ANOTHER ACCOUNT OF THE DANI-SMILLIE THEOREM

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Abstract. This is yet another account of the proof of the Dani-Smillie theorem, inspired by approaches of Coudene and Hubbard-Miller. This makes no claims on originality, but hopefully helps to make these ideas and approaches more accessible.

1. Introduction

Let $V$ be a finite area surface of negative curvature. It is well known from the work of Furstenberg that the horocycle flow $h_t^+ : M \to M$ on the three dimensional unit tangent bundle $M = SV$ of a finite area surface $V$ of negative curvature is ergodic with respect to the normalized Haar measure $\nu$. Moreover, in the case of a compact surface then the horocycle flow is also uniquely ergodic and also minimal, i.e., every orbit is dense.

In the case that $V$ has finite area, but is not compact, then the horocycle flow is neither minimal nor uniquely ergodic. More precisely, there are families of closed horocycles which orbit cusps (and are parameterized by their height up the cusp). Each of these supports a closed $h^+$-invariant measure, which is also ergodic.

**Theorem 1** (Classification of measures). Let $f \in C_c(M)$ be a compactly supported continuous function. Assume that $\mu$ is an ergodic $h^+$-invariant probability which is not supported on a closed $h^+$-orbit then $\mu = \nu$.

We also recall a stronger equidistribution result.

**Definition 2.** We say that a point $x \in M = SV$ has an equidistributed orbit if for any compactly supported function $f : M \to \mathbb{R}$ we have that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(h_t^+ x) dt = \int f d\mu.$$ 

The main result we recall is the following due to Dani and Smillie [2]:

**Theorem 3** (Equidistribution of orbits). There are precisely two distinct possibilities for each $x \in M$. Either

1. $x$ lies on a periodic horocycle; or
2. $x$ has an equidistributed orbit.

There are also interesting results for infinite area periodic surfaces due to Babillot-Ledrappier.

**Remark 4.** The equidistribution result is stronger than the classification result. To see this, let $\nu$ be an ergodic probability measure which is not supported on a closed horocycle Choose a typical non-periodic point $x \in G/\Gamma$ which is typical respect to $\nu$ then

$$\int f d\nu = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(h_t^+ x) dt = \int f d\mu,$$

by the Birkhoff ergodic theorem applied to $\nu$ and Theorem 3, respectively.

There are already many accounts of the proof of these results. For the classification result we follow the approach of Coudene [1] the classification result. For the stronger Equidistribution result we follow the approach of Hubbard and Miller [3].

I am grateful to Yves Coudene for correspondence about his article [1].
Let us begin by recalling the definitions of the horocycle and geodesic flows.

2.1. The horocycle flow(s). Let $G = \text{PSL}(2, \mathbb{R})$ be the space of $2 \times 2$-matrices with determinant one. We can write $M = G/\Gamma$ where $\Gamma \subset G$ is a lattices (i.e., a discrete subgroup with the quotient having finite area). For example, we can let $\Gamma = \text{SL}(2, \mathbb{Z})$, or any principle congruence subgroup.

Definition 5. The (unstable) horocycle flow $h^+_t : G/\Gamma \to G/\Gamma$ can then be written

$$h^+_t(g\Gamma) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g\Gamma.$$

The ergodicity of horocycle flows is well known:

Lemma 6. The normalized Haar measure $\mu$ is invariant and ergodic for the horocycle flow.

Proof. The invariance follows easily since $h^+_t \in G$ and the Haar $\mu$ is invariant under the group action. The proof of ergodicity is given in the Appendix. $\square$

2.2. Geodesic flows. Complementing the horocycle(s) flow is the geodesic flow.

Definition 7. The geodesic flow $g_t : M \to M$ is defined by

$$g_t(g\Gamma) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} g\Gamma \text{ for } t \in \mathbb{R}.$$

The ergodicity of geodesic flows is well known and is stated for completeness (although we will not need it):

Lemma 8. Let $\mu$ be the normalized Haar measure. The geodesic flow is ergodic.

Proof. The invariance follows easily since $g_t \in G$ and the Haar $\mu$ is invariant under the group action. The proof of ergodicity uses the classical Hopf method. $\square$

The well known and useful relationship between the geodesic flow and horocycle flows is simply stated.

Lemma 9. For $s, t \in \mathbb{R}$, we have that $g_th^+_s g_{-t} = h^+_s e^t$ and $g_th^-_s g_{-t} = h^-_s e^t$.

