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1. Introduction

Complex analysis is one of the classical branches in mathematics with roots in the 19th century and just prior. Complex analysis, in particular the theory of conformal mappings, has many physical applications and is also used throughout analytic number theory. In modern times, it has become very popular through a new boost from complex dynamics and the pictures of fractals produced by iterating holomorphic functions. Another important application of complex analysis is in string theory which studies conformal invariants in quantum field theory.

1.1. A little history. The study complex numbers arose from try to find solutions to polynomial equations. Al-Khwarizmi (780-850) in his book Algebra had solutions to quadratic equations of various types. Under the caliph Al-Mamun (reigned 813-833) Al-Khwarizmi became a member of the House of Wisdom in Bagdad.

The first to solve the polynomial equation \( x^3 + px = q \) was Scipione del Ferro (1465-1526). On his deathbed, del Ferro confided the formula to his pupil Antonio Maria Fiore, who subsequently challenged another mathematician Nicola "Tartaglia" Fontana (1500-1557) to a mathematical contest on solving cubics. (The name Tartaglia means "stammerer" a symptom of injuries acquired aged 12 during the french attack on his home town of Bresca). The night before the contest, Tartaglia rediscovered the formula and won the contest. Tartaglia in turn told the formula (but not the proof) to an influential mathematician Gerolamo Cardano (1501-1576), provided he signed an oath to secrecy. However, from a knowledge of the formula, Cardano was able to reconstruct the proof. Later, Cardano learned that del Ferro, not Tartaglia, had originally solved the problem and then, feeling under no further obligation towards Tartaglia, proceeded to publish the result in his Ars Magna (1545). Cardano was also the first to introduce complex numbers \( a + \sqrt{b} \) into algebra, but had misgivings about it. In the Ars magna he observed, for example, that the problem of finding two numbers that add to 10 and multiply to 40 was satisfied by \( 5 + \sqrt{-15} \) and \( 5 - \sqrt{-15} \) but regarded the solution as both absurd and useless. Cardan was also said to have correctly predicted the exact date of his own death (but it has also been claimed that he achieved this by committing suicide!).

Rene Descartes (1596-1650), the mathematician and philosopher coined the term imaginary: "For any equation one can imagine as many roots [as its degree would suggest], but in many cases no quantity exists which corresponds to what one imagines."

Abraham de Moivre (1667-1754), a protestant, left France to seek religious refuge in London at eighteen years of age. There he befriended Isaac Newton. In 1698 he mentions that Newton knew, as early as 1676 of an equivalent expression to what is today known as de Moivres theorem (and is probably one of the best known formulae) which states that:

\[
(cos(\theta) + i \sin(\theta))^n = cos(n\theta) + i \sin(n\theta)
\]

where \( n \) is an integer. (De Moivre, like Cardan, is famed for predicting the day of his own death. He found that he was sleeping 15 minutes longer each night and summing the arithmetic progression, calculated that he would die on the day that he slept for 24 hours.)

Leonhard Euler (1707-1783) introduced the notation \( i = \sqrt{-1} \) in his book Introductio in analysin innitorum in 1748, and visualized complex numbers as points with rectangular coordinates, but did not give a satisfactory foundation for complex numbers. In contrast, there are indications that Carl Friedrich Gauss (1777-1855). had been in possession of the geometric representation of complex numbers since
1796, but it went unpublished until 1831, when he submitted his ideas to the Royal Society of Gottingen. It was Gauss who introduced the term complex number.

Joseph-Louis Lagrange (1736-1813) showed that a function is analytic if it has a power-series expansion. However, it was Augustin-Louis Cauchy (1789-1857) who really initiated the modern theory of complex functions in an 1814 memoir submitted to the French Academie des Sciences.

Although the term analytic function was not mentioned in his memoir, the concept is present there. The memoir eventually published in 1825. In particular, contour integrals appear in this memoir (although Poisson had written a 1820 paper with a path not on the real line). Cauchy also gave proofs of the Fundamental Theorem of Algebra (1799, 1815) which, as we will see, has analytic proof. In summary, Cauchy, gave the foundation for most of the modern ideas in the field, including:

1. integration along paths and contours (1814);
2. calculus of residues (1826);
3. integration formulae (1831);
4. Power series expansions (1831); and
5. applications to evaluation of definite integrals of real functions

The Cauchy-Riemann equations (actually dating back to d’Alembert 1752, then Euler 1757, d’Alembert 1761, Euler 1775, Lagrange 1781) are also usually attributed to Cauchy 1814-1831 and Riemann 1851.

Cauchy resigned from his academic positions in France in 1830 rather than to swear an oath of allegiance to the new government. However, he felt able to resume his career in France in 1848, when the oath was finally abolished.

Regarding subsequent work, Karl Weierstrass (1815-1897) formulated analyticity in terms of existence of a complex derivative, which is the perspective taken in most textbooks, and Georg Riemann (1826-1866) made fundamental use of the notion of conformality (previously studied by Euler and Gauss). Later contributions were made by Poincaré to conformal maps and Teichmüller to quasiconformal maps.

1.2. Notation. We denote by \( \mathbb{C} \) the complex numbers. If \( z \in \mathbb{C} \) then we can write \( z = x + iy \) where \( x, y \in \mathbb{R} \). We denote the real and imaginary parts by \( x = Re(z) \) and \( y = Im(z) \). We then write the absolute value as \( |z| = \sqrt{x^2 + y^2} \).

Occasionally it is useful to write complex numbers in radial coordinates, i.e., \( z = re^{i\theta} \) where \( r > 0 \) and \( 0 \leq \theta < 2\pi \).

We denote by \( \overline{z} = x - iy = re^{-i\theta} \) the complex conjugate. If \( z, w \in \mathbb{C} \) then we write

\[ [z, w] = \{\alpha z + (1 - \alpha)w : 0 \leq \alpha \leq 1\} \]

for the line segment joining them.
Exercise 1.1. Show that the map $\phi : \mathbb{C} \to M_2(\mathbb{R}) (= 2 \times 2 \text{ real matrices})$

$$\phi : x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$ is a monomorphism.

1.3. Some useful reference books.

(1) R. Churchill and J. Brown, Complex Variables and Applications (ISBN 0-07-010905-2). This is a fairly readable account including much of the material in the course.

(2) I. Stewart and D. Tall, Complex Analysis (ISBN 0-52-128763-4). This is a popular and accessible book.

(3) L. Alhfors, Complex Analysis: an Introduction to the Theory of Analytic Functions of One Complex Variable (ISBN 0-07-000657-1). This is a classic textbook, which contains much more material than included in the course and the treatment is fairly advanced.

(4) S. Krantz and R. Greene, Function Theory of One Complex Variable (ISBN 0-82-183962-4). This is a nice textbook, which contains much more material than included in the course.


   The first chapter gives a nice summary of some of the ideas in the course. The rest of the book is very interesting, but too geometric for this course.

2. A few basic ideas

2.1. The Riemann sphere. It is convenient to add an extra point to $\mathbb{C}$. In order to accommodate the extra point $\infty$, we need to extend the complex plane by adding this point in to get the Riemann sphere.

We denote by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. There is a natural “stereographic” projection between the sphere (minus the “north pole” $(1, 0, 0)$) and the complex plane

$$\pi : \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} - \{(1, 0, 0)\} \to \mathbb{C}$$

defined by

$$\pi(x_1, x_2, x_3) = z := \left(\frac{x_1}{1 - x_3}\right) + i \left(\frac{x_2}{1 - x_3}\right).$$

In particular, $0$ is the image of the south pole $(0, 0, -1)$, and the unit circle $|z| = 1$ is the image of the equator $x_3 = 0$.

Definition 2.1. A circle on the sphere corresponds to the intersection of $x_1^2 + x_2^2 + x_3^2 = 1$ with a plane $ax + by + cz = d$. 
The following is easy to prove.

**Lemma 2.2.** The stereographic projection of a circle on the sphere is either a circle or a line in \( \mathbb{C} \).

**Proof.** The image of the intersection under the projection can be written as

\[
a(z + \overline{z}) - ib(z - \overline{z}) + c(|z|^2 - 1) = d(|z|^2 + 1)
\]

If we write \( z = x + iy \) then

\[
(d - c)(x^2 + y^2) - 2ax - 2by + (d + c) = 0.
\]

**Case I:** If \( c = d \) then this is the equation of a straight line.

**Case II:** If \( c \neq d \) then

\[
x^2 + y^2 - \frac{2ax}{d-c} - \frac{2by}{d-c} + \frac{(d+c)}{d-c} = 0
\]

which we can rearrange as

\[
\left(x - \frac{a}{d-c}\right)^2 + \left(y - \frac{b}{d-c}\right)^2 = \frac{a^2 + b^2 + (c^2 - d^2)}{(d-c)^2}.
\]

It only remains to show that \( a^2 + b^2 + c^2 - d^2 > 0 \) to see this is the equation of a circle. However,

\[
|a_0| = |ax + by + cz| \leq \sqrt{x^2 + y^2 + z^2} \sqrt{a^2 + b^2 + c^2}
\]

by the usual Cauchy-Schwarz inequality and we are done. \( \square \)

Note that in the proof there is an equality in the last line only when \( (a, b, c) = \lambda(x_1, x_2, x_3) \) for some \( \lambda \), i.e., the plane is tangent to the sphere.

**Remark 2.3.** The inverse images of a points \( z \in \mathbb{C} \) is a triple \((x_1, x_2, x_3)\) lying on the sphere and satisfying \( |z|^2 = \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1 + x_3}{1-x_3} \) (since \( x_1^2 + x_2^2 + x_3^2 = 1 \)). We can then write

\[
x_1 = \frac{z + \overline{z}}{1 + |z|^2} \quad \text{since} \quad 1 - x_3 = \frac{2}{|z|^2 + 1}.
\]

Similarly,

\[
x_2 = \frac{z - \overline{z}}{1 + |z|^2} \quad \text{and} \quad x_3 = \frac{|z|^2 - 1}{1 + |z|^2}.
\]

We deduce that

\[
(x_1, x_2, x_3) = \left( \frac{z + \overline{z}}{1 + |z|^2}, \frac{z - \overline{z}}{1 + |z|^2}, \frac{|z|^2 - 1}{1 + |z|^2} \right).
\]

This completes the proof.

**Exercise 2.4.** Prove the converse, i.e., the preimage of a circle or a straight line in \( \mathbb{C} \) is a circle on the Riemann sphere.
Remark 2.5. To associate the appropriate topology to \( \hat{\mathbb{C}} \) we take the usual open sets in \( \mathbb{C} \) plus the complements of compact sets union with the point \( \infty \). This means that we can interpret \( z_n \to z \) in the usual sense if \( z \neq \infty \). However, we say that \( z_n \to +\infty \) if for every \( K > 0 \) we have there exists \( N > 0 \) such that \( |z| > K \). Then \( \mathbb{C} \) is homeomorphic to the ball minus the "north pole". The appropriate metric for \( \hat{\mathbb{C}} \) comes from the standard metric on the ball, i.e.,

\[
d((x_1, x_2, x_3), ((x'_1, x'_2, x'_3))) = \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}
\]

\[
= \sqrt{2(1 - x_1 x'_1 + x_2 x'_2 + x_3 x'_3)}.
\]

If \( z := \pi((x_1, x_2, x_3)) \) and \( w := \pi((x'_1, x'_2, x'_3)) \) then we can define a natural metric on \( \hat{\mathbb{C}} \cup \{\infty\} \) by

\[
d(z, w) = d((x_1, x_2, x_3), ((x'_1, x'_2, x'_3))) = \frac{|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}
\]

where \( (x_1, x_2, x_3), (x'_1, x'_2, x'_3) \neq (0, 0, 1) \) (i.e., \( z, w \neq \infty \)) and

\[
d(z, \infty) := d((x_1, x_2, x_3), (0, 0, 1)) = \frac{2}{\sqrt{1 + |z|^2}}.
\]

Exercise 2.6. Check all the above formulae.

2.2. Möbius maps. We can consider \( a, b, c, d \in \mathbb{C} \) with \( ad \neq bc \).

Definition 2.7. We define a map \( f : \hat{\mathbb{C}} - \{-\frac{d}{c}\} \to \mathbb{C} \) by \( f(z) = \frac{az + b}{cz + d} \). (If \( ad = bc \) then \( f(\hat{\mathbb{C}}) = \{\infty\} \), a single point).

Remark 2.8. We can also assume without loss of generality that \( ad - bc = 1 \), since we see that replacing \( a, b, c, d \) to \( \lambda a, \lambda b, \lambda c, \lambda d \) gives the same map. We will adopt this convention when convenient.

Observe that if \( c = 0 \) then \( f(\infty) = \infty \), i.e., \( \infty \) is a fixed point. On the other hand, if \( c \neq 0 \) then letting \( z \mapsto -d/c \) we see that \( f(-d/c) = \infty \). Letting \( z \to +\infty \) we can write that \( f(\infty) = a/c \).

Lemma 2.9. If \( c \neq 0 \) then the Möbius map \( f \) is invertible, and its inverse is also a Möbius map (and thus \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is a homeomorphism).

Proof. If \( f(z) = \frac{az + b}{cz + d} \) with \( ad - bc = 1 \) then we can define \( f^{-1}(z) = \frac{dz - b}{cz + a} \). We can then explicitly check \( f \circ f^{-1}(z) = z \) and \( f^{-1} \circ f(z) = z \). For example,

\[
f(f^{-1}(z)) = \frac{a \left( \frac{dz - b}{cz + a} \right) + b}{c \left( \frac{dz - b}{cz + a} \right) + d}
\]

\[
= \frac{a (dz - b) + b (-cz + a)}{c (dz - b) + d (-cz + a)}
\]

\[
= \frac{(ad - bc) z}{(ad - bc)} = z
\]

Lemma 2.10. If \( f_1 \) and \( f_2 \) are Möbius maps then so is the composition \( f_1 \circ f_2 \).

Proof. This follows immediately by substitution. (Exercise)

In particular, the set of Möbius maps forms a group.
Example 2.11. There are four fundamental examples of Möbius transformations \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \).

a) **Translations:** \( z \mapsto z + b = \frac{1}{z} \) where \( b \in \mathbb{C} \). These are often called parabolic transformations.

b) **Rotations:** \( z \mapsto az = \frac{\sqrt{a\bar{z} + 0}}{0 + \sqrt{a}} \) where \(|a| = 1\). These are often called elliptic transformations.

c) **Expansions (and Contractions):** \( z \mapsto \lambda z = \frac{\sqrt{\lambda z + 0}}{0 + \sqrt{\lambda}} \) with \( \lambda > 1 \) (or \( 0 < \lambda < 1 \)) and \( \lambda \) real. These are often called hyperbolic transformations.

d) **Inversions:** \( z \mapsto \frac{1}{z} \).

We can use this to deduce the following.

**Lemma 2.12.** Every Möbius map can be written as a composition of Möbius maps of the above type.

**Proof.** Every Möbius map is a combination of three types of maps \( f(z) = Az, f(z) = z + B \) and \( f(z) = 1/z \), where \( A, B \in \mathbb{C} \), since we can write

\[
\frac{az + b}{cz + d} = \frac{a}{c} \left( cz + d \right) + \left( b - \frac{ad}{c} \right) = \frac{a}{c} + \left( b - \frac{ad}{c} \right)
\]

which is a composition of there

\[
z \mapsto cz \mapsto cz + d \mapsto \frac{1}{cz + d} \mapsto \left( b - \frac{ad}{c} \right) \left( \frac{1}{cz + d} \right) \mapsto \left( b - \frac{ad}{c} \right) \left( \frac{1}{cz + d} \right) + \frac{a}{c} \]

\[\square\]

The follow result is very useful. Consider circles in the complex plane of the form \( C := \{ z \in \mathbb{C} : |z - z_0| = r \} \) where \( z_0 \in \mathbb{C} \) and \( r > 0 \).

**Theorem 2.13.** The image of a circle or straight line under a Möbius transformation is a circle or straight line.

**Proof.** It suffices to consider three different types of Möbius transformations:

1. If \( f(z) = Az \) where \( A \in \mathbb{C} \) with \( A \neq 0 \) then the image \( f(C) \) is a circle.
2. If \( f(z) = z + B \) where \( B \in \mathbb{C} \) then the image \( f(C) \) is a circle.
3. If \( f(z) = 1/z \) then the image \( f(C) \) is a circle. We first observe if \( z = x + iy \) lies on a circle centred at \( z_0 = x_0 + iy_0 \) of radius \( r > 0 \) then

\[
(x - x_0)^2 + (y - y_0)^2 = r^2 \hspace{1cm} (1)
\]

Consider the set of \( z \) such that

\[
\frac{|z - p|}{|z - q|} = k \hspace{1cm} (2)
\]

where \( p, q \in \mathbb{C} \) and \( k > 0 \). If \( p = u + iv \) and \( q = s + it \) then this becomes:

\[
(x - u)^2 + (y - v)^2 = k^2 ((x - s)^2 + (y - t)^2) \hspace{1cm} (3)
\]

In particular, we can rewrite (1) in terms of (3) by choosing \( p, q, k \) such that

\[
x_0 = \frac{(u - k^2 s)}{1 - k^2} \hspace{1cm} y_0 = \frac{v - k^2 t}{1 - k^2} \hspace{1cm} r^2 = \frac{u^2 + v^2 - k^2 (s^2 + t^2)}{1 - k^2} + \left( \frac{u - sk^2}{1 - k^2} \right) + \left( \frac{v - tk^2}{1 - k^2} \right) > 0.
\]
Finally we can see that if \( z \) satisfies (2) then
\[
\frac{|z - p|}{|z - q|} = k \iff \frac{\frac{1}{z} - \frac{1}{p}}{\frac{1}{z} - \frac{1}{q}} = \frac{q}{p}
\]
eq k
\]
i.e., a version of (3) (with \( p \) replaced by \( \frac{1}{p} \), \( q \) replaced by \( \frac{1}{q} \) and \( k \) replaced by \( \frac{q}{p}k \)). In particular, the image of the circle given by (2) is a circle given by (3).

\[ \Box \]

**Theorem 2.14.** For distinct \( z_1, z_2, z_3 \in \hat{\mathbb{C}} \) and distinct \( w_1, w_2, w_3 \in \hat{\mathbb{C}} \) there exists a unique M"obius map \( f: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that \( f(z_i) = w_i \), for \( i = 1, 2, 3 \).

