gamma) \leq T\} = \int_2^{e^T} \frac{du}{\log u} + O(e^{(1-\epsilon)T}) \text{ as } T \rightarrow +\infty$$

5.4 Pairs of pants: an example of analytic extension for $Z(s)$

Having extolled the (very real) virtues of the spectral approach for compact surfaces of constant negative curvature, we will now consider the case of a non-compact infinite area surface for it is more appropriate to use a dynamical, rather than a spectral approach.

A pair of pants V (or three funnelled surface) is a classical example of a surface of constant curvature $\kappa = -1$, but unlike the previous setting has three infinite area funnels. Equivalently, we

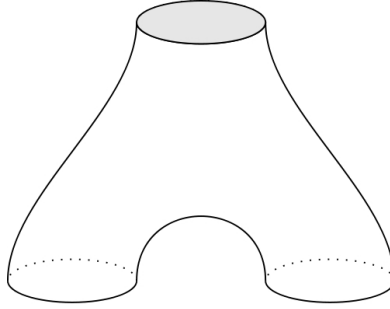


Figure 13: A pair of pants: A surface of constant curvature, but with three geodesic boundary components

can consider this as a sphere with three geodesic boundary components by choosing the closed geodesics around each of the funnels.

In this context we have the following result:

Theorem 5.10. *There exists $0 < \delta < 1$ such that $Z(s)$ converges to a non-zero analytic function for $\operatorname{Re}(s) > \delta$. Moreover,*

1. $Z(s)$ has a simple zero at $s = \delta$; and
2. $Z(s)$ has an analytic extension to \mathbb{C} .

Unlike the case of compact surfaces, the location of the other zeros in $\operatorname{Re}(s) < \delta$ is very mysterious, and we will return to this problem later.

We can give a sketch of a proof of Theorem 5.10 (particularly of the second part) which uses a transfer operator approach. This can vaguely be thought of as using Ruelle transfer operators as a natural generalisation of the matrix in the case of graph zeta functions.

5.4.1 Maps on the boundary

We begin by associating to the surface an expanding map on four arcs on the unit circle.

1. **Möbius maps:** We can associate to the pair of pants two Möbius maps $a, b : \mathbb{D} \rightarrow \mathbb{D}$ which generate the covering transformations for V and preserve that unit disk \mathbb{D} (and are isometric with respect to the Poincaré metric) and extend to the boundary $S^1 = \partial\mathbb{D}$. In particular, we can write $V = \mathbb{D}^2/\Gamma$. We can then associate to each element of $\Gamma_0 = \{a, b, a^{-1}, b^{-1}\}$ their isometric circles (i.e., for any such Möbius map $g \in \Gamma_0$ we write $C(g) = \{z \in \overline{\mathbb{D}} : |g'(z)| = 1\}$).
2. **Isometric circles:** The four arcs $C(a), C(b), C(a^{-1}), C(b^{-1}) \subset \overline{\mathbb{D}}$ are disjoint and each meets the boundary $\partial\mathbb{D}$ orthogonally. Moreover, observe that if $g \in \Gamma_0$ then $gC(g) = C(g^{-1})$ since $(g^{-1}g)'(z) = (g^{-1})'(gz)g'(z) = 1$ and thus if $|g'(z)| = 1$ then $|(g^{-1})'(gz)| =$

Which tend to infinite at a rate given by Weyl's law

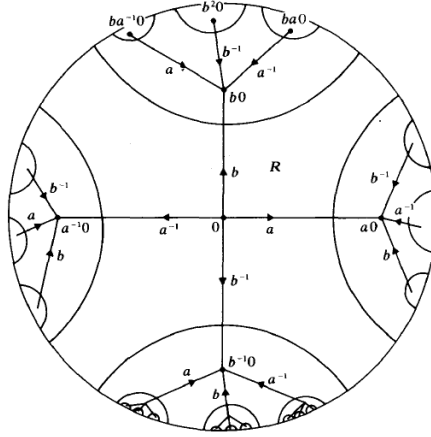


Figure 14: The Möbius maps associated to a Pair of Pants

1. In particular, we have that

$$\begin{aligned} aC(a) &= C(a^{-1}) \\ bC(b) &= C(b^{-1}) \\ a^{-1}C(a^{-1}) &= C(a) \\ b^{-1}C(b^{-1}) &= C(b) \end{aligned}$$

3. **The transformations:** For each $g \in \Gamma_0$, we denote by $I(g)$ the natural arc of ∂D which has the two endpoints $\partial D \cap C(g)$. The arcs $I(a), I(b), I(a^{-1}), I(b^{-1})$ are disjoint. We then have well defined maps

$$\begin{aligned} T_a : I(a^{-1}) \cup I(b) \cup I(b^{-1}) &\rightarrow I(a^{-1}) \text{ defined by } T_a(z) = az \\ T_b : I(a) \cup I(a^{-1}) \cup I(b^{-1}) &\rightarrow I(b^{-1}) \text{ defined by } T_b(z) = bz \\ T_{a^{-1}} : I(a) \cup I(b) \cup I(b^{-1}) &\rightarrow I(a) \text{ defined by } T_{a^{-1}}(z) = a^{-1}z \\ T_{b^{-1}} : I(a) \cup I(b^{-1}) \cup I(a^{-1}) &\rightarrow I(b^{-1}) \text{ defined by } T_{b^{-1}}(z) = b^{-1}z \end{aligned}$$

These maps can be assumed to be contracting.