Proof. This follows from the definitions and the matrix identities

$$\begin{pmatrix} 1 & s e^t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & s e^{-t} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

$\square$

2.3. Geometry of cusps. We also require some geometric estimates. In particular, we have a property of cusps:

Lemma 10. We have that:

1. Each cusp is isometric to $\{z = x + iy : y \geq 1, |x| \leq a^2\}$ for some $a > 0$ with the Poincaré metric; and
2. The cusps are disjoint and have area $a^2$.

In particular, since $\mu(M) < +\infty$ there are only finitely many cusps. Let us denote by $M_0 \subset M$ the complement of the union of the cusps. The main property we require is the following dichotomy.

Lemma 11. If $x \in M$ then either:
\( \phi_t(x) \to +\infty \) as \( t \to +\infty \); or

(2) there exists \( t_k \to +\infty \) such that \( \phi_{t_k}(x) \) lies in the compact part \( M_0 \).

3. Proof of Theorem 1

We follow the approach of Coudene.

3.1. Families of averages of functions. Fix a continuous compactly supported function \( f \in C_c(\mathbb{R}) \). We can introduce a family of functions \( (\mathcal{M}_t f) \in C_c(M) \).

\[
(\mathcal{M}_t f)(x) = \int_0^1 f(g_{-\log \cdot} \circ h_s(x)) ds.
\]

The most important property of the family \( \{\mathcal{M}_t(f(x)) : t > 0\} \) is the following.

**Lemma 12.** The family \( \{\mathcal{M}_t(f(x)) : t > 0\} \) is equicontinuous.

**Proof.** This follows by competing the thickening up of the horocycles segments \( h^+_{[0,1]}(x) \) and \( h^*_{[0,1]}(x') \) for nearby points \( x, x' \in M \). More precisely, let us define

\[
D_x(\epsilon) := \{g_t h_s(x) : |s|, |t| \leq \epsilon\} \text{ and } V_x(\epsilon) := h^*_{[0,1]} D_x(\epsilon)
\]

then by continuity

\[
\left| \mathcal{M}_t f(x) - \frac{1}{V_x(\epsilon)} \int_{V_x(\epsilon)} f(g_{-\log \cdot}) d\lambda \right| \to 0 \text{ as } \epsilon \to 0.
\]

In particular,

\[
\left| \frac{1}{V_x(\epsilon)} \int_{V_x(\epsilon)} f(g_{-\log \cdot}) d\lambda - \frac{1}{V_x'(\epsilon)} \int_{V_x'(\epsilon)} f(g_{-\log \cdot}) d\lambda \right|
\]

\[
\leq \left\| f(g_{-\log \cdot}) \right\|_2 \left\| \frac{V_x'(\epsilon)}{V_x(\epsilon)} - \frac{V_x'(\epsilon)}{V_x(\epsilon)} \right\|_2 \to 0 \text{ as } \epsilon \to 0
\]

uniformly in \( x, x' \in M \), using the Cauchy-Schwartz inequality, \( g \)-invariance of \( \mu \) and the fact that \( \mu \) is non-atomic.

\[\square\]

Furthermore, by a change of variables using Lemma 9 we can relate these to the averages by

\[
\frac{1}{t} \int_0^t f(h_{it}^+ x) dt = \mathcal{M}_t f(g_{log t} x) \text{ for } t > 0
\]

(1)

3.2. Topological aspects. Let \( f : M \to \mathbb{R} \) be a compactly supported set. By Lemma 12 and the Arzela-Ascoli theorem (for the compactification of \( V \) by adding a point at infinity) we immediately get the following.

**Lemma 13.** We can choose a continuous function \( \overline{f} : M \to \mathbb{R} \) and a subsequence \( t_k \) so that \( \mathcal{M}_{t_k} \) converges to \( \overline{f} \) on any compact set.
3.3. **Proof of Theorem.** Let \( \nu \) be a \( h_i^+ \)-invariant ergodic probability measure and assume that \( \nu \) is not supported on a closed horocycle. We need to show that \( \nu = \mu \). We break the proof up into a number of easy steps.