**Proof.** We first prove existence and then uniqueness.

**Existence.** Consider the case that \( z_1, z_2, z_3 \neq \infty \). Let
\[
S(z) = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}
\]
then we see that \( S(z_1) = 1, S(z_2) = 0, S(z_3) = \infty \). Consider
\[
T(z) = \frac{(z - w_2)(z_1 - w_3)}{(z - w_3)(z_1 - w_2)}
\]
then we see that \( T(w_1) = 1, T(w_2) = 0, T(w_3) = \infty \). If we define \( f(z) := S^{-1} \circ T(z) \) then we can then observe that \( f(z_1) := T^{-1} \circ S(z_1) = w_i \) (\( i = 1, 2, 3 \)).

In the case that \( z_1 = \infty \) then we let \( S(z) = \frac{z - z_2}{z_1 - z_3} \) and similarly for \( T \).

In the case that \( z_2 = \infty \) then we let \( S(z) = \frac{z - z_1}{z_3 - z_2} \) and similarly for \( T \).

In the case that \( z_3 = \infty \) then we let \( S(z) = \frac{z - z_1}{z_2 - z_3} \) and similarly for \( T \).

**Uniqueness.** We can assume without loss of generality that \( w_1 = 1, w_2 = 0, w_3 = \infty \). (Otherwise we can additionally compose with a M"obius map \( g: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) taking \( w_1, w_2, w_3 \) to \( 1, 0, \infty \), respectively, and then we can apply the following argument to show that \( f_1 \circ g^{-1} = f_2 \circ g^{-1} \), which therefore gives us \( f_1 = f_2 \)). Assume that \( f_j: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) are two M"obius maps for \( j = 1, 2 \) such that \( f_j(1) = z_1, f_j(0) = z_2, f_j(\infty) = z_3 \). Since \( S(z) = f_1^{-1} \circ f_2(z) \) is a M"obius transformation we can write
\[
S(z) = \frac{az + b}{cz + d}
\]
Moreover, \( f_1^{-1} \circ f_2: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) fixes the three points \( 1, 0, \infty \). In particular, we see that
\[
S(0) = b/d = 0 \implies b = 0
\]
\[
S(\infty) = a/c = \infty \implies c = 0, \text{ and}
\]
\[
S(1) = (a + b)/(c + d) = a/d = 1
\]
from which we deduce that \( S(z) = z \) for all \( z \), i.e., \( f_1 = f_2 \).

Let us consider a couple of examples of this result.

**Example 2.15.** Find the M"obius transformation \( f \) which maps \( -1, 0, 1 \) to the points \(-i, 1, i\). Assume that \( f(z) = \frac{az + b}{cz + d} \). Since \( f(0) = \frac{b}{d} = 1 \) we have \( b = d \) and \( f(z) = \frac{az + b}{cz + b} \).

Similarly, since \( f(-1) = \frac{-ia + b}{-ic + b} = 1 \implies ic - ib = -a + b \) and \( f(1) = \frac{ia + b}{ic + b} = i \implies ic + ib = a + b \). Adding the last two equations gives \( c = -ib \) and subtracting gives \( a = ib \). Thus
\[
f(z) = \frac{ibz + b}{-ibz + b} = \frac{iz + 1}{-iz + 1} = \frac{i - z}{i + z}
\]
(Formally, we should also multiply the coefficients by constant to get \( ad - bc = 1 \).}

*2. A FEW BASIC IDEAS* 11
Example 2.16. Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disk unit and let \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) be the upper half-plane. Find a surjective Möbius map \( f : \mathbb{H} \to \mathbb{D} \).

Actually, the map in the previous example works (as one might guess from the three points in the boundary of \( \mathbb{H} \) being mapped to three points in the boundary of \( \mathbb{D} \)). To see this, let \( z = x + iy \) with \( y > 0 \) then

\[
|f(z)| = \left| \frac{i - (x + iy)}{i + (x + iy)} \right|^2 = \left| \frac{-x + i(1 - y)}{x + i(y + 1)} \right|^2 = \frac{x^2 + (1 - y)^2}{x^2 + (1 + y)^2} < 1,
\]
i.e., \( f(z) \in \mathbb{D} \).

Example 2.17. Fix \( a \in \mathbb{C} \) and then we can define a Möbius map \( f : \mathbb{C} \to \mathbb{C} \) by

\[
f(z) = \frac{a - z}{1 - az}
\]

Notice that

\[
|f(z)|^2 = \frac{|a - z|^2}{|a - az|^2} = \frac{|a|^2 - 2Re(az) + |z|^2}{1 - 2Re(az) + |az|^2}
\]

It is easy to check that if \( |z| = 1 \) if and only if \( |f(z)| = 1 \). Moreover, if \( |f(z)| < 1 \) is equivalent to

\[
|a|^2 + |z|^2 < 1 + |az|^2 < 1 \iff (1 - |a|^2)(1 - |z|^2) > 0 \iff |z| < 1
\]

since \( |a| < 1 \). We can conclude that \( f : \mathbb{D} \to \mathbb{D} \).

2.3. Two applications.

Application 2.18 (Apollonion Circle Packings). This is related to taking three circles in the plane.

Theorem 2.19 (Apollonius). Given three mutually tangent circles \( C_1, C_2, C_3 \) there are precisely two circles \( C, C' \) tangent to all three.

Proof. We can write this in a number of steps.

Step 1: Let \( \xi \) be the point of intersection \( C_1 \cap C_2 \) of two of the circles. We can apply the Möbius transformation \( f(z) = \frac{1}{z - \xi} \) then \( f(\xi) = \infty \) and \( C_1 - \{ \xi \} \) and \( C_2 - \{ \xi \} \) are mapped to disjoint (and thus parallel) straight lines \( L_1 \) and \( L_2 \).

Since \( C_1 \) and \( C_3 \) are tangent at one point \( \eta \), say, and \( C_2 \) and \( C_3 \) are tangent at one point \( \rho \), say, then the image \( C_4 = f(C_3) \) is another circle tangent to \( L_1 \) and \( L_2 \).
Step 2: We can trivially translate $C_4$ between the two parallel lines $L_1$, $L_2$ to precisely two more circles $C_5$ and $C_6$ of the same radius such that: $C_5$ and $C_6$ are both tangent to both $L_1$ and $L_2$ and $C_3$.

Step 3: We then define $C = f^{-1}(C_5)$ and $C' = f^{-1}(C_6)$. Since again Möbius transformations take circles to circles (where straight lines are understood as circles passing through infinity) we deduce that $C, C'$ are the only two such circles tangent to $C_1, C_2, C_3$. □

Theorem 2.20. Descartes Theorem: Given four mutually tangent circles $C_1, C_2, C_3$ and $C_4$ whose radii are $r_1, r_2, r_3, r_4$ then $F(r_1, r_2, r_3, r_4) = 0$ where

$$F(r_1, r_2, r_3, r_4) = 2(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} - (\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2})^2)$$ (1)

Proof. We begin with two simple estimates. Consider a circle $E$ of radius $k$.

1. We can reflect in it a second circle $C$ of radius $r$ whose centres are separated by $d > r + k$. In particular, the circles are disjoint and have disjoint interiors. The image circle $f(E')$ with have radius $k^2 r/(d^2 - r^2)$.

2. Consider next a straight line $L$ at a distance $b > k$ from the centre of $E$. We can reflect it in the circle $E$ and the image is a circle of radius $k^2/2b$. 
Step 1. Let $C_1, C_2, C_3$ and $C_4$ be four mutually tangent circles. Choose $E$ to be a (large) circle centred at the tangency point $\xi = (x_0, y_0)$ between $C_1$ and $C_2$.

We can invert the four circles in $E$ by $f$, say, to arrive at a configuration (after some rotation and translation) with: $L_1 = f(C_1)$ a line given by $y = 1$; $L_2 = f(C_2)$ a line given by $y = -1$; and $C'_3 = f(C_3)$ and $C'_4 = f(C_4)$ being circles of radius 1 with centres $(-1, 0)$ and $(1, 0)$, respectively, i.e., the straight lines are “circles” with radius $r_1 = r_2 = \infty$ and the other two circles have radius $r_3 = r_4 = 1$. In particular, we observe that it satisfies (1).

Step 2. We can apply the two estimates to the lines $L_1, L_2$ and the circles $C'_3, C'_4$ to obtain

$$r(C_3) = \frac{k^2}{x_0^2 - 2x_0 + y_0^2}, \quad r(C_4) = \frac{k^2}{x_0^2 + 2x_0 + y_0^2}, \quad r(C_2) = \frac{k^2}{2(y_0 - 1)} \quad \text{and} \quad r(C_1) = \frac{k^2}{2(y_0 + 1)}$$

The result follows by substituting. \qed

Proceeding inductively we can construct circle packings.

Exercise 2.21. If a Möbius map $f(z) = \frac{az + b}{cz + d}$ maps real numbers to real numbers then show that $a, b, c, d$ are all real numbers.

Show that $f : \mathbb{H} \to \mathbb{H}$ is a bijection.

Application 2.22 (Hyperbolic Half-plane). The upper half plane $\mathbb{H}$ has the Poincaré metric written as $(dx^2 + dy^2)/y^2$, i.e., it is similar to the usual Euclidean
metric, except at $z = x + iy$ the distance is scaled by $1/y$. We claim that this is preserved under Möbius maps.

We want to show that any path $\sigma$ has the same length as its image $f \sigma$. If we let $f(z) = \frac{az + b}{cz + d}$ then we can write

$$f'(z) = \frac{a}{cz + d} - \frac{c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} = \frac{1}{(cz + d)^2}$$

and

$$\text{Im}(g(z)) = \text{Im}\left(\frac{az + b}{cz + d}\right) = \frac{1}{|cz + d|^2} \text{Im}(z)$$

By the chain rule

$$\text{length}(g \gamma) = \int \frac{|(g \circ \sigma'(t))|}{\text{Im}g \circ \sigma(t)} dt$$

$$= \int \frac{|(g'(\sigma'(t))||\sigma'(t)|}{\text{Im}g \circ \sigma(t)} dt$$

$$= \int \frac{1}{|c\sigma(t) + d|^2} |\sigma'(t)| c\sigma(t) + d|^2 \frac{1}{\text{Im}c\sigma(t)} dt$$

$$= \int |\sigma'(t)| dt = \text{length}(\gamma)$$

Alternatively, if $z_1, z_2 \in \mathbb{H}$ then can show that

$$d(z_1, z_2) = 2 \tanh^{-1} \left| \frac{z_1 - z_2}{z_1 - \bar{z}_2} \right|$$

and show that this is preserved by Möbius maps, i.e.,

$$|g(z_1) - g(z_2)| = \left| \frac{az_1 + b}{cz_1 + d} - \frac{az_2 + b}{cz_2 + d} \right|$$

$$= \left| \frac{(az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d)}{(cz_1 + d)(cz_2 + d)} \right|$$

$$= \left| \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)} \right|$$

Thus

$$\left| \frac{g(z_1) - g(z_2)}{g(z_1) - g(z_2)} \right| = \left| \frac{z_1 - z_2}{z_1 - \bar{z}_2} \right| \frac{(cz_1 + d)(cz_2 + d)}{(cz_1 + d)(cz_2 + d)} = 1$$

2.4. Classification of Möbius maps and their behaviour. We would like to understand better the behaviour of different types of Möbius maps. We begin with a simple result.

**Lemma 2.23.** Möbius maps other than the identity must fix at least one point and at most two.

**Proof.** Assume that $f(z) = \frac{az + b}{cz + d} = z$ then this is equivalent to $cz^2 + (d - a)z - b = 0$. The solution to this quadratic equation is

$$z = \left( (a - d) \pm \sqrt{(a - d)^2 + 4bc} \right) / 2c \in \mathbb{C}.$$  

Unless $f$ is the identity ($a = c$ and $b = d = 0$) this has two roots (possibly a single root repeated if $(a - d)^2 + 4bc = 0$).

We would like to classify Möbius maps up to conjugacy.

**Definition 2.24.** We say that two Möbius maps $f_1, f_2$ are conjugate if $f_2 = g^{-1}f_1g$ for some Möbius map $g$.

**Definition 2.25.** We can associate to the Möbius map $f(z) = \frac{az + b}{cz + d}$ (satisfying $ad - bc = 1$) the value $\text{tr}(f) := a + d$. 


Lemma 2.26. The value $tr(f)$ is preserved by conjugacy, i.e., if $f_1, f_2$ are conjugate then $tr(f_1) = tr(f_2)$

Proof. Since every Möbius map $g$ can be written as a composition of basic Möbius transformations (translation, inversion and rotation) it suffices to show the result where $g$ is of each type.

Translations: if $g(z) = z + \beta$ then we can explicitly write

$$f_2(z) = g^{-1}f_1g(z) = \frac{a(z - \beta) + b}{c(z - \beta) + d + \beta}$$

$$= \frac{a(z - \beta) + b + \beta(c(z - \beta) + d)}{c(z - \beta) + d}$$

and we see that $tr(f_2) = (a + \beta c) + (d - \beta c) = a + d = tr(f_1)$.

Inversions: If $g(z) = 1/z$ then

$$f_2(z) = g^{-1}f_1g(z) = \frac{a/z + b}{c/z + d}$$

$$= \frac{a + bz}{c + dz}$$

and we see that $tr(f_2) = d + a = tr(f_1)$.

Rotations: If $g(z) = \alpha z$ then

$$f_2(z) = g^{-1}f_1g(z) = \frac{1}{\alpha} \left( \frac{a\alpha z + b}{c\alpha z + d} \right)$$

$$= \frac{az + b/\alpha}{c\alpha z + d}$$

and we see that $tr(f_2) = a + d = tr(f_1)$.

Definition 2.27. We say that a rational map is parabolic if it has precisely one fixed point in $\hat{\mathbb{C}}$.

In particular, being parabolic is easily seen to be a conjugacy invariant (i.e., if $f_1$ has a single fixed point $z_0$, say, then $f_2$ has a single fixed point $g^{-1}z_0$. In particular, if $f_1$ has a fixed point $f_1(z_0) = z_0$ we can choose $g$ so that $g(\infty) = z_0$ and then the conjugate map $f := f_2$ fixes $\infty$.

Lemma 2.28. Let $f$ be a Möbius map with $f(\infty) = \infty$.

1. Then $f$ is of the form $f(z) = \alpha z + \beta$;
2. $f$ has a second distinct fixed point (i.e., it is not parabolic) if and only if
   $\alpha \neq 1$;
3. if the second fixed point is 0 (i.e., $f(0) = 0$) then $f(z) = \alpha z$.

Proof. (1) Since $f(\infty) = \infty$ this immediately implies that $f_2(z) = \alpha z + \beta$, for some $\alpha, \beta \in \mathbb{C}$.

(2) If $f$ has a second point $z_0$, say, (other than $\infty$) then $f(z_0) = \alpha z_0 + \beta = z_0$, i.e., $z_0 = \frac{-\beta}{\alpha - 1}$ with $\alpha \neq 1$. In particular, if $\alpha \neq 1$ then $f$ has a second fixed point (in addition to $\infty$). Conversely, if $\alpha = 1$ then $f$ has no other fixed points except $\infty$ since $f(z) = z + \beta$ is a straightforward translation. (In the degenerate case $\beta = 0$ this is just the identity.)

(3) Since $f(0) = 0$ we see that $\beta = 0$ and the result follows.
Remark 2.29. In fact Möbius transformations \( f_1, f_2 \) are conjugate if and only if \( \text{tr}(f_1)^2 = \text{tr}(f_2)^2 \).

The non-identity Möbius transformations are commonly classified into three types:

1. parabolic (conjugate to \( z \mapsto z + \beta \) with a single fixed point \( \infty \));
2. elliptic (conjugate to \( z \mapsto \alpha z \) with \(|\alpha| = 1\) with points \( 0, \infty \)); and
3. loxodromic (everything else).

The hyperbolic transformations are those conjugate to the expanding and contracting maps \( z \mapsto \lambda z \) (with \( 0 < \lambda < 1 \) or \( 1 < \lambda < +\infty \)) being a subclass of the loxodromic ones.

Lemma 2.30. Let \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a non-trivial Möbius transformation. It is

1. a parabolic Möbius transformation if and only if \( \text{tr}(f) = \pm 2 \);
2. an elliptic Möbius transformation if and only if \( -2 < \text{tr}(f) < 2 \);
3. a hyperbolic Möbius transformation if and only if \( \text{tr}(f) > 2 \).

Proof. Assume that \( f(\infty) = \infty \) (else replace by a conjugate map with this property).

Case I: \( f \) is parabolic. By definition, in this case \( f \) has no other fixed points and we see from the lemma see that \( f(z) = z + \beta \). This always corresponds to \( \text{tr}(g) = 1 + 1 = 2 \) (or \(-2\) since we recall that multiplying \( a, b, c, d \) by \(-1\) gives the same map).