The first advantage of the above coding is that we can write the lengths of closed geodesics in terms of the derivatives of compositions of the maps at their unique fixed points. More precisely, given any cyclically reduced string of generators $g = g_1 \cdots g_n \in \langle a, b \rangle$ we can associate a contacting map

$$T_g := T_{g_1} \circ \cdots \circ T_{g_n} : I(g_1^{-1}) \rightarrow I(g_1^{-1}).$$

Let $x(g)$ be the unique fixed point then we can see by a simple calculation that:

Lemma 5.11. *The length of the corresponding closed geodesic is $-\log |T'_g(x(g))|$.*

This construction is described on pages 337-338 of MR0873435, Series, Caroline, Geometrical Markov coding of geodesics on surfaces of constant negative curvature. Ergodic Theory Dynam. Systems 6 (1986), no. 4, 601-625.

For example, by considering the axis of the hyperbolic translation corresponding to the geodesic, say

5.4.2 The Ruelle transfer operator

We next want to introduce a classical Ruelle transfer operator. Let \mathcal{B} be the Banach space of bounded analytic functions on a small complex neighbourhood U , say, the union of the four arcs.

Definition 5.12. *We want to define a family of operators $\mathcal{L}_s : \mathcal{B} \rightarrow \mathcal{B}$, where $s \in \mathbb{C}$, by*

$$\mathcal{L}_s w(z) = \sum_{g \neq g_0} e^{-s \log |T'_g(z)|} w(T_g z) \text{ for } z \in I(g_0)$$

where the sum is over three of the four elements $g \in \Gamma_0$ where we exclude the one g_0 for which z lies in the arc $I(g_0)$. This makes the operator well defined.

The operator is well defined providing the neighbourhood U is sufficiently small. The key feature of these operators on this Banach space is the following strong spectral properties.

Lemma 5.13 (Grothendieck, 1955). *The operator \mathcal{L}_s , (and its powers \mathcal{L}_s^n , $n \geq 1$) are trace class (i.e., there are only countably many non-zero eigenvalues and their sum is well defined).*

Perhaps one might think of the operators \mathcal{L}_s as somehow replacing the matrices we had before for graphs. We can easily check by explicit calculations the following:

Lemma 5.14 (Ruelle, 1976). *We have that*

$$\text{trace}(\mathcal{L}_s^n) = \sum_{g=g_1 \cdots g_n} \frac{e^{-s(\log |T'_g(x(g))|)}}{1 - |T'_g(x(g))|}$$

and then we can write

$$Z(s) = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \text{trace}(\mathcal{L}_s^n) \right).$$

This provides the analytic extension. The other results follow from the properties of the eigenvalues of \mathcal{L}_s .

Remark 5.15. If one thinks of trace class operators as being natural generalisations of matrices A (i.e., finite dimensional operators) then the equation for $Z(s)$ above is rather reminiscent of the easy matrix identity

$$\frac{1}{\det(I - A)} = \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \text{trace}(A^n) \right).$$

5.4.3 The zeros for $Z(s)$ for a pair of pants.

In contrast to the case of compact surfaces (where the spectral theory of the Laplacian is available) much less is known about the location of the zeros of $Z(s)$ in this case than in the compact surface. Indeed experimental evidence shows that the situation is very different.

There are a small number of interesting results on the locations of the zeros.

A very nice reference for this in the simpler case of continued fractions is MR0418168, Mayer, Dieter H., On a zeta function related to the continued fraction transformation. Bull. Soc. Math. France 104 (1976), no. 2, 195-203.

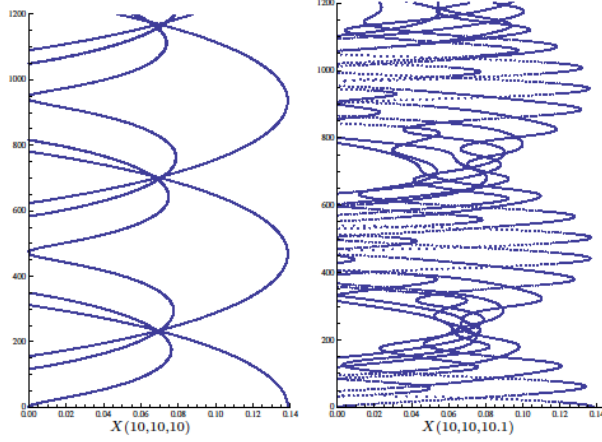


Figure 15: Brothwick's experimental plot of the location of zeros for $Z(s)$ for two different pairs of pants.

1. For any $\sigma < \delta$, we have

$$\text{Card}\{\text{zeros } z : T \leq |Im(z)| \leq T + 1, \sigma \leq Re(z) \leq \delta\} = O(T^\delta)$$

(Guillope-Lin-Zworski).

2. There is a zero free region $\{s \in \mathbb{C} : Re(s) > \delta - \epsilon\} - \{\delta\}$, for some $\epsilon > 0$ (Naud)

Moreover, it is conjectured that there are only finitely many zeros in any half-plane $\frac{\delta}{2} + \epsilon < Re(s)$.

Related estimates appear in work of Bourgain-Gamburd-Sarnak and Bourgain-Kontorovich.

Remark 5.16. This application of transfer operators to $Z(s)$ is illustrative of how useful they can be in situations where they can be applied to get accurate numerical estimates of values. Similar methods have been applied to computing the Hausdorff Dimension of hyperbolic Julia sets and the limit sets of Schottky groups.