**Step 1:** Consider the set \( \Lambda \subset M \) of geodesic trajectories which enter the compact part \( M_0 \) infinitely often. This is easily seen to be \( h^+ \)-invariant (using lemma 9). Thus by the ergodicity hypothesis for \( \nu \) we see that either \( \nu(\Lambda) = 1 \) or \( \nu(\Lambda) = 0 \). Moreover, we claim that \( \nu(\Lambda) = 1 \), since otherwise \( \nu(\Lambda) = 0 \) and then, by Lemma 11, the measure is supported on the set of vectors corresponding to a geodesic going to infinity, but these vectors are all associated to periodic horocycle, contradicting the original hypothesis on \( \nu \).

**Step 2:** The Birkhoff ergodic theorem applied to \( \nu \) gives that \( \lim_{T \to \infty} \frac{1}{T} \int_0^T f(h_i^+ t) dt = \int f d\nu \), a.e. \( (\nu) \ x \in \Lambda \).

**Step 3:** We can choose \( x \in \Lambda \) and \( t_k \to +\infty \) such that:

1. \( g_{\log t_k} x \) converges to some \( x_0 \in M_0 \), say (by Lemma 11); and
2. \( \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(h_i^+ t) dt = \int f d\nu \) (by Step 2).

**Step 4:** Using (1) we can write identity \( \frac{1}{T} \int_0^T f(h_i^+ t) dt = \mathcal{M}_{t_k} f(g_{\log t_k} x) \) for the point \( x \) in Step 3 (1), and Lemma 13, to deduce \( \bar{f}(x_0) = \int f d\nu \).

**Step 5:** Since the Haar measure \( \mu \) is ergodic and \( \mu(\Lambda) = 1 \), the previous argument applies equally well to \( \mu \) to deduce that \( \int f d\mu = f(x_0) = \int f d\nu \). This completes the proof.

4. **Equidistribution Theorem**

   To prove the stronger equidistribution theorem we first need to use a little more geometry.

4.1. **Boxes and Cubes.** We first need to define the complementary horocycle flows.

**Definition 14.** The (stable) horocycle flow \( h_i^- : G/\Gamma \to G/\Gamma \) can then be written

\[
h_i^-(g\Gamma) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} g\Gamma.
\]

We want to define small two dimensional boxes which are transverse to the orbits of the flow \( h^+ \), and then push these boxes along the \( h^+ \)-orbits to get three dimensional cubes.

**Definition 15.** Given \( x \in M \) and \( \epsilon, \delta > 0 \) sufficiently small we define

\[
B(x, \epsilon, \delta) = \{ g_i h_i^- (x) : |t| \leq \delta, |r| \leq \epsilon \}.
\]

Given \( x \in M \) and \( \epsilon, \delta > 0 \) we define

\[
C(x, \epsilon, \delta, \eta) = h_i^+(B(x, \epsilon, \delta)) = \{ h_s^+ g_i h_i^- (x) : |t| \leq \delta, |r| \leq \epsilon, |s| \leq \eta \}.
\]

4.2. **Stable holonomy.** We can fix a point \( x \in M \) and \( s > 0 \). Consider the image \( x' = h_i^+ x \) under the unstable horocycle flow. We can consider the corresponding holonomy map \( H : U \to U' \) between a neighbourhood \( U \) of \( x \) in \( B(x, \epsilon, \delta) \) and a neighbourhood \( U' \) of \( x' \) in \( B(x', \epsilon, \delta) \) (i.e., \( H(\xi) = U' \cap h_{[0,s+\delta+\eta]}(\xi) \), for suitable \( \eta = \eta(\xi) \)).

If \( y \in B(x, \epsilon, \delta) \) then we can write \( y = g_i h_i^- (x) \in B(x, \epsilon, \delta) \), say, with natural coordinates on the box. If the corresponding point \( y' = H(y) \in B(x', \epsilon, \delta) \) lying on the same \( h^+ \)-orbit can be written as \( y' = h_{\sigma_s(y)}(y) \). This defines a function \( \sigma_s : B(x, \epsilon, \delta) \to \mathbb{R}^+ \) giving the transition time between sections of the horocycle flow.

We have the following concrete estimate for \( \sigma_s \).
Remark 17. If \( \epsilon > 0 \), then \( y = g_{x}x \) lies on the same geodesic orbit and so \( \sigma_{s} = se^{t} \).

Lemma 16 leads naturally to the following estimate on the derivative.