Case II: \( f \) is not parabolic. In this case, \( f \) has a second fixed point. Assume that \( f(0) = 0 \) (else we can replace \( f \) by a conjugate map with this property) and then by the lemma we can write \( f(z) = \alpha z \). Then we require \( a = d^{-1} = \sqrt{\alpha} \) (to satisfy \( ad - bc = ad = 1 \)).

We can now consider the different values of the trace \( \text{tr}(f) = a + \frac{1}{a} \).

1. If \( \text{tr}(f) = \pm 2 \) then \( a + 1/a = \pm 2 \) and so \( a(a^2 \pm 2a + 1) = (a \pm 1)^2 = 0 \). But \( a = \pm 1 \) which means that \( f(z) = z \), the identity map, which is excluded by the non-triviality assumption.

The proofs of parts (2) and (3) are slightly jumbled up.

(a) More generally, if \( a = re^{i\theta} \) then

\[
\text{tr}(f) = a + \frac{1}{a} = re^{i\theta} + \frac{e^{-i\theta}}{r} = (r + 1/r) \cos \theta + i(r - 1/r) \sin \theta
\]

If we assume that the trace in (1) is real (i.e., \( \text{Im}(\text{tr}(f)) = (r - 1/r) \sin \theta = 0 \) then this is equivalent to having either \( r = 1/r (= 1) \) or \( \theta = 0, \pi \) or both. In the first case, if \(|a| = r = 1\) then \( f \) is elliptic, and \( \text{tr}(f) = 2 \cos \theta \in (-2, 2) \). In the second case, if \( \theta = 0 \) or \( \pi \) then \( f \) is hyperbolic and \( \text{tr}(f) = r + 1/r \geq 2 \).

(b) If we assume that the trace in (1) has a non-zero imaginary part (i.e., \( \text{tr}(f) \not\in \mathbb{R} \)) then this is equivalent to requiring both \( r \neq 1/r \) and \( \theta \neq 0, \pi \). But then since \(|a| = r \neq 1\) this means that the map is not elliptic (as well as not being parabolic). Thus by definition it is loxodromic.

Remark 2.31. A loxodromic Mobius map is a composition of a hyperbolic and elliptic maps. In addition, a loxodromic Mobius map has that \( \text{tr}(f) \in \mathbb{C} - [-2, 2] \).

Exercise 2.32. Show that if \( f \) is hyperbolic then \( f^n(z) \) converges to one of the fixed points.
2.5. cross-ratios of quadruples of complex numbers. We begin with the definition of a very elegant expression.

**Definition 2.33.** Given four distinct complex numbers $z_0, z_1, z_2, z_3 \in \hat{\mathbb{C}}$ we define the cross ratio by

$$(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}.$$

Recall that we have already proved a theorem to the effect that can find a (unique) Möbius map $f(z) = \frac{az + b}{cz + d}$ that maps any three distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, say, to the reference points $1, 0, \infty$ in the same order.

**Lemma 2.34.** Let $z_0, z_1, z_2, z_3 \in \hat{\mathbb{C}}$ be four distinct points. We have that

$$(z_0, z_1, z_2, z_3) = f(z_0)$$

i.e., the image of $z_0$ under the unique Möbius map which takes $z_1, z_2, z_3$ to $1, 0, \infty$.

The following is trivial.

**Example 2.35.** $(z, 1, 0, \infty) = z$ since the associated Möbius map is simply $f(z) = z$.

The next lemma shows that the cross ratio is preserved by Möbius maps.

**Lemma 2.36.** If $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is (another) Möbius map then

$$(z_0, z_1, z_2, z_3) = (gz_0, gz_1, gz_2, gz_3).$$

**Proof.** Exercise. □

The Möbius transformation has an interesting geometric interpretation:

**Theorem 2.37.** If $z_1, z_2, z_3$ are three distinct points in $\hat{\mathbb{C}}$ then let $C$ denote the unique circle (or line, if one of the point is equal to $\infty$) passing through them. Then $(z_0, z_1, z_2, z_3) \in \mathbb{R}$ if and only if $z_0 \in C$.

**Proof.** Let $f(z) = (z, z_1, z_2, z_3)$ be the associated Mobius map, then we want to show that $f^{-1}(\mathbb{R} \cup \{\infty\}) = C$. Suppose $w \in f^{-1}(\mathbb{R} \cup \{\infty\})$ or, equivalently, that $f(w) \in \mathbb{R} \cup \{\infty\}$.

**Case 1** ($f(w) = \infty$): In this case since

$$f(z) = \frac{z - z_2}{z - z_3}$$

we deduce that this is equivalent $w = z_3 \in C$.

**Case 2** ($f(w) \neq \infty$): In this case $w \neq z_3$ then since by assumption $f(w) \in \mathbb{R}$ we see that $f(w) = \overline{f(w)}$ and thus

$$\frac{aw + \overline{b}}{cw + \overline{d}} = \frac{aw + b}{cw + d} \iff (\overline{a} \overline{c} - \overline{a}c)|w|^2 + (\overline{a}d - \overline{b}c)w + (\overline{b}c - \overline{a}d)\overline{w} + (\overline{d}d - \overline{d}b) = 0.$$  

This brings us to two subcases.

**Case 2a** ($a\overline{c} = \overline{a}c$): In this case the equation is of the form

$$A w - \overline{A}w + i k = 0$$

where $k$ is real and $A = \overline{a}d - \overline{b}c$. Rewriting this as

$$iAw - i\overline{A}w + k = 0$$

this is the equation of straight line (cf. the usual form $Bw + \overline{B}w + C = 0$).

**Case 2b** ($a\overline{c} \neq \overline{a}c$): If $a\overline{c} = \overline{a}c$ then we can write

$$|w|^2 + \frac{(\overline{a}d - \overline{b}c)}{a\overline{c} - \overline{a}c} w + \frac{(\overline{b}c - \overline{a}d)}{a\overline{c} - \overline{a}c} \overline{w} = \frac{d\overline{d} - \overline{d}b}{a\overline{c} - \overline{a}c} = 0.$$
Completing the square we can write
\[ |w - (\frac{ad - be}{ac - bc})|^2 = |w|^2 + \frac{(ad - be)}{ac - bc} w + \frac{(bc - ad)}{ac - bc} \frac{w^2 + (ad - be)^2}{ac - bc} = \frac{a \overline{a} - bc}{ac - bc} + \frac{(ad - bc)^2}{ac - bc} |^2 = R^2. \]

In particular, this is seen to be the equation of a circle of radius \( R \) provided the right hand side is positive. We can rewrite this as
\[ \frac{ad - db}{a \overline{c} - c \overline{c}} \frac{(ad - bc)(ac - \overline{b}a)}{ac - bc} = \frac{|a|^2|d|^2 + |b|^2|c|^2 - \overline{abc}d - a \overline{bcd}}{|ac - bc|} \]

Thus it suffices to observe that we can write the numerator as
\[ (ad - bc)(\overline{a}d - \overline{b}c) = |ad - be|^2 > 0 \]
using that \( ad - bc \neq 0 \).

In either case we see that \( f^{-1}(\mathbb{R} \cup \{\infty\}) \) is a circle or straight line \( \square \).

As an immediate corollary we have another presentation of the proof that Möbius transformation's preserve circles (although the proof is essentially the same as before).

**Corollary 2.38.** If \( f \) is a Möbius transformation then it maps circles (or lines) to circles (or lines).

**Proof.** Let \( C \) be a circle (or line) determined by three distinct points \( z_1, z_2, z_3 \in \mathbb{C} \). Then
\[ z_0 \in C \iff (z_0, z_1, z_2, z_3) \in \mathbb{R} \]
\[ \iff (f(z_0), f(z_1), f(z_2), f(z_3)) \in \mathbb{R} \]
\[ \iff (f(z_0), f(z_1), f(z_2), f(z_3)) \in f(C) = C' \]
where \( C' \) is a circle (or line). In particular, the circle (or line) for \( (z_1, z_2, z_3) \) is mapped to the circle \( (f(z_1), f(z_2), f(z_3)) \). \( \square \)

Finally, the following is an interesting geometric interpretation of cross ratios.

**Remark 2.39 (Cross ratios and hyperbolic geometry).** We let
\[ \mathbb{H}^3 = \{ (z, t) : z = x + iy \in \mathbb{C}, t > 0 \} \]
denote the three dimensional upper half-space with metric
\[ ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2} \]
We can consider a geodesic \( \gamma : \mathbb{R} \to \mathbb{H}^3 \) with end points
\[ z_1 = \lim_{t \to -\infty} \gamma(t) := \gamma(-\infty) \quad \text{and} \quad z_2 = \lim_{t \to \infty} \gamma(t) := \gamma(+\infty) \]
More precisely, for any two distinct \( t_1 < t_2 \) the curve \( [t_1, t_2] \supset t \mapsto \gamma(t) \) is the shortest path between \( \gamma(t_1) \) and \( \gamma(t_2) \) in \( \mathbb{H}^3 \). It appears a semi-circular arc in \( \mathbb{H}^3 \).

Given \( z_3, z_4 \in \mathbb{C} \) we can consider the geodesic \( \gamma' \) with end points \( z_3 = \gamma'(-\infty) \) and \( z_4 = \gamma'(\infty) \). We denote the Hausdorff distance between the curves \( \gamma \) and \( \gamma' \) by
\[ d(\gamma, \gamma') = \inf_{t_1, t_2 \in \mathbb{R}} d(\gamma(t_1), \gamma'(t_2)) \]
The connection between the cross ratio of the four complex numbers \( z_1, z_2, z_3, z_4 \) (i.e., the end points) and the geometry is the following.

**Lemma 2.40 (Fenchel and Ahlfors).** \( |(z_1, z_2, z_3, z_4)| = \tanh(d(\gamma, \gamma')) \)
The argument of \((z_1, z_2, z_3, z_4)\) also has a geometric interpretation. It corresponds to the change in the angle by parallel transporting a frame along the geodesic realizing this distance.

3. Analyticity

We want to recall the definition(s) of analyticity of functions \(f : U \to \mathbb{C}\).

**Definition 3.1.** Assume that \(U \subset \mathbb{C}\) is an open set, i.e., for every \(z_0 \in U\) there exists \(\epsilon > 0\) such that

\[
B(z_0, \epsilon) = \{ z \in \mathbb{C} : |z - z_0| < \epsilon \} \subset U.
\]

It is usually also convenient to additionally assume that:

1. \(U\) is path connected (i.e., for any two points \(z, w \in U\) we can find a continuous path \(\gamma : [0, 1] \to U\) such that \(\gamma(0) = z\) and \(\gamma(1) = w\)).

2. \(U\) is simply connected (i.e., any closed path \(\gamma : [0, 1] \to U\) with \(\gamma(0) = \gamma(1)\) can be contracted to a single point or, equivalently, \(U\) is homeomorphic to the unit disk).

We will assume that \(U\) always has these properties, unless we explicitly state otherwise, and frequently call it a domain.

We will present three equivalent definitions once we have introduced three new lots of notation.

3.1. Ingredients for the first definition (using power series). We want to start by recalling the more intuitive definition of analytic functions.
**Definition 3.2.** Let \( z_0 \in \mathbb{C} \) and let \( (a_n)_{n=0}^\infty \) be a sequence of complex numbers. We say that an infinite series \( \sum_{n=0}^\infty a_n(z-z_0)^n \) has a radius of convergence \( 0 \leq R \leq +\infty \) (at \( z_0 \)) if
\[
\frac{1}{R} = \limsup_{n \to +\infty} |a_n|^{1/n}.
\]
(1)

In particular, we have the following:

**Lemma 3.3.** Assume that \( R > 0 \). For any \( 0 < r < R \) the series
\[
f_{z_0}(z) := \sum_{n=0}^\infty a_n(z-z_0)^n
\]
converges (uniformly) on \( B(z_0, r) \) and defines a function.

**3.2. Ingredients for the second definition (using complex differentiability).** We begin by what it means for a function to be differentiable as a complex function.

**Definition 3.4.** Let \( U \) be a domain. We say that \( f : U \to \mathbb{C} \) is complex differentiable at \( z_0 \) if the limit
\[
f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}
\]
exists, i.e., there exists \( f'(z_0) \in \mathbb{C} \) such that for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( |z - z_0| < \delta \) then
\[
|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \epsilon|z - z_0|.
\]

**Remark 3.5.** The key point is that \( z \) can approach \( z_0 \) in "all directions" as a complex number.

It is easy to see the following result.

**Lemma 3.6.** If \( f \) is complex differentiable at \( z_0 \) then it is continuous at \( z_0 \).

**Proof.** Given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( |z - z_0| < \delta \) then we have that
\[
|f'(z_0) - \frac{f(z) - f(z_0)}{z - z_0}| < \epsilon.
\]
In particular, for \( |z - z_0| < \delta \) we can bound
\[
|f(z) - f(z_0)| \leq (|f'(z_0)| + \epsilon)|z - z_0| \leq (|f'(z_0)| + \epsilon)\delta,
\]
which can be made arbitrarily small by choosing \( \delta > 0 \) appropriately small. \( \square \)

**3.3. Ingredients for the third definition (using the Cauchy-Riemann Equations).** We can write a complex function \( f : U \to \mathbb{C} \) in terms of its real and imaginary parts
\[
f(x + iy) = u(x, y) + iv(x, y)
\]
where \( z = x + iy \in U \) and \( u(x, y), v(x, y) \in \mathbb{R} \).
Definition 3.7. We say that \( f : U \to \mathbb{C} \) satisfies the Cauchy-Riemann equations if for each \( z_0 = x_0 + iy_0 \in U \) the partial derivatives

\[
\frac{\partial u}{\partial x}(x_0, y_0), \frac{\partial u}{\partial y}(x_0, y_0), \frac{\partial v}{\partial x}(x_0, y_0) \text{ and } \frac{\partial v}{\partial y}(x_0, y_0)
\]

all exist and satisfy

\[
\begin{align*}
\frac{\partial u}{\partial x}(x_0, y_0) &= \frac{\partial v}{\partial y}(x_0, y_0) \\
\frac{\partial u}{\partial y}(x_0, y_0) &= -\frac{\partial v}{\partial x}(x_0, y_0)
\end{align*}
\]

for each \( z_0 = x_0 + iy_0 \in U \). Equivalently, we can write \( f(z) = u(x,y) + iv(x,y) \) and

\[
i \frac{\partial f}{\partial x} = i \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \quad \text{and} \quad \frac{\partial f}{\partial y} = \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial x} \right)
\]

using the Cauchy-Riemann equations. Thus we see that

\[
i \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}
\]

Remark 3.8. We can write this as \( \frac{\partial f}{\partial \bar{z}} = 0 \) using the Wirtinger derivatives

\[
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right).
\]

4. Definitions of analyticity

We are now in a position to give the equivalent definitions of analyticity.

4.1. The main result. We say that a function \( f : U \to \mathbb{C} \) is analytic if it satisfies any (and thus all) of the following three equivalent conditions.

Theorem 4.1. The following are equivalent:

1. For each \( z_0 \in U \) we can choose \( \epsilon > 0 \) and \( (a_n)_{n=0}^\infty \) such that the function \( f(z) \) can be represented as a convergent power series about the point \( z_0 \), i.e.,

\[
f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n
\]

for \( z \in B(z_0, \epsilon) \subset U \) and with radius of convergence \( R > \epsilon > 0 \);

2. The function \( f : U \to \mathbb{C} \) is complex differentiable at each \( z_0 \in U \);

3. The partial derivatives \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \) all exist, are continuous and satisfy the Cauchy-Riemann equations.

We are not yet in a position to prove the theorem completely (since we will still need to recall the Cauchy theorem for integrals). However, we will proceed with the proofs of the parts that we can prove, and then we will return to the remaining parts once we have established Cauchy’s Theorem.
Proof. (1) $\implies$ (2). The proof of part (2) assuming part (1) is easy. More
precisely, given $z_0 \in U$ we can write down its expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for $z \in B(z_0, r)$ in a sufficiently small neighbourhood of $z_0$. (i.e., $r < R$)

Let $R > 0$ be the radius of convergence of a series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$.

We can choose $0 < r < R$ then for $h \in \mathbb{C}$ sufficiently small

$$\frac{f(z_0 + h) - f(z_0)}{h} - a_1 = \frac{f(z_0 + h) - a_0}{h} - a_1$$

$$= \sum_{n=2}^{\infty} \frac{a_n h^{n-1}}{g_n(h)} = \sum_{n=2}^{\infty} g_n(h)$$

(1)

where we denote

$$g_n(h) = \begin{cases} a_n h^{n-1} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

for $n \geq 2$. In the disc $\{h \in \mathbb{C} : |h| < r\}$ we want to show that

$$\lim_{h \to 0} \sum_{n=2}^{\infty} g_n(h) = 0,$$

(i.e., $\sum_{n=2}^{\infty} g_n(h)$ is continuous at 0) from which we deduce, together with (1), that $f(z)$ is differentiable at $z_0$. For $n \geq 2$, we can bound $|g_n(h)| \leq |a_n|r^{n-1}$. Now since $r < R$ we have that

$$\sum_{n=2}^{\infty} |a_n|r^{n-1} < +\infty$$

By uniform convergence we can interchange the summation and the limit to get

$$\lim_{h \to 0} \sum_{n=2}^{\infty} g_n(h) = \lim_{h \to 0} \sum_{n=2}^{\infty} a_n h^{n-1} = 0.$$

(2) $\implies$ (1). To prove (1) assuming (2) requires the Cauchy theorem, so we will
return to this later (after we recall that theorem).
(2) \implies (3). If \( f(z) = u(x,y) + iv(x,y) \) is a complex differentiable function of a complex number \( z = x + iy \in \mathbb{C} \) then the derivative at \( z_0 = x_0 + iy_0 \) is given by
\[
\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)
\]

If we choose \( h = t \in \mathbb{R} \) to be real then this becomes
\[
\lim_{t \to 0} \frac{f(z_0 + t) - f(z_0)}{t} = \frac{\partial f}{\partial x}(z_0)(= f'(z_0))
= \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0)
\]

If we choose \( h = it \) where \( t \in \mathbb{R} \) to be purely imaginary then this becomes
\[
\lim_{t \to 0} \frac{f(z_0 + it) - f(z_0)}{it} = \frac{1}{i} \frac{\partial f}{\partial x}(z_0)(= f'(z_0))
= -i \left( \frac{\partial u}{\partial y}(x_0,y_0) + i \frac{\partial v}{\partial y}(x_0,y_0) \right)
\]

Complex differentiability means that there is only one derivative and we can equate the real parts and imaginary parts of these identities and write
\[
\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0)
\]
(from the real parts) and
\[
\frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0)
\]
(from the imaginary parts). This corresponds to the Cauchy Schwartz identities.