6 Ruelle zeta function for Anosov flows

We now come to the more general setting of the Ruelle zeta function for Anosov flows. This includes, as a special case, the zeta function for geodesic flows on manifolds with negative sectional curvatures.

6.1 Definitions

We recall the definition of an Anosov flow on a compact manifold M .

Definition 6.1. We say that a C^∞ flow $\phi_t : M \rightarrow M$ is Anosov if

1. there is a $D\phi_t$ -invariant splitting $TM = E^0 \oplus E^s \oplus E^u$ such that

- (a) E^0 is one dimensional and tangent to the flow direction;
(b) $\exists C, \lambda > 0$ such that $\|D\phi_t|E^s\| \leq Ce^{-\lambda t}$ and $\|D\phi_{-t}|E^u\| \leq Ce^{-\lambda t}$ for $t > 0$,
2. The flow is transitive (i.e., there exists a dense orbit).

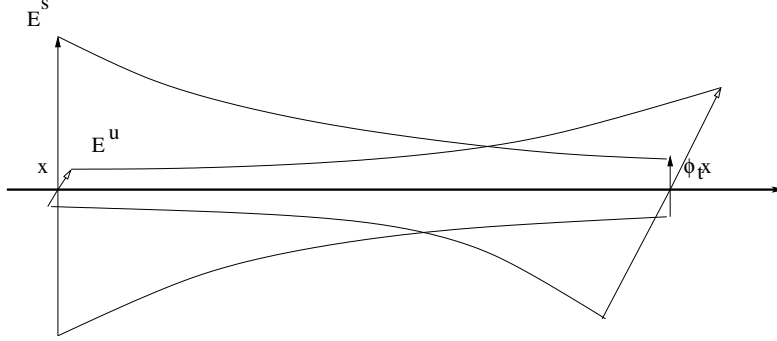


Figure 16: The hyperbolicity transverse to the orbit of an Anosov flow

The main example is the geodesic flow on a surface of (variable) negative curvature.

Example 6.2 (Geodesic flow). *Let V be a compact surface with curvature $\kappa(x) < 0$, for $x \in V$. Let $M = SV = \{(x, v) \in TV : \|v\|_x = 1\}$ denote the unit tangent bundle then we let $\phi_t : M \rightarrow M$ denote the geodesic flow, i.e., $\phi_t(v) = \dot{\gamma}(t)$ where $\gamma : \mathbb{R} \rightarrow V$ is the unit speed geodesic with $\dot{\gamma}(0) = (x, v)$. Moreover, the closed orbits for the geodesic flow correspond to closed geodesics on V .*

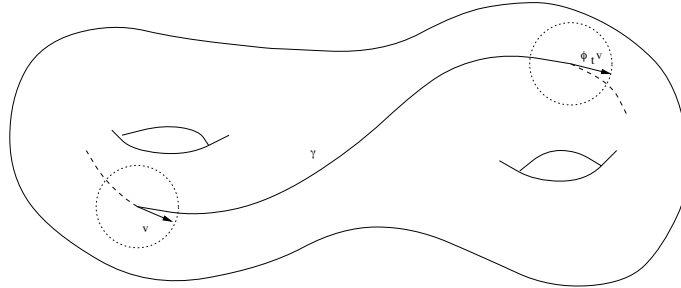


Figure 17: The geodesic flow on a negatively curved surface is an example of an Anosov flow

In particular, the study of Anosov flows therefore includes the extension of geodesic flows from constant to variable negative curvature.

Definition 6.3. *We say that τ is a closed orbit of least period $\lambda(\tau)$ if there exists $x \in \tau$ with $\phi_{\lambda(\tau)}x = x$, with $\lambda(\tau) > 0$ the least such value.*

We can denote by $N(T) = \text{Card}\{\tau : \lambda(\tau) \leq T\}$ the number of periodic orbits τ of (prime) period $\lambda(\tau) \leq T$. The following result is the continuous analogue of Lemma 4.3

Lemma 6.4 (Sinai, 1966). *The number $N(T)$ grows exponentially fast, i.e.,*

$$h(\phi) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log N(T) > 0$$

This leads naturally to the following definition.

Definition 6.5. *The value $h(\phi)$ is called the topological entropy of the Anosov flow.*

By analogy with the product form of the zeta function for diffeomorphisms we define the zeta function for an Anosov flow as follows:

Definition 6.6. *We define the Ruelle zeta function for an Anosov flow by*

$$\zeta(s) = \prod_{\tau} (1 - e^{-s\lambda(\tau)})^{-1}.$$

It is easy to see from Lemma 6.4 that the zeta function $\zeta(s)$ converges on the half-plane $\operatorname{Re}(s) > h(\phi)$.

Remark 6.7. The definition can be viewed as the natural analogue of Definition 2.1 of the Riemann zeta function. In that definition we can replace the primes by closed orbits τ and the value p by $e^{\lambda(\tau)}$ and then this formally gives the expression for the zeta function of Ruelle.

Remark 6.8. The “extra” product over n in the definition of the Selberg zeta function is essentially a convenience in presenting results which are specific to that setting. However, it can easily be related to the definition of the Ruelle zeta function. Formally, we can write

$$\zeta(s) = \frac{Z(s+1)}{Z(s)} \text{ and } Z(s) = \prod_{n=0}^{\infty} \zeta(s+n)^{-1}.$$

6.2 Basic properties of the Ruelle zeta function

There are old results which are analogues of the first properties of the Riemann zeta function.