**Corollary 18** (s-dependence of \( \sigma_{s}(y) \)). For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( y \in B(x, \delta/T, \delta) \) and \( 0 \leq s < T \) we have that

\[
1 - \frac{\partial \sigma_{s}(y)}{\partial s} < \epsilon.
\]

**Proof.** We can write

\[
\frac{\partial \sigma_{s}(y)}{\partial s} = e^{t} \frac{\partial}{\partial s} \left( \frac{s}{1 - sr} \right) = e^{t} \left( \frac{1}{1 - sr} + \frac{sr}{(1 - sr)^{2}} \right) = e^{t} \left( \frac{1}{1 - sr} \right).
\]

Since \( |r| \leq \delta/T, |t| \leq \delta \) and \( |s| \leq T \) we have that

\[
\frac{\partial \sigma_{s}(y)}{\partial s} \in \left[ \frac{e^{-\delta}}{(1 + \delta)^{2}}, \frac{e^{\delta}}{(1 - \delta)^{2}} \right] \subset [1 - \epsilon, 1 + \epsilon]
\]

provided \( \delta \) is sufficiently small. \( \square \)

4.3. **Bounds on the holonomy.** We now have a bound on the holonomy

**Lemma 19** (Bounds on holonomy). There exists a constant \( C > 0 \) such that for all \( T > 0, 0 < \delta < \frac{1}{2} \) and \( y \in B(x, \delta/T, \delta) \), we have that \( d(h_{x}^{+}x, h_{y}^{+}(y)) \leq C\delta \) where \( 0 \leq s \leq T \).

**Proof.** We can denote

\[
x' = g_{-\log T}(x) \quad \text{and} \quad y' = g_{-\log T}(y)\\
x'' = h_{y}^{+}(x') \quad \text{and} \quad y'' = H(y') \in B(x'', \delta, \delta)\\
x''' = h_{x}^{+}(x) \quad \text{and} \quad y''' = h_{y}^{+}(y).
\]

If we apply by \( g_{-\log T} \) to the cube \( C(\delta, \delta/T, s) \) we get the new cube \( C(\delta, \delta, \delta) \). Moreover, \( d(x', y') \leq 2\delta \). The \( C(\delta, \delta, \delta)-\)cube has a uniformly Lipschitz foliation by \( h_{x}^{+}\)-orbits and thus there exists \( C > 0 \) such that

\[
d(h_{x}^{+}x, h_{y}^{+}(y)) \leq d(x'', y'') \leq Cd(x', y') \leq C\delta.
\]

This complete the proof. \( \square \)
Corollary 20. Let \( f \in C_c(M) \) be a compactly supported function. For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( T > 0 \), \( 0 \leq s \leq T \), \( x \in M \), \( y \in B(x, \delta / T, \delta) \) we have that
\[
|f(h_s^+ x) - f(h_{\sigma_s(y)}^+ y)| < \epsilon
\]

Proof. This follows from uniform continuity of \( f \). \( \square \)

4.4. Convergence through good times. We will only require the following lemma for the normalised Haar measure \( \mu \).

Lemma 21 (Egoroff’s theorem). Let \( \nu \) be a \( h^+ \)-invariant ergodic probability measure. For any \( \eta > 0 \) there exists a closed subset \( Y \subset M \) with \( \nu(Y) > 1 - \eta \) such that for all \( \epsilon > 0 \) and \( f \in C_c(M) \) there exists \( T_0 > 0 \) such that for all \( x \in Y \):
\[
\left| \frac{1}{T} \int_0^T f(h_s^+ x) ds - \int f d\nu \right| < \epsilon.
\]

Proof. By the Birkhoff ergodic theorem
\[
f_T(x) := \frac{1}{T} \int_0^T f(h_s^+ x) ds - \int f d\nu \to 0
\]
for a.e.\( (\nu) x \in M \). For each \( m \geq 1 \) we can choose \( T_m > 0 \) such that
\[
\nu \left( \left\{ x \in M : |f_T(x)| < \frac{1}{m}, \forall T \geq T_m \right\} \right) \geq 1 - \eta / 2^m.
\]
Thus \( Y = \cap_{m=1}^\infty X_m \) satisfies \( \nu(Y) \geq \sum_{m=1}^\infty \nu(X_m^c) \geq 1 - \eta \) and the lemma follows easily \( \square \)

Assume that \( x \in M \) is not on a closed \( h^+ \)-orbit, then \( x \in \Lambda \) by Lemma 11 and thus there exist \( T_n \to +\infty \) such that \( g_{T_n} x \in M_0 \).