We postpone the proof of continuity of the derivatives (until after we recall Cauchy’s theorem).
(3) \implies (2). We can write
\[
\frac{f(z) - f(z_0)}{z - z_0} = \frac{(u(x, y) - u(x_0, y_0)) + i(v(x, y) - v(x_0, y_0))}{z - z_0}.
\]

For each \(|x - x_0|, |y - y_0|\) sufficiently small we can write
\[
|u(x, y) - u(x_0, y_0)| = |z - z_0| \left| \epsilon_1(x_0, y_0) + \epsilon_2(x, y) \right|,
\]
and the mean value theorem, where \(\epsilon_1(\cdot), \epsilon_2(\cdot)\) are bounds on the partial derivatives.

Since \(|x - x_0|, |y - y_0| \leq |z - z_0|\) then for \(|z - z_0|\) sufficiently small we can bound the differences by
\[
|u(x, y) - u(x_0, y_0)| \leq |u(x, y) - u(y, y_0)| + |u(x, y) - u(x, y_0)|
\]
and
\[
|v(x, y) - v(x_0, y_0)| \leq |v(x, y) - v(x_0, y_0)| + |v(x, y) - v(x_0, y_0)|
\]
by using the triangle inequality
\[
|u(x, y) - u(x_0, y_0)| \leq |u(x, y) - u(y, y_0)| + |u(x, y) - u(x, y_0)|
\]
\[
|v(x, y) - v(x_0, y_0)| \leq |v(x, y) - v(x_0, y_0)| + |v(x_0, y_0) - v(x_0, y_0)|
\]
and the mean value theorem, where \(\epsilon_1(\cdot), \epsilon_2(\cdot)\) are bounds on the partial derivatives.

Moreover, continuity of the partial derivatives means that \(\epsilon_1(\cdot), \epsilon_2(\cdot)\) \to 0 as \(z \to z_0\).

Since \(|x - x_0|, |y - y_0| \leq |z - z_0|\) then for \(|z - z_0|\) sufficiently small we can bound the differences by
\[
|u(x, y) - u(x_0, y_0)| \leq |u(x, y) - u(y, y_0)| + |u(x, y) - u(x, y_0)|
\]
and
\[
|v(x, y) - v(x_0, y_0)| \leq |v(x, y) - v(x_0, y_0)| + |v(x_0, y_0) - v(x_0, y_0)|
\]
by using the triangle inequality
\[
|u(x, y) - u(x_0, y_0)| \leq |u(x, y) - u(y, y_0)| + |u(x, y) - u(x, y_0)|
\]
and the mean value theorem, where \(\epsilon_1(\cdot), \epsilon_2(\cdot)\) are bounds on the partial derivatives.

Moreover, continuity of the partial derivatives means that \(\epsilon_1(\cdot), \epsilon_2(\cdot)\) \to 0 as \(x, y \to (x_0, y_0)\).

By the Cauchy-Riemann equations we have that
\[
(x - x_0) \frac{\partial u}{\partial x} (x_0, y_0) + (y - y_0) \frac{\partial u}{\partial y} (x_0, y_0) + i \left( (x - x_0) \frac{\partial v}{\partial x} (x_0, y_0) + (y - y_0) \frac{\partial v}{\partial y} (x, y) \right)
\]
\[
= (x - x_0) \left( \frac{\partial u}{\partial x} (x_0, y_0) + i \frac{\partial v}{\partial x} (x_0, y_0) \right) + (y - y_0) \left( \frac{\partial u}{\partial y} (x_0, y_0) + i \frac{\partial v}{\partial y} (x, y) \right)
\]
\[
= (x - x_0) \left( \frac{\partial u}{\partial x} (x_0, y_0) + i \frac{\partial v}{\partial x} (x_0, y_0) \right) + (y - y_0) \left( \frac{\partial u}{\partial y} (x_0, y_0) + i \frac{\partial v}{\partial y} (x, y) \right)
\]
\[
= (x - x_0) \left( \frac{\partial u}{\partial x} (x_0, y_0) + i \frac{\partial v}{\partial x} (x_0, y_0) \right) + (y - y_0) \left( \frac{\partial u}{\partial y} (x_0, y_0) + i \frac{\partial v}{\partial y} (x, y) \right)
\]
\[
= \frac{z - z_0}{(x - x_0) + i(y - y_0)} \left( \frac{\partial u}{\partial x} (x_0, y_0) + i \frac{\partial v}{\partial x} (x, y) \right)
\]
using the Cauchy-Riemann equations. Comparing (1) and (2) we have that
\[
\frac{f(z) - f(z_0)}{z - z_0} = \left( \frac{\partial u}{\partial x} (x_0, y_0) + i \frac{\partial v}{\partial x} (x_0, y_0) \right) + \frac{|z - z_0|}{z - z_0} (\epsilon_1(\cdot) + i \epsilon_2(\cdot))
\]
\[
\to \frac{\partial u}{\partial x} (x_0, y_0) + i \frac{\partial v}{\partial x} (x_0, y_0) (= f'(z_0))
\]
as \(z \to z_0\).
Example 4.2 (Three ways to see \( f(z) = e^z \) is analytic at \( z = 0 \)). We let \( f : \mathbb{C} \to \mathbb{C} \) be the exponential map \( f(z) = e^z \) where \( z = x + iy \in \mathbb{C} \).

1. We can write \( f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \);
2. We have a complex derivative \( f'(z) = e^z \) (which makes sense as a power series); and
3. We can \( f(z) = u + iv = e^x \cos y + ie^x \sin y \). In particular,
   \[
   \frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.
   \]

The continuity of the partial derivatives in part (3) of the theorem is crucial. Without the assumption that partial derivatives are continuous, then satisfying the Cauchy-Riemann equations is not enough to deduce that \( f \) is complex differentiable.

Example 4.3. If \( f : \mathbb{C} \to \mathbb{C} \) is defined by

\[
f(z) = \begin{cases} 
0 & \text{if } z = 0 \\
\exp(-1/z^4) & \text{otherwise}
\end{cases}
\]

Thus all the expressions in the Cauchy-Riemann equations are zero, but one can show that \( f \) is not differentiable.

5. Integrals and Cauchy’s Theorem

The most useful theorem in Complex analysis is probably Cauchy’s theorem.

5.1. Integration on piecewise continuous curves. Given a continuous curve \( \gamma : [a, b] \to \mathbb{C} \) we can take the real and imaginary parts \( \gamma(t) = u(t) + iv(t) \), where \( u, v : [a, b] \to \mathbb{R} \) are real valued.

We say that \( f(t) \) is differentiable if \( u, v : [a, b] \to \mathbb{R} \) are differentiable and then we define

\[ f'(t) = u'(t) + iv'(t). \]

Definition 5.1. If \( f : [a, b] \to \mathbb{C} \) is continuous then we define

\[ \int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt. \]

Moreover, for a piecewise continuous function \( f(t) \) on \( a = a_0 < a_1 < \cdots < a_n = b \) we can still define the integral by

\[ \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(dt). \]

Lemma 5.2. For \( f, g \) piecewise continuous on \([a, b]\) and \( \alpha, \beta \in \mathbb{C} \) we have that:

1. \( \int_a^b (\alpha f(t) + \beta g(t))dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt \)
2. \( |\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt \)
5. INTEGRALS AND CAUCHY'S THEOREM

Proof. The first part follows easily from the definitions. For the second part, assume that \( \int_a^b f(t) \, dt \neq 0 \) (otherwise the result is trivial) and we can write \( re^{i\theta} = \int_a^b f(t) \, dt \). Thus \( r = |\int_a^b f(t) \, dt| = \int_a^b e^{-i\theta} f(t) \, dt \in \mathbb{R} \). Therefore,

\[
    r = \text{Re} \left( \int_a^b e^{-i\theta} f(t) \, dt \right) = \int_a^b \text{Re} \left( e^{-i\theta} f(t) \right) \, dt \leq \int_a^b |e^{-i\theta} f(t)| \, dt = \int_a^b |f(t)| \, dt
\]

and thus

\[
    |\int_a^b f(t) \, dt| \leq \int_a^b |f(t)| \, dt
\]

□

5.2. parameterization of the curve of integration. If \( \phi : [c,d] \rightarrow [a,b] \) is differentiable map with continuous positive derivative so that \( \phi(t) \) is strictly increasing then

\[
    \int_a^b f(s) \, ds = \int_c^d f(\phi(t)) \phi'(t) \, dt
\]

by the substitution law and the change of variables law with \( s = \phi(t) \).

Definition 5.3. A contour (or curve or arc or path) is a continuous image by \( z : [a,b] \rightarrow \mathbb{C} \) of a closed interval \( [a,b] \) in \( \mathbb{C} \) with the orientation provided by the ordering on \( [a,b] \). Thus \( z(a) \) is the first point on the contour and \( z(b) \) is the last point on the contour.

Any continuous map giving the curve and orientation is called a parameterization. The contour is called smooth if there is a parameterization \( z \) such that \( z \) is differentiable in its interval of definition \( [a,b] \) and if \( z'(t) \) is continuous in \( [a,b] \).

The contour is called piecewise smooth if there exists a parameterization \( z : [a,b] \rightarrow \mathbb{C} \) for the contour and a partition \( a = a_0 < a_1 < \cdots < a_n = b \) such that \( z|[a_i, a_{i+1}] \) is smooth.

A contour is simple if can be given by an injective parameterization (i.e., \( z(a) = z(b) \implies a = b \)).

A contour is closed and simple if it can be given by an injective parameterization \( z : [a,b] \rightarrow \mathbb{C} \) (i.e., \( z(a) = z(b) \) and or \( z(a) = z(b) \)).

Remark 5.4. A piecewise smooth curve \( \gamma \) may be given by many parameterizations. If \( \phi : [c,d] \rightarrow [a,b] \) is strictly increasing then \( w : [c,d] \rightarrow \mathbb{C} \) defined by \( w(t) = z(\phi(t)) \) is another parameterization.

In complex analysis we are usually concerned with piecewise smooth curves and piecewise smooth parameterizations. A piecewise smooth curve \( \gamma \) may be given by many parameterisations, i.e., if \( \phi : [c,d] \rightarrow [a,b] \) is strictly increasing then
$w : [c, d] \to \mathbb{C}$ defined by $w(t) = z(\phi(t))$ is also piecewise smooth if $z$ is piecewise smooth and $\phi$ is differentiable with continuous and positive derivative.

**Definition 5.5.** We say that two parameterizations $z$ and $w$ are equivalent if there exists such a $\phi$.

**Exercise 5.6.** Show that this is an equivalence relation on parameterizations.

**Hint.** Clearly we can write $w \circ \phi^{-1} = z$ and $\phi^{-1}$ is continuous with positive derivative. Transitivity follows simply. □

If $z : [a, b] \to \gamma \subset \mathbb{C}$ is a piecewise smooth parameterization of $\gamma$ then we obtain a piecewise parameterization of $\gamma$ (in reverse) by $z_- : [-b, -a] \to \mathbb{C}$ with $z_-(t) = z(-t)$. Let us denote by $-\gamma$ the reverse of $\gamma$.

**Definition 5.7.** If $\gamma_1$ is piecewise smooth and $\gamma_2$ is piecewise smooth and if the last point of $\gamma_1$ is the same as the first point of $\gamma_2$ then we define the composition $\gamma_1 \gamma_2$ as follows:

If $z_1 : [a, b] \to \gamma_1 \subset \mathbb{C}$ is a parameterization of $\gamma_1$ and $z_2 : [c, d] \to \gamma_2 \subset \mathbb{C}$ is a parameterization of $\gamma_2$ we define $z_3 : [a, b + d - c] \to \gamma_3 \subset \mathbb{C}$ by

$$z_3(t) = \begin{cases} 
  z_1(t) & \text{if } t \in [a, b] \\
  z_2(\phi(t)) & \text{if } t \in [b, b + d - c]
\end{cases}$$

where $\phi$ is the translation $\phi(t) = t + (b - c)$.

We are interested in integration around a piecewise smooth simple contour.

**Definition 5.8.** Given smooth curves $z_i : [a_i, a_{i+1}] \to \gamma_i \subset \mathbb{C}$ for

$$a = a_0 < a_1 < \cdots < a_n = b$$

we define

$$\int_\gamma f(z)dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)dz.$$

(Once we define the integral for smooth contours we can then extend it to piecewise smooth contours.) If $\gamma$ is a smooth contour parameterized by $z : [a, b] \to \gamma \subset \mathbb{C}$ then we define

$$\int_\gamma f(z)dz := \int_{a}^{b} f(z(t))z'(t)dt$$

**Exercise 5.9.** Show that $\int_{-\gamma} f(w)dw = -\int_{\gamma} f(z)dz$

**Proof.** We can write $\int_{-\gamma} f(z)dz = \int_{-b}^{-a} f(z(-t))z'(-t)dt$. Substituting $s = -t$ this becomes: $-\int_{a}^{b} f(z(s))z'(s)ds = -\int_{\gamma} f(z)dz$. □
Remark 5.10. More generally, integration with respect to arc length along a simple contour $\gamma$ is rectifiable if it is given by a continuous map $z : [a, b] \to \mathbb{C}$ which is of bounded variation, i.e., the summations

$$\sum_{i=0}^{n-1} |z(a_{i+1}) - z(a_i)|,$$

ranging over all partitions $a = a_0 < a_1 < \cdots < a_n = b$, is bounded. The supremum of these numbers is by definition the length of $\gamma$. In particular, if $\gamma$ is smooth represented by $z : [a, b] \to \mathbb{C}$ then $\gamma$ is rectifiable and $l(\gamma) = \int_a^b |z'(t)| dt$.

The following upper bound is useful.

**Lemma 5.11 (Integration along a straight line).** If $\gamma$ is piecewise smooth and $f$ is continuous on $\gamma$ and $|f(z)| \leq M$ on $\gamma$ then

$$\left| \int_\gamma f(z) dz \right| \leq Ml(\gamma).$$

**Proof.** Let $z : [a, b] \to \gamma \subset \mathbb{C}$ be a parameterization. Then

$$\left| \int_\gamma f(z) dz \right| = \left| \int_a^b f(z(t))z'(t) dt \right|$$

$$\leq \int_a^b |f(z(t))||z'(t)| dt$$

$$\leq M \int_a^b |z'(t)| dt = Ml(\gamma)$$

as required. \qed

It is useful to illustrate this by a couple of examples.

**Example 5.12 (Integration of $z^n$ along a straight line).** Let $\gamma = [w_1, w_2]$ and consider the affine parameterization $z : [0, 1] \to \mathbb{C}$ given by

$$z(t) = w_1(1 - t) + w_2t, \text{ for } 0 \leq t \leq 1,$$

then $z'(t) = (w_2 - w_1)$ and by definition

$$\int_{[w_1, w_2]} f(z) dz = \int_0^1 f(w_1 + t(w_2 - w_1)) (w_2 - w_1) dt.$$

Consider the specific example $f(z) = z^n$, then if $n \neq -1$ then

$$\int_0^1 (w_1 + t(w_2 - w_1))^n (w_2 - w_1) dt = \frac{w_2^{n+1}}{n+1} - \frac{w_1^{n+1}}{n+1}.$$

**Example 5.13 (Integration around a simple closed curve).** Let $\gamma$ be the $\theta$-arc in the unit circle with $z : [0, \theta] \to \mathbb{C}$ given by

$$z(t) = e^{it} \text{ for } 0 \leq t \leq \theta$$
then \( z'(t) = ie^{it} \) and by definition
\[
\int_{\gamma} f(z)dz = \int_0^t e^{nit}ie^{it}dt.
\]

Consider the specific example \( f(z) = z^n \) then
\[
\int_{\gamma} f(z)dz = \int_0^t e^{nit}ie^{it}dt = \frac{i}{(n+1)i} \left[ e^{i(n+1)t} \right]_0^t = e^{i(n+1)t} \frac{n+1}{n+1} - 1
\]
if \( n \neq -1 \). On the other hand, if \( n = -1 \) then one can easily see that
\[
\int_{\gamma} \frac{1}{z}dz = \int_0^t e^{-it}ie^{it}dt = \int_0^t idt = \theta i
\]
In particular, if \( \theta = 2\pi \) then
\[
\int_{\gamma} z^n dz = \int_0^{2\pi} e^{nit}ie^{int}dt = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1. \end{cases}
\]

### 5.3. Theorems of Gousart and Cauchy.

The last example is a special case of the following important theorem.

**Theorem 5.14 (Gousart’s Theorem).** Let \( f : U \to \mathbb{C} \) be an analytic function. Let \( \gamma \subset U \) be a closed simple curve then \( \int_{\gamma} f(z)dz = 0 \).