Theorem 6.9 (Ruelle, 1978; Parry-Pollicott, 1983). *The Ruelle zeta function is non-zero and analytic on $\operatorname{Re}(s) > h(\phi)$. Moreover,*

Ruelle’s result was that $h(\phi)$ is a simple pole, a result that appears as an exercise in his book *Thermodynamic Formalism* which Parry and I couldn’t solve. The following recollection appears in David Ruelle’s article, “Dynamical Zeta Functions and Transfer Operators”, from the Notices Amer. Math. Soc. (September 2002): “Having obtained the above nontrivial but apparently useless result, I put it as Exercise 7(c) on page 101 in my book *Thermodynamic Formalism*. A few years later (December 29, 1982) Bill Parry of Warwick wrote to me about very interesting results on Axiom A flows he had obtained with his student Mark Pollicott. These results used Exercise 7(c), which unfortunately he had been unable to do. Could I help? By the time I had (painfully) managed to reconstruct the solution of the exercise I received another letter: 13 Jan 83

Dear David, We’ve finally managed to do your exercise! So ignore my last letter.

Sincerely, Bill Parry.”

What Ruelle didn’t know is that our solution to the exercise was incorrect, and so we used his answer after all

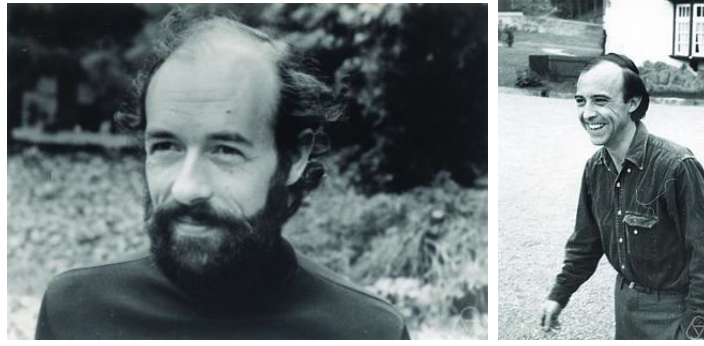


Figure 18: Ruelle (1935-) and Parry (1934-2006)

1. $\zeta(s)$ has a simple pole at $s = h(\phi)$; and
2. $\zeta(s)$ has an analytic extension to a neighbourhood of $\{s \in \mathbb{C} : \operatorname{Re}(s) = h(\phi)\} - \{h(\phi)\}$ (when the flow is weak-mixing, e.g., geodesic flows)

However, as in the case of the derivation of the Prime Geodesic Theorem (Corollary 5.4) and the proof of the Prime number theorem we have a corresponding Prime Orbit Theorem. For convenience we make the following assumption: *The periods of closed orbits are not all natural number multiples of a single constant, i.e., there is no $a > 0$ such that $\{\lambda(\tau)\} \subset a\mathbb{N}$.* This is a generic condition called *topological weak mixing* and holds, for example, for any geodesic flow on a negatively curved manifold.

Theorem 6.10 (Prime orbit theorem). *For any topologically weak mixing Anosov flow*

$$\operatorname{Card}\{\tau : \lambda(\tau) \leq T\} \sim \frac{e^{h(\phi)T}}{h(\phi)T} \text{ as } T \rightarrow +\infty$$

One can develop the approach to proving Theorem 6.9 little to get the following slightly stronger result:

Theorem 6.11 (Pollicott, 1986). *There exists $\epsilon > 0$ such that the zeta function $\zeta(s)$ has a meromorphic extension to a slightly large half-plane $\operatorname{Re}(s) > 1 - \epsilon$*

However, this is as far as this classical method of “symbolic dynamics” could get us and therefore a new idea was needed.

The basic steps were the following:

1. Choose codimension-one Markov sections transverse to the flow (with associated flow boxes);
2. The flow boxes are foliated by stable/contracting manifolds;
3. The flow boxes can be collapsed along the stable/contracting directions; and finally
4. The Poincaré map reduces to an expanding map.

The advantage of the expanding map is that there are many preimages, which can be used to approximate the periodic orbits; and the dual (transfer) operator to the expanding map has good spectral properties (e.g., a spectral gap). However, the regularity of the stable foliations determines the regularity of the function space, and thus the size of the essential spectrum of the operator.

6.3 Further extensions

Using more modern techniques one can get a full extension to all of \mathbb{C} .

Theorem 6.12 (Giulietti-Liverani-Pollicott, 2012; Dyatlov-Zworski, 2013). *Let $\phi_t : M \rightarrow M$ be a C^∞ Anosov flow. The zeta function $\zeta(s)$ has a meromorphic extension to \mathbb{C} .*

Similar results were previously known under stronger hypotheses:

1. If the Anosov flow $\phi_t : M \rightarrow M$ is C^ω and has stable and unstable foliations which are C^ω then $\zeta(s)$ has a meromorphic extension to \mathbb{C} (Ruelle, 1976); and stronger still
2. If the Anosov flow $\phi_t : M \rightarrow M$ is C^ω then $\zeta(s)$ has a meromorphic extension to \mathbb{C} (Rugh, Fried 1996).

Thus our main result is really to go from C^ω to C^∞ .

We briefly recall the strategy of the poof of Theorem 6.12.