Proposition 22 (Convergence through good times). For any \( f \in C_c(M) \) we have that
\[
\lim_{n \to +\infty} \frac{1}{T_n} \int_0^{T_n} f(h_t^+ x) dt = \int f d\nu
\]

Proof. Given \( \epsilon > 0 \) choose \( \delta > 0 \) so that Corollaries 18 and 20 apply. Let \( \eta := \inf_{y \in M_0} \mu(C_y(\delta, \delta, \delta)) > 0 \), using the compactness of \( M_0 \). Thus since \( g_{T_n} x \in M_0 \) we can bound
\[
\eta \leq \mu(C_{g_{T_n} x}(\delta, \delta, \delta)) = \mu(C_x(\delta / T_n, \delta, \delta T_n))
\]
by the \( g \)-invariance of \( \mu \). We can now apply Egoroff’s theorem (Lemma 21) to choose \( Y \subset M \) with \( \nu(Y) > 1 - \eta \), and therefore
\[
\nu(Y \cap C_x(\delta / T_n, \delta, \delta T_n)) > 0.
\]
Choose $y_n \in Y \cap C_x(\delta/T_n, \delta, \delta T_n)$ and $z_n \in B_4(\delta/T, \delta)$ on the same $h^+$-orbit. Fix $f \in C_c(M)$, and assume without loss of generality that $\|f\|_\infty \leq 1$. By the triangle inequality we can bound
\[
\left| \frac{1}{T_n} \int_0^{T_n} f(h^+_t x) dt - \int f d\mu \right| \leq \frac{1}{T_n} \int_0^{T_n} f(h^+_t x) dt - \frac{1}{T_n} \int_0^{T_n} f(h^+_t z_n) dt
\]
\[
\stackrel{\epsilon}{\leq} \left(\text{by corollary 20}\right)
\]
\[
+ \frac{1}{T_n} \int_0^{T_n} f(h^+_t z_n) dt - \frac{1}{T_n} \int_0^{T_n} f(h^+_t z_n) \frac{\partial \sigma_t(z_n)}{\partial t} dt
\]
\[
\stackrel{\epsilon}{\leq} \left(\text{since } |\frac{\partial \sigma_t}{\partial t} - 1| < \epsilon \text{ by Corollary 18}\right)
\]
\[
+ \frac{1}{T_n} \int_0^{T_n} f(h^+_t z_n) dt - \frac{1}{T_n} \int_0^{T_n} f(h^+_t y_n) dt
\]
\[
\stackrel{\epsilon}{\leq} \left(\text{since } z_n, y_n \text{ close on same } h^+_t\text{-orbit}\right)
\]
\[
+ \frac{1}{T_n} \int_0^{T_n} f(h^+_t y_n) dt - \int f d\mu
\]
\[
\stackrel{\epsilon}{\leq} \left(\text{since } y_n \in Y, \text{ by uniform convergence}\right)
\]
\[
\leq 5\epsilon.
\]

\[\square\]

\textbf{Remark 23.} The convergence through good times is actually enough to deduce Theorem 1 using the argument in Remark 4.

4.5. Uniform convergence. To complete the proof of Theorem 3, assume for a contradiction we can find $x$ not supported on a closed horocycle (i.e., $x \in \Lambda$) and $f \in C_c(M)$ for which we do not have $\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(h^+_t x) dt = \int f d\mu$. If we define probability measure $\nu_t$ by $\int gd\nu_t = \frac{1}{T} \int_0^T g(h^+_t x) ds$ for $t > 0$ and $g \in C_c(M)$ then we can deduce that there exists a $h^+$-invariant measure $\nu \neq \mu$ and a subsequence sequence $t_k \to +\infty$ such that $\nu_{t_k} \to \nu$ in the weak-star topology. If we can show that $\nu$ is a probability measure then this would contradict Theorem 1, and the proof would be done.

Let $M_0^\delta \subset M$ be the compact subset corresponding to the compact subset of $V$ bounded by closed horocycles of length $\delta > 0$. The next lemma shows that the proportion of any orbit in $M_0^\delta$ can be made arbitrarily large, by choosing $\delta > 0$ sufficiently small.