There are two basic approaches to proving this. One approach uses Stokes’ theorem, but for variety we will describe the other approach (at least in the context of \( \gamma \) being a rectangle).

**Proof.** We begin with a proof of this result in the case that \( \gamma \) is the boundary of a rectangle \( R_0 = [a,b] \times [c,d] \). In particular
\[
\partial R_0 = [a,b] \times \{c\} \cup \{b\} \times [c,d] \cup [b,a] \times \{d\} \cup \{a\} \times [d,c].
\]

We proceed as follows.

**Step 1.** We can subdivide \( R_0 \) into four similar sub-rectangles:
\[
\begin{align*}
R_0^{(1)} &= [a,(a+b)/2] \times [c,(c+d)/2] \\
R_0^{(2)} &= [(a+b)/2,b] \times [c,(c+d)/2] \\
R_0^{(3)} &= [a,(a+b)/2] \times [(c+d)/2,d] \\
R_0^{(4)} &= [(a+b)/2,b] \times [(c+d)/2,d]
\end{align*}
\]
and then we can write
\[
\int_{\partial R_0} f(z)dz = \sum_{i=1}^4 \int_{\partial R_0^{(i)}} f(z)dz.
\]
(Where the integrals along the same curves in different directions cancel).
5. INTEGRALS AND CAUCHY’S THEOREM

In particular, we can bound

\[ \left| \int_{\partial R_0} f(z) \, dz \right| \leq 4 \max_{1 \leq i \leq 4} \left| \int_{\partial R_0^{(i)}} f(z) \, dz \right|. \]

Let us denote by \( R_1 \) the subrectangle which maximizes the last term.

**Step 2.** We can repeat this iteratively to construct a sequence of subrectangles

\[ R_0 \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots \]

such that \( R_n, n \geq 1 \), is a sub-rectangle of size \( \frac{(b-a)}{2^n} \times \frac{(d-c)}{2^n} \) and we can bound

\[ \left| \int_{\partial R_0} f(z) \, dz \right| \leq 4^n \left| \int_{\partial R_n} f(z) \, dz \right|. \] (a)

We can choose \( z_0 \in \bigcap_{n=0}^{\infty} R_n \) (which implicitly uses compactness of \( R_0 \)).

We now want to use that \( f(z) \) is complex differentiable (at \( z_0 \)). In particular, for any \( \epsilon > 0 \) we can choose \( \delta > 0 \) so that providing \( |z - z_0| < \delta \) then

\[ |f(z) - f(z_0) + f'(z_0)(z - z_0)| \leq \epsilon |z - z_0|. \] (b)

Provided \( n \) is sufficiently large that the diameter

\[ \text{diam}(R_n) = \frac{1}{2^n} \sqrt{(b-a)^2 + (d-c)^2} \] (c)

of \( R_n \) is smaller than \( \delta \) then we have from (b) and (c) that for all \( z \in R_n \) we can write that

\[ |f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon \times \text{diam}(R_n) \]

\[ \leq \frac{\epsilon}{2^n} \sqrt{(b-a) + (d-c)} \] (d)
Moreover, since

\[
\text{length}(\partial R_n) = \frac{1}{2^n} (2(b - a) + (d - c))
\]

we can use (a), (d) and (e) to bound

\[
\left| \int_{\partial R_n} (f(z) - f(z_0) - f'(z_0)(z - z_0)) \, dz \right|
\leq \sup_{z \in R_n} \{|f(z) - f(z_0) - f'(z_0)(z - z_0)|\} \times \frac{\text{length}(\partial R_n)}{2^n \sqrt{(b-a)^2 + (d-c)^2}}
\leq \frac{\epsilon}{4^n} \left( \sqrt{(b-a)^2 + (d-c)^2} (2(b-a) + (d-c)) \right)
\leq \frac{\epsilon}{4^n} \left( \frac{C}{4^n} \epsilon \right) = C \epsilon
\]

**Step 3.** Moreover, we have the following estimates:

For each rectangle $R_n$:

1. We can evaluate $\int_{\partial R_n} 1 \, dz = 0$;
2. We can evaluate $\int_{\partial R_n} (z - z_0) \, dz = 0$.

**Proof.** The first part is easy, since the contribution to the integrals from opposite sides cancel. For the second part, we can simply explicitly do the integral. \(\square\)

Comparing (a) and (f) and the above claim we can write:

\[
\left| \int_{\partial R_0} f(z) \, dz \right| \leq 4^n \left| \int_{\partial R_n} f(z) \, dz \right|
\]
\[
= 4^n \left| \int_{\partial R_n} (f(z) - f(z_0) - f'(z_0)(z - z_0)) \, dz - \int_{\partial R_n} (f(z_0) - f'(z_0)(z - z_0)) \, dz \right|
\]
\[
\leq 4^n \left( \frac{C}{4^n} \epsilon \right) = C \epsilon
\]

Thus, since $\epsilon$ can be chosen arbitrarily small, we finally deduce that $\int_{\partial R_0} f(z) \, dz = 0$, as required. \(\square\)

**Remark 5.15.** For more general simple curves we can approximate $\gamma$ by a piecewise linear curve $\gamma'$ consisting of horizontal and vertical lines. We can then fill in the region inside the curve by rectangles and apply the result above.

**Corollary 5.16.** Let $f : U \to \mathbb{C}$ be analytic (i.e., complex differentiable). Let $\gamma \subset U$ be a simple closed curve containing $z_0 \in U$. Then $\int_{\gamma} \left( \frac{1}{z-z_0} \right) \, dz = 2\pi i$. 
Proof. We can assume for simplicity that $z_0 = 0$ (otherwise we simply change to the translated function for $z \mapsto f(z - z_0)$). We can also assume that $\gamma$ contains the unit disk $D$, since otherwise we can scale by $z \mapsto f(zR)$, for large enough $R$.

In the previous section we saw that the result holds with $\gamma$ replaced by the unit circle $\gamma_0$. However, this suffices, because we can change the integral from one along $\gamma$ to one along $\gamma_0$ by introducing an extra arc $c$ joining the two. Then the integrals differ by $\int_{c\gamma_0^{-1}\gamma} \frac{1}{z-z_0} dz = 0$, since the function is analytic inside the closed curve $c\gamma_0^{-1}\gamma$. \end{proof}

We now deduce Cauchy’s Theorem from this.

Theorem 5.17 (Cauchy’s Theorem). Let $f : U \to \mathbb{C}$ be an analytic function (i.e., complex differentiable). Let $\gamma \subset U$ be a simple closed curve. Assume that $z_0$ lies inside the curve $\gamma$, then

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-z_0} dz$$

Proof. Step 1. Consider the function

$$g(z) = \begin{cases} f(z) - f(z_0) & \text{if } z \neq z_0 \\ f'(z) & \text{if } z = z_0 \end{cases}$$

Moreover, by continuity of $f$ at $z_0$,

$$g(z)(z - z_0) = \lim_{z \to z_0} (f(z) - f(z_0))(z - z_0)) = 0$$

as $z \to z_0$.

Given $\epsilon > 0$ we can choose $\delta > 0$ such that for $|z - z_0| < \sqrt{2}\delta$ we have

$$|g(z) - g(z_0)| \leq \epsilon |z - z_0|.$$ 

Step 2. Let $z_0 = x_0 + iy_0$ Given a rectangle $R$ we want to divide it into nine subrectangles, including

$$R' = [x_0 - \delta, x_0 + \delta] \times [y_0 - \delta, y_0 + \delta]$$
then cancellations between the integrals on the boundaries of the eight rectangles and applying Gousart’s theorem to each of these we get
\[ \int_{\partial R} g(z) \, dz = \int_{\partial R'} g(z) \, dz \]

Moreover, since for \( z \in \partial R' \) we can write
\[ |g(z) - g(z_0)| \leq \frac{\epsilon}{|z - z_0|} \leq \frac{\epsilon}{\delta} \]
we can write
\[ \left| \int_{\partial R'} g(z) \, dz \right| \leq \text{length}(\partial R') \left( \sup_{z \in \partial R'} |g(z)| \right) \leq \frac{8\delta}{\delta} \left( \sup_{z \in \partial R'} |f(z)| \right) \]
which can be made arbitrarily small be choice of \( \epsilon \).

**Step 3.**

We now apply Gousart’s Theorem’s to deduce that
\[ 0 = \int_{\gamma} g(z) \, dz = \int_{\gamma} \left( \frac{f(z) - f(z_0)}{z - z_0} \right) \, dz = \int_{\gamma} \left( \frac{f(z)}{z - z_0} \right) \, dz - f(z_0) \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} \, dz \]

\[ \square \]

**5.4. Immediate applications of Cauchy’s Theorem.** Cauchy’s theorem has a number of immediate corollaries.

**Corollary 5.18.** Let \( f : U \to \mathbb{C} \) be analytic and let \( U \) be a simply connected domain. Let \( z_1, z_2 \in U \). Then for any piecewise smooth curve \( \gamma \) connecting \( z_1 \) to \( z_2 \) the integral \( \int_{\gamma} f(z) \, dz \) is independent of \( \gamma \).

**Proof.** Let \( \gamma_1, \gamma_2 \) be two piecewise smooth curves leading from \( z_1 \) to \( z_2 \) contained in \( U \) then \( \gamma_1 \cup -\gamma_2 \) is a closed curve and thus
\[ \int_{\gamma_1 \cup -\gamma_2} f(z) \, dz = 0 = \int_{\gamma_1} f(z) \, dz - \int_{\gamma_2} f(z) \, dz \]
and the result follows. \[ \square \]

**Corollary 5.19.** There is a formula for the derivatives of the form
\[ f'(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^2} \, dz \] (\( * \))

where \( z_0 \) is inside the simple closed curve \( \gamma \). More generally, the \( k \)th derivative takes the form
\[ f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{k+1}} \, dz \] (\( ** \))

**Proof.** We need to justify differentiating under the integral sign. Using Cauchy's theorem we can write, (for sufficiently small \( h \in \mathbb{C} \)):
\[ \frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{h} \left( \frac{1}{(z - z_0 - h)} - \frac{1}{(z - z_0)} \right) f(z) \, dz \]
\[ = \frac{1}{2\pi i} \int_{\gamma} \left( \frac{1}{(z - z_0 - h)(z - z_0)} \right) f(z) \, dz \]

We claim that the limit exists as \( h \to 0 \) and that it equals
\[ f'(z_0) := \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^2} \, dz \]
To see this we can bound
\[ \left| \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0) \right| \leq \left| \frac{h}{2\pi i} \int_{\gamma} \left( \frac{1}{(z - z_0 - h)(z - z_0)^2} \right)f(z)dz \right| \]
\[ \leq \frac{|h|}{2\pi} \left( \frac{\text{length}(\gamma)}{d(z_0, \gamma) - |h|d(z_0, \gamma)^2} \right) \sup_{z \in \gamma} |f(z)| \]
where \( d(z_0, \gamma) = \inf_{z \in \gamma} |z_0 - z| \) and observe that the Right Hand Side tends to zero as \( |h| \to 0 \).

The higher derivative formula comes similarly (cf. next application). □

**Application 5.20** (Completing the equivalence of the definitions of analyticity I: Writing analytic functions as power series). We claim that a complex differentiable function always has a power series expansion.

Let \( f : U \to \mathbb{C} \) be a complex differentiable function function. Let \( z_0 \in U \) and choose \( \epsilon > 0 \) such that \( B(z_0, \epsilon) = \{ z \in \mathbb{C} : |z - z_0| < R \} \subset U \)

Choose \( z \in B(z_0, \epsilon) \) and then \( \rho := |z_0 - z| < \epsilon \) and let \( r \) be such that \( \rho < r < \epsilon \).

Then by Cauchy’s theorem
\[ f(z) = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{\xi - z} d\xi \]
where \( C(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| = r \} \) is the circle of radius \( r \) about \( z_0 \).

For \( \rho < |\xi - z_0| = r < \epsilon \) we have that
\[ \frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} \]
\[ = \frac{1}{(\xi - z_0) \left( 1 - \frac{z - z_0}{\xi - z_0} \right)} \]
\[ = \frac{1}{(\xi - z_0)} \left( 1 + \left( \frac{z - z_0}{\xi - z_0} \right) + \left( \frac{z - z_0}{\xi - z_0} \right)^2 + \cdots \right) \]
which is uniformly convergent on the circle \( C(z_0, r) \) since \( \frac{|z - z_0|}{\xi - z_0} = \frac{r}{\rho} < 1 \). So by Cauchy’s theorem integrating around \( C(z_0, r) \) gives
\[ f(z_0) = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{(\xi - z_0)} d\xi \]
\[ = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\xi)}{\xi - z_0} d\xi + (z_0 - z_0) \frac{1}{2\pi i} \int_{C(a, r)} \frac{f(\xi)}{(\xi - z_0)^2} d\xi \]
\[ + (z_0 - z_0)^2 \frac{1}{2\pi i} \int_{C(a, r)} \frac{f(\xi)}{(\xi - z_0)^3} d\xi + \cdots \]

In particular, we can write this as
\[ f(z_0) = a_0 + a_1 (z_0 - a) + a_2 (z - a)^2 + \cdots \]
where
\[ a_n = \frac{1}{2\pi} \int_{C(a,r)} \frac{f(z)}{(z - z_0)^{n+1}} dz \]
and we know the series has Radius of convergence \( R = (\limsup_{n \to \infty} |a_n|^{1/n})^{-1} > r \). Thus \( f(z) \) is a power series about \( a \).

**APPLICATION 5.21** (Completing the equivalence of the definitions of analyticity II: Continuity of the partial derivatives). Let \( f : U \to \mathbb{C} \) be complex differentiable. For \( z_0 \in U \) we can \( \gamma \subset U \) be a simple closed curve containing \( z_0 \). We see from (*) that \( f'(z_0) \) depends continuously on the point \( z_0 \). Furthermore, since complex differentiability implies that for \( f(x + iy) = u(x, y) + iv(x, y) \) we can write
\[ f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - i \frac{\partial u}{\partial y} \]
from which we can deduce that the partial derivatives are continuous.

**COROLLARY 5.22.** Let \( f : U \to \mathbb{C} \) be analytic and let \( z_0 \) be a unique zero (of multiplicity one). Let \( \gamma \subset U \) be a simple closed curve. Then
\[ z_0 = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz \]

**PROOF.** We can write \( f(z) = (z - z_0)g(z) \), where \( g : U \to \mathbb{C} \) where \( g(z) \neq 0 \). The function \( \frac{f(z)}{g(z)} \) is of the form \( \frac{1}{z - z_0} + \psi(z) \) where \( \psi : U \to \mathbb{C} \) is analytic and given by \( \psi(z) = g'(z)/g(z) \). Thus, by Cauchy’s theorem we can write
\[ \frac{1}{2\pi i} \int_{\gamma} z \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz + \frac{1}{2\pi i} \int_{\gamma} \psi(z) dz = z_0. \]

**APPLICATION 5.23** (Perturbation Theory for eigenvalues of matrices). Assume that \( A \) is a matrix and \( \lambda_A \) is a simple eigenvalue. Then letting \( f_A(z) = \det(zI - A) \) then we see that \( z_0 \) is a simple zero for \( f(z) \). By the previous corollary, we see that
\[ \lambda_A = \frac{1}{2\pi i} \int_{\gamma} \frac{f_A'(z)}{f_A(z)} dz \]

However, it is easy to see that the coefficients of the polynomials \( f_A(z) \) and \( f_A'(z) \) depend smoothly on the entries in the matrix \( A \). This is true on \( \gamma \) and thus we can deduce the same for \( \lambda_A \).

**5.5. Converse to Cauchy’s Theorem.** The following is a converse to Cauchy’s Theorem which is often useful in proofs of later results.

**THEOREM 5.24** (Moreira’s Theorem). Let \( U \) be a domain and let \( f : U \to \mathbb{C} \) be continuous and satisfy \( \int_{\partial \Delta} f(z) dz = 0 \) for all simple closed curves \( \Delta \subset U \). Then \( f \) is analytic.

**PROOF.** Fix a point \( z_0 \in U \). We can define \( F : U \to \mathbb{C} \) by associating to any \( z \in U \) a piecewise smooth curve \( \gamma \) parameterized by \( z : [0,1] \to \gamma \subset \mathbb{C} \) where \( z(0) = z_0 \) and \( z(1) = z \) and then defining
\[ F(z) = \int_{\gamma} f(z) dz. \]
We first observe that since $U$ is a (simply connected) domain this is well defined, since if $\gamma_1$ is a second path from $z_0$ to $z$ then we have that
\[
\int_{\gamma} f(z)dz - \int_{\gamma_1} f(z)dz = \int_{\gamma} f(z)dz + \int_{-\gamma_1} f(z)dz = \int_{\gamma(-\gamma_1)} f(z)dz = 0
\]
since $\gamma(-\gamma_1)$ is a piecewise closed curve, and $F: U \to \mathbb{C}$ is analytic, and thus we can apply Cauchy’s theorem.

We can now deduce that $F'(z) = f(z)$, i.e., $f$ is the complex derivative of $F$. But if $F$ it is complex differentiable then it is analytic, and thus all of the derivatives exist, and us $f$ is also complex differentiable.

\[\square\]

6. Properties of analytic functions

We want to find some way to understand better properties of analytic functions. We will begin with properties of individual functions and move onto families of functions. However, first we have to recall some basic facts.

6.1. Logarithms and roots. Consider the exponential function $z \mapsto e^z$ for $z \in \mathbb{C}$. This is analytic for every $z \in \mathbb{C}$. We would like to define the inverse map, the logarithm, in the complex plane but this has a few complications that need to be addressed.