Step 1. Motivated by work of Gouezel-Liverani (and Baladi-Tsujii) for Anosov diffeomorphisms: *We define simple operators on complicated Banach spaces.* Consider the one parameter family of operators $\mathcal{L}_t : C^0(M) \rightarrow C^0(M)$ defined by

$$\mathcal{L}_t w(x) = w(\phi_t x) \text{ for } t > 0.$$

Step 2. To eliminate the effect of the flow direction we want to “integrate away” the flow direction: For each $s \in \mathbb{C}$ let $R(s) : C^0(M) \rightarrow C^0(M)$, be defined by

$$R(s)w(x) = \int_0^\infty e^{-st} \mathcal{L}_t w(x) dt \text{ where } w \in C^0(M).$$

for $\text{Re}(s) > 0$, say. (This corresponds to the resolvent of the generator for the Anosov flow).

Step 3. We need to replace $C^0(M)$ by “better spaces” \mathbb{B} of special distributions, i.e., one for which there is less spectrum (i.e., *complicated* Banach spaces).

Theorem 6.13. *For each $k \geq 1$ we can choose \mathbb{B}_k appropriately so that that $R(s) : \mathcal{B}_k \rightarrow \mathcal{B}_k$ has only isolated eigenvalues for $\text{Re}(s) > -k$.*

Step 4. We want to use the operator \mathcal{L}_t to extend $\zeta(s)$. We need to make sense of

$$\text{“det}(I - R(s)) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \text{tr}(R(s)^n) \right) \text{”}$$

for operators $R(s)$ which are not trace class (using suitable approximations).

Step 5. Finally, we need to relate $\zeta(s)$ to “ $\text{det}(I - R(s))$ ”, which in fact involves having to repeat the previous steps all over again for spaces of differential forms in place of functions, in order to fix up the identities.

In fact the proof also works well when the flow is only finitely differentiable. If we consider a C^k Anosov flow $\phi_t : M \rightarrow M$ ($1 \leq k < +\infty$) then we get the following extension.

The paper of Giulietti-Liverani-Pollicott was written between 2007-2012. The paper of Dyatlov-Zworski appeared as a preprint 5 days before these lectures began and uses semi-classical analysis to construct the anisotropic spaces used in both proofs

Theorem 6.14 (Giulietti-Liverani-Pollicott). *Let $\lambda > 0$ be the value in the definition of the Anosov flow then we have that $\zeta(s)$ has a meromorphic extension to $\operatorname{Re}(s) > h(\phi) - \lambda[\frac{k}{2}]$.*

In particular, a consequence of the main theorem is the following generalization of Selberg's theorem (originally proved using the Selberg trace formula approach).

Corollary 6.15 (of the Main Theorem). *For compact manifolds with sectional curvatures having variable negative curvature, $\zeta(s)$ extends meromorphically to the entire complex plane \mathbb{C} .*

Consider the special case of a compact surface of *constant negative curvature* $\kappa = -1$. We then recover the following:

Corollary 6.16 (Selberg, 1956; Ruelle, 1976). *For surfaces of constant negative curvature, $\zeta(s)$ extends meromorphically to the entire complex plane \mathbb{C} .*

6.4 The Riemann hypothesis for the Ruelle zeta function

6.4.1 The case for surfaces

We recall the “Riemann hypothesis” for geodesic flows on negatively curved surfaces.

Theorem 6.17 (Dolgopyat, 1998). *Let V be a compact negatively curved surface V . There exists $\epsilon > 0$ such that $\zeta(s)$ has an analytic zero-free extension to $\operatorname{Re}(s) > h(\phi) - \epsilon$, except for a simple pole at $s = h$.*

This has the following consequences for the number $N(T)$ of closed orbits τ for the geodesic flow with $\lambda(\tau) \leq T$.

Corollary 6.18. *Under the above hypotheses we can estimate*

$$N(T) = \int_2^{e^{h(\phi)T}} \frac{1}{\log u} du + O(e^{(h(\phi)-\epsilon)T})$$

6.4.2 The case for surfaces

Theorem 6.17 has the following partial generalization to higher dimensional manifolds V .

Theorem 6.19 (Giulietti, Liverani and P., 2012). *Let V be a compact manifold for which the (negative) sectional curvatures are $\frac{1}{9}$ -pinched. There exists $\epsilon > 0$ such that $\zeta(s)$ has an analytic zero-free extension to $\operatorname{Re}(s) > h - \epsilon$, except for a simple pole at $s = h$.*

The result also holds at the level of contact Anosov flows which are $\frac{1}{3}$ -bunched.

By analogy with what we have seen before several times, Theorem 6.19 has immediate consequences for improving the Prime Orbit Theorem:

Corollary 6.20. *Under the above hypotheses we can estimate that the number $N(T)$ of closed orbits τ for the geodesic flow with $\lambda(\tau) \leq T$ by*

$$N(T) = \int_2^{e^{hT}} \frac{1}{\log u} du + O(e^{(h-\epsilon)T})$$

This can be viewed as a generalisation of previous results:

1. This generalizes the theorem of Selberg and Huber from 1956-1959 (from the geodesic flow on manifolds V of *constant* negative curvature(s) to that of variable negative curvature(s)); and
2. This partly generalizes the theorem of Margulis from 1969 (for the geodesic flow on manifolds V of variable negative curvature(s), but with no error estimate).

We do not know if Corollary 6.20 remains for geodesic flows without the pinching condition (or even for any weak-mixing Anosov flows).