\textbf{Lemma 24.} For each $\epsilon > 0$ there exists $\delta > 0$ such that for $x \in \Lambda$ we have that
\[
\lim_{t \to +\infty} \frac{1}{t} \int_0^t \chi_{M^\delta}(h^+_t x) ds > 1 - \epsilon.
\]

\textbf{Proof.} The idea of the proof is to show that any portion of the $h^+_t x$ orbit outside of $M_0^\delta$ is matched by a much longer portion inside $M_0^\delta - M_0^{\delta_0} \subset M_0^\delta$, for some fixed $\delta_0 > 0$. Let us assume that the cusp lifts to a strip $\{z = x + iy : y \geq 1, |x| \leq a^2\}$ in $\mathbb{H}^2$ then by considering the Poincaré metric we see that the closed horocycle of length $\delta$ lifts to the Euclidean line $y = 1/(2\delta a^2)$. Lift the horocycle orbit $h^+tx$ to a circle in $\mathbb{H}^2$ touching the real axis at 0, say, and intersecting the imaginary axis at $iy$, say, with $y > 1/\delta$. Applying the lift of horocycle flow to $iy$ we see that the imaginary part of the orbit at time $t$ has value $ty/(1 + t^2y^2)$. To determine the time $t_\delta > 0$ till the horocycle orbit re-enters $M_0^\delta$ requires only solving $1/\delta = y/(1 + t_\delta^2 y^2)$,
i.e., \( t_{\delta} = \sqrt{(\delta y - 1)/y} \). Thus comparing this with the time \( t_{\nu} > 0 \) till the same horocycle orbit re-enters \( M_0^{\delta'} \) gives \( \sqrt{(\delta y - 1)/(\delta' y - 1)} \to 0 \) as \( \delta \to 0 \), uniformly in \( y \).

We first need to show that \( \nu(M) = 1 \). Given \( \epsilon > 0 \) we first choose \( \delta > 0 \) using Lemma 24 and we can then choose a compactly supported function \( f : M \to [0,1] \) such that \( f|_{M_{\delta}^{\delta}} = 1 \). Thus

\[
\nu(M) \geq \int f d\nu \geq \liminf_{T \to +\infty} \frac{1}{T} \int_0^T f(h_t^+ x) dt \\
\geq \liminf_{T \to +\infty} \frac{1}{T} \int_0^T \chi_{M_{\delta}^{\delta}}(h_t^+ x) dt \geq 1 - \epsilon.
\]

Since \( \epsilon > 0 \) can be chosen arbitrarily small we conclude that \( \nu(M) = 1 \).

Finally, we need to check that \( \nu(\Lambda) = 1 \), to avoid the possibility that \( \nu \) is again supported on a closed orbit. Assume for a contradiction that \( 3\epsilon = \nu(\Lambda^c) > 0 \) then we can choose a compact set \( K \subset \Lambda^c \) with \( \nu(K) > \frac{2}{3} \nu(\Lambda^c) = 2\epsilon \).

Let \( \delta > 0 \) be chosen according to Lemma 24 then by Lemma 11 we see that for sufficiently large \( T > 0 \) we have that \( g_T(K) \cap M_{\delta}^{\delta} = \emptyset \). We can then choose \( U \supset K \) to be an open neighbourhood such that \( g_T(U) \cap M_{\delta} = \emptyset \). Since by assumption, \( \nu_n \) converges to \( \nu \) in the weak star topology, we have that

\[
\frac{\lambda(\{0 \leq t \leq t_n : h_t^+(x) \in U\})}{T_n} > \frac{1}{2} \nu(K) > \epsilon,
\]

where \( \lambda \) denotes Lebesgue measure on the horocycle orbit. But by Lemma 24, for sufficiently large \( t_n \) the proportion of its orbit outside of \( M_{\delta}^{\delta} \), and thus disjoint from \( U \), at least equal to \( 1 - \epsilon \). This gives a contradiction.

5. Possible Generalizations

It is natural to ask about possible generalization of these results.

For example, we could assume that \( M \) is a finite area surface with variable negative curvature in the compact part \( M_0 \), say, but that the cusps still have constant negative curvature, say, so that the geometric estimates still apply. We would still have that there are transverse stable and unstable foliations are well defined one-dimensional foliations \( \mathcal{F}^s = \{W^s(x)\} \) and \( \mathcal{F}^u = \{W^u(x)\} \) with leaves given by

\[
W^s(x) = \{y \in M : d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to +\infty\} \text{ and } \\
W^u(x) = \{y \in M : d(\phi_{-t} x, \phi_{-t} y) \to 0 \text{ as } t \to +\infty\}.
\]