**Lemma 6.1.** Let $z = x + iy$ and $w = u + iv$. Then $e^z = e^w$ when there exists $n \in \mathbb{Z}$ such that $y = v + 2\pi n$.

**Proof.** If $e^{x+iy} = e^{w} = e^{u+iv}$ then taking the modulus gives $|e^z| = e^u = e^x = |e^w|$ gives that $u = x$. This leaves that $e^{iy} = e^{iv}$ and thus $y - v$ is an integral multiple of $2\pi$.

In particular, $\exp : z \mapsto e^z$ maps any strip $\mathbb{R} \times [\alpha, \alpha + 2\pi)$ bijectively onto $\mathbb{C} - \{0\}$. However, the inverse to $\exp(\cdot)$ is not, strictly speaking, a well defined function since there are many points mapped to a single point. However, by the lemma these points all differ by values $2\pi in$, $n \in \mathbb{Z}$.
We would like to write \( w = \log z \) when \( z = e^w \). If \( w = u + iv \) and \( \exp w = z \) then
\[
e^u e^{iv} = |z| \frac{z}{|z|} = |z| e^{i\theta}
\]
for some \( \theta \) since \( \frac{\pi}{2} = 1 \). Hence \( e^u = |z| \) and \( v = \theta + 2\pi n \) for any \( n \in \mathbb{Z} \).

Therefore \( u = \log |z| \) and \( w = \log |z| + i(\theta + 2\pi n) \) for some \( n \in \mathbb{Z} \).

**Definition 6.2.** For \( z \neq 0 \) define the function \( \arg(z) = \arg(z/|z|) = \theta \) if \( z/|z| = e^{i\theta} \) and \( -\pi \leq \theta < \pi \).

For \( \log(\cdot) \) there are different branches of \( \arg \) with different restricted domains. The principle branch of \( \arg \) is denoted \( \text{Arg}(z) \) and is such that \( \text{Arg}(e^{i\theta}) \) is the unique number of the form \( \theta + 2\pi n \) which lies between \( -\pi \) and \( \pi \). This is not defined on the negative real line \( (-\infty, 0] \).

Returning to the definition of \( \log \):

**Definition 6.3.** The principle branch of \( \log \) is denoted \( \text{Log}(z) \) and is of the form \( \log |z| + i\text{Arg}(z) \). This is not defined on the negative real line \( (-\infty, 0] \).

We can then use \( \exp \) and \( \text{Log} \) to define complex powers of complex numbers. In particular, we can write that \( z^w = \exp(w \cdot \text{Log}(z)) \).

**Example 6.4.** Find the value of
\[
(-1 - i)^{1+i} := \exp((1 + i)\text{Log}(-1 - i))
\]
We observe that \( \arg(-i - i) = -\frac{3}{4}\pi \) and \( \text{Log}(-i - i) = \log \sqrt{2} - \frac{3}{4}\pi i \). Thus
\[
(-1 - i)^{1+i} = \exp((1 + i)\text{Log}(-1 - i)) = \exp((1 + i) \left( \log \sqrt{2} - \frac{3}{4}\pi i \right)).
\]

**Example 6.5.** If \( \rho \in \mathbb{N} \) then we can write
\[
z^{1/\rho} = \exp \left( \frac{\log |z|}{\rho} + i\frac{\text{Arg}(z)}{\rho} \right).
\]
If \( z = re^{i\theta} \) then
\[
z^{1/\rho} = \exp((1/\rho)(\log r + i(\theta + 2\pi n))) = r^{1/\rho}e^{i(\theta + 2\pi n)/\rho} = r^{1/\rho}e^{i(\theta + 2\pi \rho n)/\rho}.
\]
These take the values \( r^{1/\rho}, r^{1/\rho}e^{2\pi \theta}, \ldots, r^{1/\rho}e^{i(\theta + 2\pi (\rho - 1))} \).

In particular
\[
z^{1/2} = \exp \left( \frac{1}{2} \log(z) \right) = \exp \left( \frac{1}{2} \log |z| + i\frac{1}{2}\arg(z) \right) = |z|^{1/2} e^{i\arg(z)/2}.
\]
More generally, for \( z^{q/p} \) we simply replace \( z \) by \( z^q \).

**Remark 6.6.** If \( \alpha \) is irrational then we can write
\[
z^\alpha = \exp \left( \alpha \log z \right) = \exp(\alpha \log r + \alpha i(\theta + 2\pi n))
\]
and the set of values with \( n \in \mathbb{Z} \) is infinite. To see this, let \( z = re^{i\theta} \) then
\[
\exp(\alpha i(\theta + 2\pi n)) = \exp(\alpha i(\theta + 2\pi m)) \iff \exp(\alpha i2\pi(n - m)) = 1 \iff \alpha 2\pi(n - m) = 1 \iff n = m.
\]
(since \( \alpha \) is irrational).
6.2. Location of zeros: Argument Principle and Rouché’s Theorem.

We can apply Gousart’s and Cauchy’s theorem to prove two useful results

**Lemma 6.7.** Suppose that \( f : U \to \mathbb{C} \) is analytic and non-zero then there exists an analytic function \( h : U \to \mathbb{C} \) such that \( e^{h(z)} = f(z) \) for all \( z \in U \).

**Proof.** Since \( f \) is analytic we say that \( f' \) is analytic and since \( f \) does not vanish then \( f'/f \) is analytic and so \( f'/f \) has a primitive \( h \) by integrating \( f \), i.e.,

\[
\frac{d}{dz} \left( f(z)e^{-h(z)} \right) = f'(z)e^{-h(z)} - f(z)h'(z)e^{-h(z)} = 0
\]

thus \( f(z)e^{-h(z)} = \text{Constant} = e^a \neq 0. \) Thus \( f(z) = e^{a+h(z)} \). \( \square \)

We can use this lemma to show the following theorem.

**Theorem 6.8 (The Argument Principle).** Assume \( f : U \to \mathbb{C} \) is analytic and \( \gamma \subset U \) is a simple closed curve. The general case is similar. Assume that \( f \) has no zeros on \( \gamma \). Then the number of zeros \( N \) inside \( \gamma \) is given by

\[
N = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz.
\]

**Proof.** Assume that \( z_0 \) is a zero of order \( n \geq 1 \) for \( f(z) \). Assume for simplicity that we have no other zeros.

Using the lemma we can then write \( f(z) = (z - z_0)^Ne^{h(z)} \) and then

\[
f'(z) = (z - z_0)^N e^{h(z)} h'(z) + N(z - z_0)^{n-1} e^{h(z)}
\]

and

\[
\frac{f'(z)}{f(z)} = \frac{h'(z)}{f(z)} + \frac{N}{z - z_0}
\]

Since the first term is analytic simple pole at \( z_0 \) we can apply Gousart’s theorem to the first term and Cauchy’s theorem to the second term to write

\[
N = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz.
\]

With more zeroes the argument naturally generalizes. \( \square \)

**Example 6.9.** Assume \( f : \mathbb{D} \to \mathbb{C} \) has a simple zero at 0 (i.e., \( N = 1 \)) and that \( \gamma = \{ z : |z| = r \} \) with \( 0 < r < 1 \). Writing

\[
f(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad f'(z) = \sum_{n=1}^{\infty} na_{n+1} z^n
\]

(with \( a_0 = 0 \) and \( a_1 \neq 0 \)) we see that

\[
\frac{f'(z)}{f(z)} = \frac{1}{z} + g(z)
\]
where \( g(z) \) is analytic on a neighbourhood of 0. By Gousart’s theorem \( \int_{\gamma} g(z) \, dz = 0 \) and a direct calculation gives
\[
\frac{1}{2\pi i} \int_{\gamma} f'(z) \frac{dz}{f(z)} = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} + \frac{1}{2\pi i} \int_{\gamma} g(z) \, dz
\]

**Application 6.10 (localizing zeros of Riemann Zeta function).** One of the major open problems in mathematics is the Riemann Hypothesis. For \( s \in \mathbb{C} \) with \( \Re(s) > 1 \) we define the Riemann zeta function
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]
(It converges since \( |\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{n^{\Re(s)}} < +\infty \). Moreover, it has an analytic extension to \( \mathbb{C} - \{1\} \), i.e., there is an analytic function \( f : \mathbb{C} - \{1\} \to \mathbb{C} \) such that \( f(s) = \zeta(s) \) for \( \Re(s) > 1 \).

**Riemann Hypothesis (Conjecture, 1859)** The only zeros for \( f(s) \) occur at \( s = -2, -4, -6, \ldots \) and on the line
\[
L := \{ s = \frac{1}{2} + it : t \in \mathbb{R} \}.
\]
This is Hilbert’s 8th problem from his famous list of 23 open problems from 1900, and one of the 7 Millenium problems from 2000 for which the Clay Institute offered a million dollars.

Computer searches for counter-examples (so far unsuccessful!) use the Argument Theorem. One considers a small closed curve \( \gamma \) away from the line \( L \) and estimates the integral \( \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz \). One tries to find a \( \gamma \) such that the integral is non-zero.

**Theorem 6.11 (Rouche’s Theorem: Nearby functions and nearby zeros).** Assume that \( f, g : U \to \mathbb{C} \) are analytic. Let \( \gamma \subset U \) be a simple closed curve and assume that neither \( f(z) \) or \( g(z) \) have zeros on \( \gamma \) and that for each \( z \in \gamma \) we have that \( |g(z)| < |f(z)| \). Then \( f \) and \( f + g \) have the same number of zeros in \( U \).

**Proof.** This uses the ”walking the dog” method.
Let \( F(z) = \frac{g(z)}{f(z)} \) then if \( N_f \) and \( N_{f+g} \) are the number of zeros inside \( \gamma \) for each of the two functions then by the Argument Principle we can write

\[
N_{f+g} = \frac{1}{2\pi i} \int_{\gamma} \frac{(f' + g')(z)}{(f + g)(z)} \, dz \quad \text{and} \quad N_f = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz.
\]

Thus

\[
N_{f+g} - N_f = \frac{1}{2\pi i} \int_{\gamma} \frac{(f' + g')(z)}{(f + g)(z)} \, dz - \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)(1 + F(z)) + f(z)F'(z)}{f(z)(1 + F(z))} \, dz - \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \left( \frac{F'(z)}{1 + F(z)} \right) \, dz.
\]

But \( F(z) \neq -1 \) for \( z \in \gamma \), since \( |F(z)| < 1 \), and so the final expression counts the number of \( z \) inside \( \gamma \) with \( F(z) = -1 \).

Finally, observe that if \( 0 \leq t \leq 1 \) we can replace \( g(z) \) by \( t g(z) \) (and \( F(z) \) by \( tF(z) \)) and providing \( t \) is small enough we have that the final expression is zero. But if we assume for a contradiction that \( N_{f+g} \neq N_f \) then it will contradict the obvious continuity of

\[
t \mapsto \frac{1}{2\pi i} \int_{\gamma} \left( \frac{tF'(z)}{1 + tF(z)} \right) \, dz
\]

\[ \square \]

**Example 6.12.** Show that all four of the zeros of \( z^4 - 7z - 1 \) lie in \( B(0, 2) = \{ z \in \mathbb{C} : |z| < 2 \} \).

Let \( f(z) = z^4 \) then all four zeros lie inside \( B(0, 2) \). In fact, \( z = 0 \) is a zero of multiplicity four.

Let \( g(z) = -7z - 1 \). Let \( \gamma = C(0, 2) = \{ z \in \mathbb{C} : |z| = 2 \} \). For \( |z| = 2 \) we have that

\[
|g(z)| = | -7z - 1 | \leq 7|z| + 1 = 15 < 16 = |z^4| = |f(z)|.
\]

Thus by Rouché’s theorem we have that \( f(z) + g(z) = z^4 - 7z - 1 \) has the same number of zeros in \( B(0, 2) \) as \( f(z) \), i.e., four.

**Example 6.13.** Show that all five of the zeros of \( z^5 + 3z^3 + 7 \) lie in \( B(0, 2) = \{ z \in \mathbb{C} : |z| < 2 \} \).

Let \( f(z) = z^5 \) then all five zeros lie inside \( B(0, 2) \). In fact, \( z = 0 \) is a zero of multiplicity five.

Let \( g(z) = 3z^3 + 7 \). Let \( \gamma = C(0, 2) = \{ z \in \mathbb{C} : |z| = 2 \} \). For \( |z| = 2 \) we have that

\[
|g(z)| = |3z^3 + 7| \leq 3|z^3| + 7 = 31 < 32 = |z^5| = |f(z)|.
\]
Thus by Rouché’s theorem we have that \( f(z) + g(z) = z^5 + 3z^3 + 7 \) has the same number of zeros in \( B(0, 2) \) as \( f(z) \), i.e., five.

**Example 6.14.** Show that \( z^4 - 7z - 1 \) has precisely one zero in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \).

Let \( f(z) = -7z - 1 \) then \( f(z) \) has precisely one zero in \( \mathbb{D} \), at \( z = -\frac{1}{7} \).

Let \( g(z) = z^4 \). Let \( \gamma = C(0, 1) = \{ z \in \mathbb{C} : |z| = 1 \} \). For \( |z| = 1 \) we have that
\[
|g(z)| = |z^4| = 1 < 6 = 7 - 1 = |7z| - 1 \leq | - 7z - 1 | = |f(z)|.
\]
Thus by Rouche’s theorem we have that \( f(z) + g(z) = z^4 - 7z - 1 \) has the same number of zeros in \( \mathbb{D} \) as \( f(z) \), i.e., one.

**Remark 6.15.** There is a more symmetric version of this theorem. If \( f_1 : U \to \mathbb{C} \) and \( f_2 : U \to \mathbb{C} \) are analytic then \( |f_1(z) - f_2(z)| < |f_1(z)| + |f_2(z)| \) for \( z \in \gamma \) then \( f_1(z) \) and \( f_2(z) \) have the same number of zeros inside the curve.

**Corollary 6.16 (Weierstrass-Hurwitz Theorem).** Assume \( f_n : U \to \mathbb{C} \) are a sequence of non-zero analytic functions. If \( f_n \) converges uniformly on compact sets to a continuous function \( f \) (i.e., if \( K \subset U \) is compact then \( \sup_{z \in K} |f(z) - f_n(z)| \to 0 \) as \( n \to +\infty \)) then:

1. \( f : U \to \mathbb{C} \) is analytic
2. \( f \) either has no zeros or is identically zero.

**Proof.** For the first part (due to Weierstrass) one only needs to show that for any simple closed curve \( \gamma \subset U \) we have that
\[
\left| \int_\gamma f(z)dz \right| \leq \int_\gamma f_n(z)dz + \int_\gamma (f(z) - f_n(z))dz = 0 + \text{length} (\gamma) \left( \sup_{z \in K} |f(z) - f_n(z)| \right).
\]
where the first term is zero by Gousart’s theorem (since \( f_n : U \to \mathbb{C} \) is analytic). Thus \( \int_\gamma f(z)dz = 0 \) for all simple closed curves, and so \( f \) is analytic by Moreira’s Theorem.

For the second part (due to Hurwitz), if \( f(z) \) isn’t identically zero then the set of zeros \( \{ z_i \}_{i=1}^N \) for \( f(z) \) is finite. We need to show that this set is empty. Assume for a contradiction it isn’t empty and let \( \gamma \) be a closed simple curve which contains one of these zeros \( z_1 \), say. By Cauchy’s theorem
\[
0 = \int_\gamma \frac{f'(z)}{f(z)}dz = 1 \frac{1}{2\pi i} \int_\gamma f'(z)dz = 1
\]
as \( n \to +\infty \) which gives a contradiction. \( \square \)

**Application 6.17.** Recall that the Riemann zeta function
\[
\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}
\]
was claimed to analytic for \( \text{Re}(s) > 1 \). To check this we can observe that
\[
f_N(s) = \sum_{n=1}^N \frac{1}{n^s} = \sum_{n=1}^N \exp (-s \log n)
\]
is analytic on \( \mathbb{C} \). For any compact set \( K \subset U := \{ s \in \mathbb{C} : \text{Re}(s) > 1 \} \), say, we have that \( \sup_{s \in K} |\zeta(s) - f_n(s)| \to 0 \). Thus by the theorem \( \zeta(s) \) is analytic on \( U \).
6.3. Liouville’s Theorem and the Fundamental Theorem of Algebra.

We begin with another application of the Taylor series version of analyticity.

**Theorem 6.18 (Liouville’s Theorem).** If a function $f : \mathbb{C} \to \mathbb{C}$ is analytic and bounded then $f$ is constant.

**Proof.** We can write $f(z)$ as a power series representation centred on 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$ 

Now we use an estimate for $a_n$ for any $R > 0$:

$$|a_n| = \left| \frac{1}{2\pi i} \int_{C(0, R)} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{M}{R^{n+1}} \cdot \frac{2\pi R}{\pi} = \frac{M}{R^n}$$

where $M$ is an upper bound for $|f(z)|$ and taking limits as $R \to \infty$ to get $a_n = 0$ except for $n = 0$, i.e., $f(z) = a_0$.

Functions which are analytic on all of $U = \mathbb{C}$ are called *entire functions.*

**Corollary 6.19 (Fundamental Theorem of algebra).** Every non-constant polynomial has a root (i.e., a zero, that is there exists $z_0$ such that $p(z_0) = 0$).

**Proof.** Suppose that the polynomial $p(z)$ does not vanish. Then $1/p(z)$ is entire and since $|p(z)| \to +\infty$ as $z \to +\infty$ we have that $1/p(z) \to 0$ as $z \to +\infty$ and hence $|1/p(z)| < \epsilon$ for $|z| > R$ and $1/p(z)$ is bounded for $|z| \leq R$ so $1/p(z)$ is bounded throughout $\mathbb{C}$ so $1/p(z)$ is constant, contradicting the fact that $p(z)$ is non-constant.