6.5 *L*-functions for closed geodesics

6.5.1 Closed geodesics null in homology

Following Katsuda and Sunada, we can associate to a closed geodesic γ on a compact manifold with negative sectional curvatures its homology class $[\gamma] \in H_1(V, \mathbb{Z})$. Let $\chi : H_1(V, \mathbb{Z}) \rightarrow \mathbb{C}$ be a character. We define an *L-function* by

$$L(s, \chi) = \prod_{\tau} (1 - \chi([\tau]) e^{-s\lambda(\tau)})^{-1}.$$

The analysis of this *L*-function allows one to get analogues of Dirichlet's theorem homotopy classes. Let $N(T)$ denote the number of closed orbits τ for the geodesic flow on a negatively curved manifold for which the least period satisfies $\lambda(\tau) \leq T$ and which are null in homology. Let b be the first Betti number.

Theorem 6.21 (Katsuda-Sunada). *There exists $C > 0$ such that*

$$N(T) \sim C \frac{e^{h(\phi)T}}{T^{b/2+1}}$$

Remark 6.22. It would be natural to expect that $L(s, \chi)$ has a meromorphic extension to \mathbb{C} and that $L(0, \chi)$ is related to the torsion.

6.5.2 Closed orbits and Knots

Finally, we mention a recent application of McMullen of equidistribution results for homology. We can consider a sequence K_1, K_2, \dots of smooth knots on a compact manifold M . Let $L_n = \cup_{i=1}^n K_i$ and let G be a finite group. A surjective homeomorphism $\rho : \pi_1(M - L_n) \rightarrow G$ determines a covering $\widetilde{M} \rightarrow M$ with Galois group G . The remaining knots give a sequence of conjugacy classes $[K_i] \subset G$.

Following Mazur, we say that that (K_i) obey a *Chebotarov law* if for any conjugacy class $C \subset G$ we have that

$$\lim_{N \rightarrow +\infty} \frac{|\{n \leq i \leq N : [K_i] = C\}|}{N} = \frac{|C|}{|G|}$$

McMullen showed the following:

Theorem 6.23 (McMullen). *Let X be a closed surface of constant curvature and let $K_1, K_2, \dots \subset M = T_1X$ be the closed orbits of the geodesic flow ordered by length. Then (K_i) obeys a Chebotarov law.*

Theorem 6.24 (McMullen). *Let $K_1, K_2, \dots \subset M$ be the closed orbits of any topologically mixing pseudo-Anosov flow on a closed 3-manifold. Then (K_i) obeys a Chebotarov law.*

7 Appendices

7.1 Proof of Theorem 3.10

We now recall the simple proof of Bass of the theorem of Ihara. Given an undirected graph with adjacency matrix A we can use the edge presentation of the closed paths to write

$$\zeta(z) = \prod_{\tau} (1 - z^{|\tau|})$$

where τ is a closed non-back tracking curve which we can identify with a sequence of edge lengths. We can associate an (oriented edge) adjacency matrix W by taking each unoriented edge e and associating to it two oriented edges denoted by e, \bar{e} , say. We can then write $\zeta(z) = \det(I - zW)^{-1}$. We can next define a new $2|\mathcal{E}| \times 2|\mathcal{E}|$ matrix C by

$$C(e, e') = \begin{cases} 0 & \text{if } e'(1) \neq e(0) \\ 1 & \text{if } e'(1) = e(0) \text{ and } e' \neq \bar{e} \end{cases}$$

By analogy with the proof of the Bowen-Lanford zeta function we see that $\zeta(z) = 1/\det(I - C)$.

Let $n = |\mathcal{V}|$ be the number of vertices and let $m = |\mathcal{E}|$ be the number of unoriented edges. We now have some simple linear algebra:

1. We can define

$$J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$$

where I_m is the $m \times m$ identity matrix;

2. Let S be a $n \times 2m$ matrix with

$$S(v, e) = \begin{cases} 1 & \text{if } e(0) = v \\ 0 & \text{otherwise} \end{cases}$$

3. Let T be a $n \times 2m$ matrix with

$$T(v, e) = \begin{cases} 1 & \text{if } e(1) = v \\ 0 & \text{otherwise} \end{cases}$$

It is a routine calculation to check that:

Following the account in the book of Terras

1. $SJ = T$ and $TJ = S$
2. $A = ST^T$ and $(q+1)I_n = SS^T = TT^T$
3. $W + J = T^T S$.

In particular, we can now check the matrix identity:

$$\begin{pmatrix} I_n & 0 \\ T^T & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1-z^2) & Sz \\ 0 & I_{2m} - Wu \end{pmatrix} = \begin{pmatrix} I_n - Az + qIz^2 & Sz \\ 0 & I_{2m} + Jz \end{pmatrix} \begin{pmatrix} I_n & 0 \\ T^T - S^T z & I_{2m} \end{pmatrix}$$

We next take determinants and write

$$(1-z^2)^n \det(I - zW) = \det(I_n - Az + qIz^2) \det(I_{2m} + Jz)$$

Finally, we only need to observe that since

$$I_{2m} + Jz = \begin{pmatrix} I_m & I_m z \\ I_m z & I_m \end{pmatrix}$$

we have that

$$\begin{pmatrix} I_m & 0 \\ -I_m z & I_m \end{pmatrix} (I + Jz) = \begin{pmatrix} I_m & I_m z \\ 0 & I_m(1-z^2) \end{pmatrix}$$

and taking determinants gives $\det(I_{2m} + Jz) = (1-z^2)^m$. Furthermore, $r-1 = m-n$ and the result follows.