We need to replace \( \mu \) by the measure of maximal entropy and show that they exist transverse measures \( \mu^+ \) and \( \mu^- \) to \( \mathcal{F}^s \) and \( \mathcal{F}^u \), respectively, such that \( g_t \mu^+ = e^{ht} \mu^+ \) and \( g_t \mu^- = e^{-ht} \mu^- \), where \( h > 0 \) is the entropy, and \( d\mu = \text{Const.} d\mu^+ \times d\mu^- \) (adapting from the compact case Margulis’ original proof [0] or Ruelle’s approach [0]). The classification result should remain unchanged (i.e., the only ergodic \( h^+ \)-invariant measures are \( \mu \) and closed orbit measures). However, the equidistribution result would probably require reparamaterising the horocycles using \( \mu^+ \).

Another direction in which one could consider generalising the result is to surfaces \( V \) of constant negative curvature \( -1 \) with non-empty geodesic boundary components. In this case \( \mu \) should now be replaced by \( d\mu = d\mu_1 \times d\mu_2 \times dt \), where \( \mu_1 \) and \( \mu_2 \) are given by the Patterson-Sullivan measure on \( \partial \mathbb{D} \). In this case the classification result should remain unchanged (i.e., the only ergodic \( h^+ \)-invariant measures are \( \mu \) and closed orbit measures). However, the equidistribution result would be less natural to formulate (since \( m^+ \) is not fully supported on the horocycle orbits).
6. APPENDIX: PROOF OF LEMMA 6

We return to the proof of ergodicity of $\mu$ for the horocycle flow, using an approach of Coudene [1]. The only geometric fact we need to establish ergodicity is the following.

Lemma 25. The horocycle flow $h^+$ is transitive (i.e., there exists $y \in M$ such that $\bigcup_{t \in \mathbb{R}} h^+_t(y) \subset M$ is dense).

We present the proof of Lemma 6 in a number of steps.

Step 1: By the Arzela-Ascoli theorem we can (again) choose a sequence $t_k \to +\infty$ such that $M_k f \to \overline{f} \in C(X)$ on compact sets.

Step 2: By Lemma 9 we can write that
\[
\| \frac{1}{t_k} \int_0^{t_k} f(h^+_s) ds - \overline{f} \circ g_{-\log t_k} \|_2 = \| (M_{t_k} \circ g_{-\log t_k} - \overline{f} \circ g_{-\log t_k}) \|_2 \\
= \| M_{t_k} f - \overline{f} \|_2 \to 0 \text{ as } k \to +\infty
\]
(A1)

using the $g$-invariance of $\mu$ and step 1.

Step 3: The von Neumann ergodic theorem applied $h^+$ gives that the averages $\frac{1}{T} \int_0^T f(h^+_t(x)) dt$ converges to a function $\hat{f} \in L^2(\mu)$, which is necessarily $h$-invariant, i.e.,
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(h^+_t(x)) dt - \hat{f}(x) \|_2 = 0
\]
(A2)

where $\hat{f} = \hat{f} \circ h_s$, for all $s \in \mathbb{R}$.

Step 4: We can now show that $\overline{f}$ (from step 1) is $h^+$-invariant. More precisely, we have that
\[
\| \overline{f} - \hat{f} \circ g_{\log t_k} \|_2 = \| \overline{f} \circ g_{-\log t_k} - \hat{f} \|_2 \to 0 \text{ as } k \to +\infty
\]
(A3)

by $g$-invariance of $\mu$, (A1) and (A2). We can then write
\[
\| \overline{f} \circ h_s - \hat{f} \circ g_{\log t_k} \|_2 = \| \overline{f} - \hat{f} \circ g_{\log t_k} \circ h_{-s} \|_2 \\
= \| \overline{f} - \hat{f} \circ h_{-st_k} \circ g_{\log t_k} \|_2 \\
= \| \overline{f} - \hat{f} \circ g_{\log t_k} \|_2 \to 0 \text{ as } k \to +\infty
\]
(A4)

by $h^+$-invariance of $\mu$, Lemma 9 and the $h^+$-invariance of $\hat{f}$ (from step 3) and (A3).

Comparing (A3) and (A4) and using the triangle inequality, gives that $\overline{f} = \overline{f} \circ h_s$, as required.

Step 5: The existence of a dense $h$-orbit gives that the continuous $h$-invariant function $\overline{f}$ is actually a constant, and thus is equal to $f d\mu$. This suffices to deduce that $\mu$ is ergodic.

REFERENCES