In particular, this implies that the polynomial can be written as:

$$p(z) = a_n z^n + \cdots + a_0 = a_n (z - z_1) (z - z_2) \cdots (z - z_n)$$

Let $z_1$ be a zero and then write

$$p(z) = a_n (z - z_1 + z_1)^n + \cdots + a_0 = a_n (z - z_1)^n + a'_n - 1 (z - z_1)^{n - 1} + \cdots + a'_1 (z - z_1) + a'_0$$

Clearly $a'_0 = 0$, by evaluation at $z_1$. So $p(z) = (z - z_1) q(z$) where $q(z)$ is a polynomial of degree $n - 1$ (and the leading coefficient of $q(z)$ is 1). Repeating this on $q(z)$ we obtain by induction

$$p(z) = a_n z^n + \cdots + a_0 = a_n (z - z_1) (z - z_2) \cdots (z - z_n).$$

6.4. Identity Theorem. The following is an application of the Taylor series theorem

**Lemma 6.20 (Isolation of zeros).** Let $r > 0$ and let $B(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$. The zeros of a non-trivial $f : D(z_0, r) \to \mathbb{C}$ cannot accumulate within a disk $D(z_0, r - \epsilon)$, where $\epsilon > 0$.

**Proof.** Assume for a contradiction that $z_k \to w \in B(z_0, r - \epsilon)$, for some $\epsilon > 0$, are a convergent sequence of zeros for $f(z)$. We can consider the restriction $f : B(w, \epsilon) \to \mathbb{C}$ and write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - w)^n$$

satisfies $a_n = 0$, for all $n \geq 0$, i.e., $f(z)$ is trivial contradicting the hypothesis. More precisely, assume for a contradiction to the claim that for some $N \geq 0$ we have that $a_0 = a_1 = \cdots = a_{N-1} = 0$ and $|a_n| > 0$. Since $f(z_k) = 0$ each $k \geq 1$ we can write

$$f(z_k) = \sum_{n=N}^{\infty} a_n (z_k - w)^n$$
If $|a_n| \leq C(2\epsilon)^n$, say, then we can bound
\[
0 = \left| \sum_{n=N}^{\infty} a_n(z_k - w)^n \right| \geq |a_n| - C \sum_{n=N}^{\infty} (r|z_k - w|)^n = |a_n| - C \frac{2|z_k - w|^N}{1 - 2|z_k - w|}.
\]
Providing $k$ is sufficiently large the right hand side is positive, giving a contradiction. 

**Theorem 6.21 (Identity Theorem).** Suppose that $f, g : U \to \mathbb{C}$ are analytic in the domain $U$ and suppose that $\{z_n\}, z_0 \in U$ and $z_n \to z_0$ with $z_n \neq z_0$ with $f(z_n) = g(z_n)$.

Proof. We have already proved this for a disk with centre $z_0$, say, by the previous lemma, we see that $f = g$ on a disk with centre $z_0$ contained in $U$.

Let $z_1 \in U$ then we can see that $f(z_1) = g(z_1)$ by parameterizing a polynomial line from $z_0$ to $z_1$. Otherwise let $w$ be a point on this line such that $f(z) = g(z)$ for all points from $z_0$ up to $w$ and for some points arbitrarily close to $w$ we have $f(z) \neq g(z)$. But $w$ is the limit of a sequence $\{w_n\}$ of points with $f(w_n) = g(w_n)$ and so $f = g$ in a disk centred at $w$.

This contradicts the definition of $w$ if $w$ is before $z_1$. Hence $z_1 = w$ and $f(z_1) = g(z_1)$.

**Corollary 6.22 (Isolation of values).** Let $f : U \to \mathbb{R}$ be analytic and non-constant. Let $\lambda \in \mathbb{C}$ then for each $z_0 \in U$ there exists a punctured disc $P(z_0, r) = \{z \in \mathbb{C} : |z_0 - z| < r\} - \{z_0\}$ such that $f(z) \neq \lambda$.

Proof. Assume for a contradiction that for every small punctured disk of radius $r = 1/n$, say, there exists $z_n \in P(z_0, 1/n)$ such that $f(z_n) = \lambda$. But this implies $f(z) = \lambda$ on $U$, i.e., $f(z)$ is constant and contradicts the hypothesis.

**Corollary 6.23 (Open Mapping Theorem).** Let $f : U \to \mathbb{R}$ be a non-constant analytic function. Then the image $f(V)$ of any open set $V \subset U$ is again an open set.

**Remark 6.24.** This is very different to case for $\mathbb{R}$. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not an open map since, for example, $f((-1, 1)) = [0, 1)$, which is not open.

Proof. Let $z_0 \in V$ then it is enough to show that there exists $\epsilon > 0$ such that $B(f(z_0), \delta) \subset f(V)$.

Let $g(z) = f(z) - f(z_0)$ then $g(z_0) = 0$. Moreover, $z_0$ must be a simple zero (i.e., multiplicity one) for $g$ otherwise the function $g(z)$ (and hence $f(z)$) would be a constant function. In particular, there is a simple closed curve $\gamma$ about $z_0$ and within $V$, containing which $z_0$ as the unique zero. Moreover, by taking the curve smaller (if necessary) we can assume that $g'(z) = f'(z)$ has no zeros within $\gamma$. By the Argument theorem we see that
\[
1 = \frac{1}{2\pi i} \int_{\gamma} \frac{g'(\xi)}{g(\xi)} d\xi = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(\xi)}{f(\xi) - f(z_0)} d\xi.
\]
Moreover, for $\gamma$ fixed, this remains true if we replace $f(z_0)$ by $f(z)$ with $|f(z) - f(z_0)| < \delta$, for sufficiently small $\delta > 0$. i.e., for such $f(z) \in B(f(z_0), \delta)$ there exists a unique $w$ inside $\gamma$, and thus in $V$.

**6.5. Families of functions and Montel’s Theorem.** Recall that a set $K \subset \mathbb{C}$ is compact if and only if it is closed and bounded.

**Definition 6.25.** Let $\{f_n\}_{n=1}^{\infty}$ be a family of continuous functions $f_n : U \to \mathbb{C}$, for $n \geq 1$. We say that this family is *normal* if there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges uniformly on every compact subset of $U$ to a continuous function $f : U \to \mathbb{C}$.
\( C \), i.e., for every compact subset \( K \subset U \) we have that \( \sup_{z \in K} |f_n(z) - f(z)| \to 0 \) as \( n \to +\infty \).

We can also adopt the convention that we allow uniform convergence to \( \infty \) (consistent with the functions being viewed as being valued in the Riemann Sphere).

In the case of analytic functions, we can get that the family is normal by just assuming the sequence is uniformly bounded.

**Theorem 6.26 (Montel’s Theorem).** Let \( f_n : U \to \mathbb{C} \), \( n \geq 1 \), be a family of analytic functions. Assume that there is a positive constant \( M > 0 \) such that
\[
|f_n(z)| \leq M \text{ for all } z \in U, n \geq 1
\]
then \( \{f_n\}_{n=1}^\infty \) is a normal family. Moreover, the limiting function \( f : U \to \mathbb{C} \) is analytic.

The proof is based on a classical result in real analysis. We recall the following definition.

**Definition 6.27.** Let \( K \) be a compact set. Let \( \{f_n\}_{n=1}^\infty \) be a family of continuous functions \( f_n : K \to \mathbb{C} \), for \( n \geq 1 \). We say that this family is equicontinuous if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( z, w \in K \) with \( |z - w| < \delta \) then \( |f_n(z) - f_n(w)| < \epsilon \) for all \( n \geq 1 \).

The corresponding result from real analysis we need is the following.

**Theorem 6.28 (Arzela-Ascoli).** Any equicontinuous family \( f_n : K \to \mathbb{C} \) has a subsequence \( (f_{n_k}) \) which is uniformly convergent to a continuous function \( f : K \to \mathbb{C} \) (i.e., \( \sup_{z \in K} |f_{n_k}(z) - f(z)| \to 0 \) as \( n_k \to +\infty \)).

**Proof.** Let \( z_0 \in U \) and let \( r > 0 \) be so that \( \overline{B(z_0, r)} \subset U \). Then \( \mathbb{C} - U \) and \( \overline{B(z_0, r)} \) are disjoint open sets. We can choose \( \eta > 0 \) such that if \( w \in \mathbb{C} - U \) and \( z \in \overline{B(z_0, r)} \) then \( |z - w| > \eta \).
If \( z \in \overline{B(z_0, \eta)} \) then we can consider restriction \( f_n : \overline{B(z_0, r)} \rightarrow \mathbb{C} \) and by Cauchy’s Theorem

\[
|f'(z)| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^2} \, dz \right| \leq \frac{M}{\rho}
\]

Thus for any \( z_1, z_2 \in \overline{B(z_0, \eta)} \) we have that

\[
|f_n(z_1) - f_n(z_2)| \leq \left( \sup_{|z| \in \overline{B(z_0, \eta)}} |f(z)| \right) |z_1 - z_2|.
\]

In particular, we see that the restrictions \( f_n : \overline{B(z_0, \eta)} \rightarrow \mathbb{C}, n \geq 1 \), form an equicontinuous family. Given \( \epsilon > 0 \) we simply choose \( \delta = \frac{\epsilon}{M} > 0 \).

If \( K \subset U \) is compact we can cover it be finitely many such balls and we again deduce that \( f_n : K \rightarrow \mathbb{C}, n \geq 1 \), forms an equicontinuous family. We can then invoke the Arzela-Ascoli theorem to deduce that there is a convergent subsequence \( f_{n_k} : K \rightarrow \mathbb{C} \), for \( k \geq 1 \).

The analyticity of the function \( f : U \rightarrow \mathbb{C} \) follows from the Hurwitz-Weiestrauss theorem.

\[ \square \]

**Remark 6.29.** There is a stronger version which says that \( f_n : U \rightarrow \mathbb{C}, n \geq 1 \), is a normal value if they all omit the same three values.

**Application 6.30 (Julia sets).** Let \( f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) be a polynomial (or, more generally, a rational map).

**Definition.** We say that \( z \in \mathbb{C} \) is in the Fatou set \( \mathcal{F} \) if there exists \( \epsilon > 0 \) such that the compositions \( \{ f^n : B(z_0, \epsilon) \rightarrow \mathbb{C} : n \geq 1 \} \) is a normal family.

We define the Julia set \( \mathcal{J} \) to be the complement of the Fatou set \( \mathcal{J} = \mathbb{C} - \mathcal{F} \).

**Example.** If \( f(z) = z^2 \) then \( f^n(z) = z^{2^n} \). Thus on \( U_\infty = \{ z \in \mathbb{C} : |z| > 1 \} \) we have that \( f^n(z) \rightarrow +\infty \), i.e., the family of iterates converge to \( \infty \). On the other hand, on \( U_0 = \{ z \in \mathbb{C} : |z| < 1 \} \) we have that \( f^n(z) \rightarrow 0 \), i.e., the family of iterates converge to \( \infty \). Thus the Fatou set contains \( U_\infty \cup U_0 \). But if \( |z| = 1 \) then in any disk \( B(z, \epsilon) \) those points also inside the unit disk will converge to 0 while those points also outside the unit disk will converge to \( \infty \). i.e., \( f^n(z) \) doesn’t converge to an analytic function on \( B(z, \epsilon) \).

Therefore \( \mathcal{F} = U_\infty \cup U_0 \) and \( \mathcal{J} = C(0, 1) \). The Julia set is closed, bounded and \( f(J) = J \).

**Theorem 6.31.** The Julia set consists precisely of those points \( z \in \mathbb{C} \) such that for every \( \epsilon > 0 \) there exists \( w \in B(z, \epsilon) \) such that \( f^n(w) \rightarrow +\infty \).

**Proof.** Assume that for every \( \epsilon > 0 \) there exists \( w \in B(z, \epsilon) \) such that \( f^n(w) \rightarrow +\infty \). Since \( f^n(z) \in J \) is bounded, we cannot have \( f^n : B(z, \epsilon) \rightarrow \mathbb{C} \) isn’t normal and thus \( z \in \mathcal{J} \).
Conversely, given \( z \) if there exists \( \epsilon > 0 \) such that for all \( w \in B(z, \epsilon) \) we have that \( f^n(w) \) is bounded. Thus we can apply Montel’s theorem to deduce that \( f^n : B(z, \epsilon) \to \mathbb{C} \) is normal, i.e., \( z \notin \mathcal{J} \). \( \square \)

Exercise: If \( f \) is a polynomial and \( z \in \mathbb{C} \) then either \( f^n(z) \to \infty \) or \( \{f^n(z) : n \geq 0\} \) is a bounded set.

**Theorem 6.32.** The Julia set is equal to the closure of the repelling periodic points.

The proof uses the other version of Montel’s theorem.

### 7. Riemann Mapping Theorem

One of the most interesting results in complex analysis is the Riemann Mapping Theorem.

#### 7.1. Riemann Mapping Theorem

The most elegant theorem in complex analysis is the following.

**Theorem 7.1 (Riemann Mapping Theorem).** Let \( U \subset \mathbb{C} \) be an open subset homeomorphic to \( \mathbb{D} \). Then there exists an analytic bijection \( f : U \to \mathbb{D} \).

**Proof.** The proof is quite elaborate, so we break it down into steps. We assume for simplicity that \( U \) is bounded, i.e., \( U \subset B(z_0, R) \) for some \( R > 0 \).

**Step 1:** Fix \( z_0 \in U \) and let \( \mathcal{F} \) be the family of analytic functions \( f : U \to \mathbb{D} \) such that

1. \( f \) is a bijection; and
2. \( f(z_0) = 0 \).

To show that \( \mathcal{F} \neq \emptyset \) we can construct \( f : U \to \mathbb{D} \) by

\[
f(z) = \frac{(z - z_0)}{2R}
\]

then

\[
|f(z)| = \left| \frac{(z - z_0)}{2R} \right| \leq \frac{|z| + |z_0|}{2R} \leq \frac{R + R}{2R} \leq 1
\]

and by construction we see that \( f(z_0) = 0 \) and thus \( f \in \mathcal{F} \).

**Step 2:** Since the functions in \( \mathcal{F} \) are analytic and bounded (i.e., for \( f \in \mathcal{F} \) and \( |f(z)| \leq 1 \) we can apply Montel’s theorem to deduce that the family is normal.

**Step 3:** We define

\[
M = \sup \{|f'(z_0)| : f \in \mathcal{F}\}.
\]

**Claim:** We want to show that this is finite (i.e., \( M < +\infty \)).
Proof of Claim: Choose \( r > 0 \) sufficiently small that \( B(z_0, r) \subset U \). By Cauchy’s theorem we have that for any \( f \in \mathcal{F} \),
\[
|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^2} \, dz \right| \leq \frac{1}{r} \left( \sup_{z \in \overline{B}} |f(z)| \right) \leq \frac{1}{r}.
\]

Thus the result follows with \( M \leq \frac{1}{r} \).

Step 4: We now want to show the supremum is realized.

Claim: There exists \( f_0 \in \mathcal{F} \) such that \( f_0'(z_0) = M \).

Proof of Claim: We can choose \( f_n \in \mathcal{F} \), \( n \geq 1 \), such that \( |f_n'(z_0)| \to M \) as \( n \to +\infty \). By Montel’s Theorem applied to \( \{f_n\} \) we can choose a subsequence \( \{f_{n_k}\}_{k=1}^{\infty} \) and an analytic function \( f : U \to \mathbb{C} \) such that \( f_{n_k} \to f \) uniformly on any compact set \( K \subset \mathbb{C} \). Therefore, we see that \( |f'(z_0)| = M \).

If we replace \( f(z) \) by \( e^{i\theta} f(z) \) then we can choose \( 0 \leq \theta < 2\pi \) so that \( f'(z_0) = M \).

Step 5: We want to show that \( f : U \to \mathbb{D} \) is injective.

Claim: \( f : U \to \mathbb{D} \) is injective.

Proof of Claim: Let \( z_1 \neq z_2 \) be distinct points in \( U \). Let \( 0 < \rho = |z_1 - z_2| \) and define \( g_k : \overline{B(z_2, \rho)} \to \mathbb{C} \) by
\[
g_k(z) = f_n(z) - f_{n_k}(z_1)
\]

Since the functions \( f_{n_k}(\in \mathcal{F}) \) are all injective, the functions \( g_k \) are all non-zero on \( \overline{B(z_2, \rho)} \) (i.e., \( \not\exists z \in \overline{B(z_2, \rho)} \) with \( f(z) = 0 \)).

By the Hurwitz-Weierstrass theorem the limit function \( g(z) = \lim_{k \to \infty} g_k(z) \) is either identically zero or has no zeros. But, it cannot be identically zero (since we assume that \( |f'(z_0)| = M > 0 \)). Thus \( f(z) \neq f(z_1) \) for all \( z \in \overline{B(z_2, \rho)} \) and thus \( f(z_2) \neq f(z_1) \).

Step 6: We want to show that \( f : U \to \mathbb{D} \) is surjective.

Claim: \( f : U \to \mathbb{D} \) is surjective.

Proof of Claim: Assume for a contradiction that \( f \) is not surjective, and thus there exists \( w \in \mathbb{D} \) which is not in the image \( f(U) \).