7.2 Selberg trace formulae

7.2.1 The Heat kernel

We define the Heat Kernel on V by

$$K(x, y, t) = \sum_n \phi_n(x) \phi_n(y) e^{-\lambda_n t}.$$

(This has a nice interpretation as the probability of a Brownian Motion path going from x to y in time $t > 0$.) In particular we can define the trace:

$$\text{tr}(K(\cdot, \cdot, t)) := \int_M K(x, x, t) d\nu(x) = \sum_n e^{-\lambda_n t}.$$

7.2.2 Homogeneity and the trace

Let us now choose the covering space to be the upper half plane $\mathbb{H}^2 = \{z = x + iy : y > 0\}$ with the Poincaré metric

$$\frac{dx^2 + dy^2}{y^2}$$

We can consider the Poincaré distance between $\bar{x}, \bar{y} \in \mathbb{H}^2$ given by

$$d(\bar{x}, \bar{y}) = \cosh^{-1} \left(1 + \frac{|\bar{x} - \bar{y}|^2}{2\text{Im}(\bar{x})\text{Im}(\bar{y})} \right).$$

The heat kernel is homogeneous, i.e., the heat kernel $\tilde{K}(\cdot, \cdot, t)$ on \mathbb{H}^2 depends only on the relative distance $d(\bar{x}, \bar{y})$ apart of $\bar{x}, \bar{y} \in \mathbb{H}^2$. Let us now fix lifts \bar{x} and \bar{y} of x and y , respectively, in some fundamental domain F . We can then write

$$K(x, y, t) = \sum_{\gamma \in \Gamma} \tilde{K}(\bar{x}, \gamma \bar{y}, t) = \sum_{\gamma \in \Gamma} \tilde{K}(d(\bar{x}, \gamma \bar{y}), t)$$

and evaluate the trace

$$\text{tr}(K(\cdot, \cdot, t)) := \int_F K(x, x, t) d\nu(x) = \int_F \tilde{K}(d(\bar{x}, \bar{x}), t) d\nu(x) + \sum_{\gamma \in \Gamma - \{e\}} \int_F \tilde{K}(d(\bar{x}, \gamma \bar{x}), t) d\nu(x).$$

Lemma 7.1. *There is a correspondence between Γ and $\{kp^n k^{-1} : n \in \mathbb{Z}, p, k \in \Gamma/\Gamma_p\}$ where p runs through the non-conjugate primitive elements of Γ and $g \in \Gamma$.*

Since ν is preserved by translation by elements of the group, we can write

$$\sum_{\gamma \in \Gamma - \{e\}} \int_F \tilde{K}(d(\bar{x}, \gamma \bar{x}), t) d\nu(x) = \sum_{n=1}^{\infty} \sum_p \underbrace{\left(\sum_{k \in \Gamma/\Gamma_p} \int_F \tilde{K}(d(k^{-1}\bar{x}, p^n k^{-1}\bar{x}), t) d\nu(x) \right)}_{= \int_{F_p} \tilde{K}(d(\bar{x}, p^n \bar{x}), t) d\nu(x)}. \quad (1)$$

where F_p is a fundamental domain for Γ_p . Without loss of generality, we can take the fundamental domain $z = x + iy$ where $1 \leq y \leq m^2$ (where $l(g) = 2 \log m$) and the integral can be explicitly evaluated

$$\frac{l(p)}{\sqrt{\cosh(l(p^n)) - 1}} \int_{\cosh l(p^n)}^{\infty} \frac{\tilde{K}(s) ds}{\sqrt{s - \cosh l(p^n)}}. \quad (2)$$

7.2.3 Explicit formulae and Trace formula

The heat kernel $\tilde{K}(\bar{x}, \bar{y}, t)$ on \mathbb{H}^2 has the following standard explicit form, which we state without proof.

Lemma 7.2. *One can write*

$$\tilde{K}(\bar{x}, \bar{y}, t) = \tilde{K}(d(\bar{x}, \bar{y}), t) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-t/4} \int_{d(\bar{x}, \bar{y})}^{\infty} \frac{u e^{-u^2/4t}}{\sqrt{\cosh(u) - \cosh(d(\bar{x}, \bar{y}))}} du$$

and, in particular,

$$\tilde{K}(0, t) = \frac{1}{(4\pi)^{3/2}} \int_0^{\infty} \frac{u e^{-u^2/4t}}{\sinh(u/2)} du.$$

We start with the first term in (1):

$$\int_M \tilde{K}(0, t) d\nu(x) = \frac{\nu(M) e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} \frac{s e^{-s^2/4t}}{\sinh(s/2)} ds.$$

The connection with the original problem comes via the standard result: The length of a closed geodesic in the conjugacy class of g is given by $l(g) = \inf_{x \in H} d(xg, x)$. Moreover, $l(g^n) = |n|l(g)$.

Lemma 7.3. *There is a bijection between nontrivial conjugacy class and closed geodesics on the surface. Moreover, $\text{tr}(g) = 2 \cosh(l(\gamma))$.*

Furthermore, we can write

$$\sum_{\gamma \in \Gamma - \{e\}} \int_M \tilde{K}(d(\bar{x}, \gamma \bar{x}), t) d\nu(x) = \sum_{[\gamma]} \int_M \tilde{K}(d(\bar{x}, \gamma \bar{x}), t) d\nu(x)$$

which allows us to write the rest of the trace explicitly:

Lemma 7.4 (McKean). *The trace takes the form*

$$\frac{\nu(M)e^{-t/4}}{(4\pi t)^{3/2}} \int_0^\infty \frac{se^{-s^2/4t}}{\sinh(s/2)} ds + \frac{e^{-t/4}}{(16\pi t)^{1/2}} \sum_{n=1}^\infty \sum_{\gamma} \frac{l(\gamma)}{\sinh(nl(\gamma)/2)} e^{(nl(\gamma))^2/4t} \quad (3)$$

where the sum is over γ if over conjugacy classes in Γ .

This is an explicit computation. Using Lemma 7.2 and (2) we have that

$$\int_{\cosh l(p^n)}^\infty \frac{\tilde{K}(s) ds}{\sqrt{s - \cosh l(p^n)}} = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{\cosh l(p^n)}^\infty \int_s^\infty \frac{se^{-s^2/4t}}{\sqrt{\cosh(u) - \cosh(s)}} du ds$$

which we can immediately evaluate as $\frac{e^{-t/4}}{\sqrt{8\pi t}} e^{-l(p^n)^2/4t}$, and we are done.

7.2.4 Generalizations and the Selberg zeta function

If we consider the logarithmic derivative of the Selberg Zeta function then it takes the form:

$$\frac{1}{s - \frac{1}{2}} \frac{Z'(s)}{Z(s)} = 2 \sum_{\gamma} \sum_{k=0}^\infty \frac{l(\gamma)}{\sinh(kl(\gamma)/2)} g_s(kl(\gamma)) \quad (4)$$

where

$$g_s(u) = \frac{1}{2s-1} \exp\left(-\left(s - \frac{1}{2}\right)|u|\right).$$

The double summation is similar to that in (3). To make these terms identical, we can consider more general choices of kernel $\tilde{K}_s(\cdot, \cdot)$ which we can then write in the form $\tilde{K}_s(x, y) = \sum_n h_s(\lambda_n) \phi_n(x) \bar{\phi}_n(y)$, for suitable $h_s(\cdot)$. Moreover, we can write

$$\sum_n h_s\left(\sqrt{\lambda_n - \frac{1}{4}}\right) = \frac{\nu(M)}{2\pi} \int_{-\infty}^\infty r \tanh(\pi r) h(r) dr + \sum_{\gamma} \sum_{k=1}^\infty \frac{l(\gamma)}{\sinh(kl(\gamma))} g_s(l(\gamma)) \quad (5)$$

where $g_s(u) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-ru} h_s(r) dr$. In particular, choosing $h_s(r) = (r^2 + (s - \frac{1}{2})^2)^{-1}$ results in $g_s(u)$ having the correct form to give the infinite summation in (4). In particular, (5) tells us that the poles for $Z'(s)/Z(s)$ (i.e., the zeros for $Z(s)$) come from the eigenvalues

Actually a slight modification is still needed to make the integrals converge, etc.

8 Further reading

8.1 Number Theory

One of my favourite references for primes numbers and the Riemann zeta function is:

- *W.J. Ellison and F. Ellison, Prime Numbers. New York: Wiley, 1985.*

Another remarkably nice small (and cheap) book on prime numbers is

- *G. Tenenbaum, M. Mendes France, Les Nombres Premiers, Que Sais-je? No. 571, PUF 1997.*

8.2 Graphs

An extremely readable reference for zeta functions and graphs is

- *A. Terras, Zeta Functions of Graphs, Audrey Terras, Cambridge Studies in Advanced Mathematics (No. 128), Cambridge University Press, 2010 .*

8.3 Geodesic flows and Selberg Zeta Function

For basic references of geodesics flows and their dynamics, a good references is:

- *T. Bedford, M. Keane, and C. Series (eds.), Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces, Oxford University Press, Oxford, 1991.*

There is a comprehensive book on the Selberg trace formula and the Selberg Zeta function:

- *D. Hejhal, The Selberg trace formula for $PSL(2, \mathbb{R})$, Lecture Notes in Mathematics, Springer, 1976*

and a more concise account in

- *A.B. Venkov, Spectral theory of Automorphic Functions and its applications, Kluwer, 1990.*

There are not many pedagogical references for the use of Ruelle operators. The main ideas are very nicely illustrated in a particular case in

- *D. Mayer, On a ζ function related to the continued fraction transformation, Bull. Soc. Math. France, 104 (1976), no. 2, 195-203.*

The interesting experimental results on the zeros of $Z(s)$ on a pair of pants can be find in the preprint

- *D. Borthwick, Distribution of resonances for hyperbolic surfaces, preprint (arXiv:1305.4850)*

8.4 Anosov flows

The older approach using symbolic dynamics is described in

- *W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, Asterisque 187-188, Soc. Math. France, 1990*

There are not (yet) many surveys on the newer approach to Anosov flows due to Liverani et al; nor the exciting recent work of Tsujii, Faure, Zworski, Nonnemacher. However, these authors write well and their recent articles and lecture notes are very interesting to read. One good starting point might be:

- Semiclassical approach for the Ruelle-Pollicott spectrum of hyperbolic dynamics", Lectures notes for summer school 13-17 May 2013, Rome
(<http://www-fourier.ujf-grenoble.fr/~faure/articles/index.html#exposes>)

Many other interesting results can be found in original articles (via Mathscinet)