We can choose a Möbius map \( \phi_w : \mathbb{D} \to \mathbb{D} \) by
\[
\phi_w(z) = \frac{z-w}{1-zw}
\]
which maps \( w \) to 0. We then consider \( \phi_w \circ f : U \to \mathbb{D} \). This map now doesn’t have 0 in its image. As we saw in a previous lemma, we can write \( \phi_w \circ f(z) = e^{k(z)} \) where \( k : U \to \mathbb{C} \) is analytic, and thus we can define \( g : U \to \mathbb{C} \) by
\[
g(z) = (\phi_w \circ f(z))^{1/2}(= e^{k(z)/2}).
\]
In particular, \( g : U \to \mathbb{D} \) is one-to-one. Finally, we introduce
\[
\phi_{g(z_0)}(z) = \frac{z - g(z_0)}{1 - zg(z_0)}
\]
and define \( \rho : U \to \mathbb{D} \) by \( \rho(z) = \phi_{g(z_0)} \circ g(z_0) \).

We can then compute
\[
\rho'(z_0) = \frac{\phi'_{g(z_0)}(g(z_0))^2 \cdot g'(z_0)}{1 - (\rho(z_0))^2}
\]  
(1)
and
\[(g(z_0)^2)' = 2g'(z_0).g(z_0)
= \phi'(f(z_0)).f'(z_0)
= (1 - |w|^2)f'(z_0).\]  

(2)

(after a little calculation.)

From (1) and (2) we find that:
\[\rho'(z_0) = \frac{1}{1 - |g(z_0)|^2} \left( \frac{1 - |w|^2}{2g(z_0)} \right) f'(z_0)
= \frac{1}{1 - |w|} \left( \frac{1 - |w|^2}{2g(z_0)} \right) f'(z_0)
= \left( \frac{1 + |w|}{2g(z_0)} \right) f'(z_0)\]

However, since \(w \neq 0\) then \(1 + |w| > 1\) and \(g(z_0) = \sqrt{|w|}\) and thus
\[\rho'(z_0) = \frac{(1 + |w|)}{\sqrt{|w|}} \left( \frac{1 + |w|}{2g(z_0)} \right) f'(z_0)\]

\[\geq \frac{1}{\sqrt{|w|}}\]

giving \(|\rho'(z_0)| > M\), a contradiction to the definition of \(M\).

However, this contradicts \(h(z)\) having maximum derivative \(M\) at \(z_0\). \[\square\]

Remark 7.2. If \(U\) is unbounded then we can apply a more elaborate argument to prove Step 1. Let \(z_0 \in \mathbb{C} - U\). The function \(\phi : U \rightarrow \mathbb{C}\) is non-vanishing and we can write \(\phi(z) = e^{k(z)} = k(z)^2\), where \(k(z) = e^{h(z)/2}\). Moreover, \(k(z)\) is one-to-one (since \(\phi(z)\) is) and there are not distinct points \(z_1, z_2\) such that \(h(z_1) = h(z_2)\) (since then \(\phi(z_1) = \phi(z_2)\)). This is an open mapping and so we can choose \(B(b, r) \subset h(U)\).

But then \(B(-b, r) \cap h(U) = \emptyset\).

We may therefore define the holomorphic function
\[f(z) = \frac{r}{2(h(z) + b)}\].

Since \(|h(z) - h(-z)| \geq r\) for \(z \in U\), it follows that \(f : U \rightarrow \mathbb{D}\). Since \(h\) is one-one then so is \(f\). Composing \(f\) with a Möbius map that preserves \(\mathbb{D}\) we can get a function which is analytic and one-on-one and bounded by 1.

Proposition 7.3 (Caratheodory’s Theorem). Let \(U \subset \mathbb{C}\) be a simply connected open set whose boundary is a simple closed curve. Let \(f : U \rightarrow \mathbb{D}\) be the analytic bijection in the Riemann mapping theorem. Then \(\phi\) extends to a homeomorphism from \(U\) to \(\mathbb{D}\).

7.2. Christoffel-Schwarz Theorem. The Christoffel-Schwarz Theorem is a more explicit version of the Riemann mapping theorem where the open region to be mapped to the unit disk is taken to be the inside of a polygon \(P\).
We recall that the map \( w = f(z) = i \frac{z}{1+z} \) maps the unit disk to the upper half plane \( \mathbb{H}^2 \).

Thus it suffices to find a map \( g : \mathbb{H} \to \mathbb{P} \) which is analytic and then \( g \circ f : \mathbb{D} \to \mathbb{P} \).

The general result is the following:

**Theorem 7.4 (Schwarz-Christoffel transform).** Let \( \mathbb{P} \) be a polygon with \( n \) sides such that the internal angles are \( \pi \alpha_1, \ldots, \pi \alpha_n \). There exist \( w_1, \ldots, w_n \in \mathbb{R} \) such that if we consider the function

\[
F(w) = \frac{C}{(w - w_1)^{1-\alpha_1}(w - w_2)^{1-\alpha_2} \cdots (w - w_n)^{1-\alpha_n}}
\]

(1)

then for \( z_0 \in \mathbb{H} \) we can choose \( A, C \) such that \( f(z) := A + \int_{z_0}^{z} F(w)dw \) is an analytic bijection from \( \mathbb{H} \) to the interior of the polygon \( \mathbb{P} \) (and which extends continuously to the boundary, taking points \( w_1, \ldots, w_k \) to the vertices of \( \mathbb{P} \)).

Here \( C \) changes the size and orientation of the image and \( A \) translates it.

**Remark 7.5.** Since the external angles of the polygon sum to \( (n - 2)\pi \) we see that

\[
\sum_{i=1}^{n} (1 - \alpha_i) = -2.
\]

**Remark 7.6.** It is often convenient to consider the case in which the point at infinity of the \( w \) plane maps to one of the vertices of the polygon (e.g., \( x_1 = \infty \) and the vertex with angle \( \alpha_1 \)). If this is done, the first factor in the formula is effectively a constant and may be regarded as being absorbed into the constant \( C \).

The following general result is often useful. Let \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) and \( \mathbb{H}_- = \{ z \in \mathbb{C} : \text{Im}(z) < 0 \} \).

**Lemma 7.7 (Schwarz-reflection principle).** If \( g : \mathbb{H} \to \mathbb{C} \) is analytic and extends continuously to \( [a, b] \subset \mathbb{R} \) such that \( g([a, b]) \subset \mathbb{R} \) then \( g \) has an extension as an analytic function to \( \mathbb{H} \cup [a, b] \cup \mathbb{H}_- \).

**Proof.** For \( z \in \mathbb{H} \) we have that \( \overline{z} \in \mathbb{H}_- \) and we define \( g(\overline{z}) = \overline{g(z)} \). To deduce analyticity we can use Moreira’s theorem. For example, if \( \gamma \) is a closed curve that crosses \( [a, b] \) then we can consider two loops \( \gamma_1, \gamma_2 \) consisting of the pieces of \( \gamma \) in each half-plane plus a subinterval of \( [a, b] \) between their intersection points. By approximating them by curves wholly in the half planes we see that \( \int_{\gamma_1} f(z)dz = 0 \) and \( \int_{\gamma_2} f(z)dz = 0 \), and thus \( \int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = 0 \). Thus Moreira’s Theorem applies and we are done. \( \square \)
Lemma 7.8 (Osgood-Caratheodory Theorem). If $g : \mathbb{P} \to \mathbb{D}$ is an analytic bijection then it extends to a continuous function $g : \overline{\mathbb{P}} \to \overline{\mathbb{D}}$ on the boundary.

Proof. Assume that $\alpha_i \leq 1$.

1. Choose one side $I_1$ of $\mathbb{P}$ and use a Möbius map $f_1 : \mathbb{P} \to \mathbb{H}$ such that $S := f_1(\mathbb{P})$ and $f_1(I_1) = [-1,1]$.

2. We can reflect $S$ in the real axis to get $\overline{S}$ and consider $U := S \cup (-1,1) \cup \overline{S}$. By the Riemann mapping theorem we can find a unique analytic bijection $f_2 : U \to \mathbb{D}$ with $f_2(0) = 0$ and $f'_2(0) > 0$. However, we see that $g_2(z) := f_2(\overline{z})$ satisfies the same conclusion and so $g_2 = f_2$ and we conclude that $f_2([-1,1]) \subset \mathbb{R}$ and then $f_2(S) = \mathbb{D}^+ = \{z \in \mathbb{D} : \text{Im}(z) > 0\}$.

3. We can choose a Möbius map $f_3 : \mathbb{D}^+ \to \mathbb{D}$. Then we set $f(z) = f_3 \circ f_3 \circ f_1(z)$. We thus see that $f$ extends continuously to $I_1$.

If we fix $z_1 \in \mathbb{P}$ then we can choose $f_3$ so that $f(z_1) = 0$ and $f'(z_1) > 0$. Thus $f$ is unique. In particular, the conclusion that $f$ extends applies to each side.

This brings us to the main part of the Schwarz-Christoffel theorem.

Theorem 7.9. Every analytic bijection $f : \mathbb{H} \to \mathbb{P}$ is necessarily of the form (1).

The existence of such maps follows from the Riemann Mapping Theorem. The mapping $f(z)$ extends continuously to the boundary. We can then extend it to $\mathbb{H}[-]$ via one of the intervals $(x_i, x_{i+1})$ by the Schwarz reflection principle.

In particular, $f(\mathbb{H})$ corresponds to a polygon $\mathbb{P}'$ obtained from $\mathbb{P}$ by reflection in the corresponding side. Using the reflection principle (for another interval
\((x_j, x_{j+1})\) gives another branch of the function, which we denote by \(f_1(z)\). However, the function \(\frac{f''(z)}{f'(z)} = \frac{f''_1(z)}{f'_1(z)}\) is uniquely defined.

We can observe that \(g(z) = (f(z) - f(x_i))^{\frac{1}{2}}\) maps the interval \((x_i - \epsilon, x_i + \epsilon)\) onto another line segment. Thus again by the reflection principle we see that \(g(z)\) is analytic. We can then write that \(f(z) = f(z_i) + g(z)^{\alpha_i}\), for \(z\) near to \(x_i\) and thus \(f''(z) \frac{f'(z)}{f'(z)} = (\alpha_i - 1) \frac{z - x_i}{z - x_i} + h(z) = \frac{f''_1(z)}{f'_1(z)}\)

where \(h_i : \mathbb{H} \rightarrow \mathbb{C}\) is analytic.

Moreover, we can write globally that \(f''(z) \frac{f'(z)}{f'(z)} = \sum_{i=1}^{k} \frac{(\alpha_i - 1)}{z - x_i} + h(z)\)

where \(h : \mathbb{H} \rightarrow \mathbb{C}\) is analytic - and \(h\) extends to an entire function. However, we can deuce that it is the constant function.

**Schwartz - Christoffel for \(D\).** We can reformulate this in terms of the unit disk (to look more like the Riemann Mapping Theorem, except that it goes in the wrong direction).

Let \(D = \{z \in \mathbb{C} : |z| < 1\}\) denote the unit disk. Let \(\mathbb{P}\) be an open bounded region in the complex plane with polygonal boundary and vertices \(V = \{v_1, \cdots, v_n\}\), say. The Schwarz-Christoffel theorem gives a (fairly) explicit formula for a holomorphic bijection \(\psi : D \rightarrow \mathbb{P}\).

**Corollary 7.10.** More precisely, the map \(\psi : D \rightarrow \mathbb{P}\) defined by:

\[
\psi(z) = A + C \int_{z_0}^{z} \left(1 - \frac{\xi}{w_1}\right)^{\alpha_1 - 1} \cdots \left(1 - \frac{\xi}{w_k}\right)^{\alpha_k - 1} d\xi,
\]

where

1. The exponents \(\alpha_1, \cdots, \alpha_k\) correspond to the internal angles of \(\mathbb{P}\);
2. the points \(W = \{w_1, \cdots, w_k\} \in \partial D\) on the unit circle correspond to preimages of the vertices \(V\);
7.3. examples.

Example 7.11 (Maps to a semi-infinite strip). Consider a semi-infinite strip $P$ in the $z$-plane. (This may be regarded as a limiting form of a triangle with vertices 0, $\pi i$, and $R$ (with $R$ real), as $R$ tends to infinity). Now $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = \pi/2$ in the limit.

Suppose we guess that $x_1 = -1$ and $x_2 = 1$ and then we are looking for the mapping $f : \mathbb{H} \to P$ with $f(-1) = \pi i$, $f(1) = 0$ (and $f(\infty) = \infty$). Then $f(z)$ is given by

$$f(z) = A + \int_{z_0}^{z} \frac{C}{\sqrt{(w-1)(w+1)}} \, dw = A + \int_{z_0}^{z} \frac{C}{\sqrt{(w^2 - 1)}} \, dw$$

In this special case the integral can be explicitly evaluated to get

$$f(z) = A + C \cosh^{-1}(w)$$

where $A, C \in \mathbb{C}$. Requiring that $f(-1) = \pi i$ and $f(1) = 0$ gives $A = 0$ and $C = 1$. Hence the Schwarz-Christoffel mapping is given by $f(z) = \cosh^{-1}(z)$.

Example 7.12 (Maps to another semi-infinite strip). Consider the semi-infinite strip

$$S = \{x + iy : -a < x < a \text{ and } y > 0\}$$

for $a > 0$. Find an explicit form for the holomorphic bijection $f : \mathbb{H} \to S$ from the upper half plane to $S$ for which extends to $f(\pm 1) = \pm a$ and $f(\infty) = \infty$. 
The strip has internal angles $\pi/2$. We take the two points to be mapped to the $\pm a$ to be $\pm 1$. By the Christoffel-Schwarz Theorem this takes the form

$$f(z) = A \int_0^z (w - 1)^{-\frac{1}{2}} (w + 1)^{-\frac{1}{2}} dw + B = A \int_0^z \frac{dw}{\sqrt{1 - w^2}} + B$$

We can evaluate the indefinite integral:

$$f(z) = B + A \sin^{-1}(z)$$

The image $-a = f(-1)$ of the first vertex $-1$ gives that

$$-a = B + A \sin^{-1}(-1) = B - \frac{A\pi}{2}.$$  

The image $a = f(1)$ of the second vertex $1$ gives that

$$a = B + A \sin^{-1}(1) = B + \frac{A\pi}{2}.$$  

from which we deduce that $B = 0$ and thus $A = 2a/\pi$ and thus

$$f(z) = \frac{2a \sin^{-1}(z)}{\pi}$$

**Example 7.13 (An equilateral triangle).** Let $f : \mathbb{H} \to \mathbb{C}$ be defined by

$$f(z) = \int_0^z \frac{dw}{(w(1-w))^{2/3}}.$$  

We claim that the image is an equilateral triangle with vertices at $0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}$. The internal angles are all equal to $\frac{\pi}{3}$. It we consider the real axis as the boundary of $\mathbb{H}$ then we see that the interval $[0, 1]$ is mapped to $[0, K]$, say, since the integrand is real. Here $K = \int_0^1 \frac{dt}{t(1-t)^{2/3}} = 5.299 \cdots$.

The interval $[1, \infty)$ is mapped to the side $[K, \frac{K}{2} + \frac{i\sqrt{3}}{2} K)$ and the interval $(-\infty, 0]$ is mapped to the side $(-\frac{K}{2} + \frac{i\sqrt{3}}{2} K, 0]$.

More generally, if the triangle with angles $\pi \alpha_1, \pi \alpha_2$ and $\pi(1 - \alpha_1 - \alpha_2)$ then let $g : \mathbb{H} \to \mathbb{C}$ be defined by

$$f(z) = A + C \int_0^z \frac{dw}{(w - 1)^{1-\alpha_1}(w + 1)^{1-\alpha_2}}.$$  

(i.e., $x_1 = -1, x_2 = 1$ (and $x_3 = \infty$). The image is a triangle with angles $\pi \alpha_1, \pi \alpha_2$ (and $\pi(1 - \alpha_1 - \alpha_2)$).
Example 7.14 (A right angle triangle). Consider the triangle $\Delta$ in $\mathbb{C}$ with vertices at $-1$, 0 and $i$ and find the analytic bijection $f : \mathbb{H} \to \Delta$ which extends to $f(-1) = i$, $f(1) = -1$ and $f(\infty) = 0$.

Show that

$$f(z) = \frac{-1 - i}{2\kappa} \int_0^z (z^2 - 1)^{-3/4} dz + \frac{i - 1}{2}$$

where

$$\kappa = \int_0^{-1} (z^2 - 1)^{-3/4} dz$$

The vertices $i$ and $-1$ have internal angles of $\frac{\pi}{4}$ and are the images of $-1$ and 1, respectively. We can use the Christoffel Schwarz Theorem to write

$$F(z) = (z + 1)^{-3/4}(z - 1)^{-3/4} = (z^2 - 1)^{-3/4}$$

We have that

$$f(z) = A \int_0^z (z^2 - 1)^{-3/4} dz + B$$

and then

$$i = A \int_0^{-1} (z^2 - 1)^{-3/4} dz + B \quad \text{and} \quad -1 = A \int_0^1 (z^2 - 1)^{-3/4} dz + B.$$ 

If we denote

$$\kappa = \int_0^1 (z^2 - 1)^{-3/4} dz \quad \left( = - \int_{-1}^0 (z^2 - 1)^{-3/4} dz \right)$$

then $-A\kappa + B = i$ and $A\kappa + B = -1$ and we can solve for $A = \frac{-1-i}{2\kappa}$ and $B = \frac{i-1}{2}$. Hence

$$f(z) = \frac{-1-i}{2\kappa} \int_0^z (z^2 - 1)^{-3/4} dz + \frac{i-1}{2}.$$ 

Example 7.15 (Squares and Rectangles). Let $g : \mathbb{H} \to \mathbb{C}$ be defined by

$$f(z) = \int_0^z \frac{dw}{(w(1-w)(\rho-w))^{1/2}}$$

where $\rho > 1$. The image is a rectangle vertices at 0, $-K, -K - iK', -iK'$. 

[Diagram of a right angle triangle with vertices at $-1, 0, i$.]

[Diagram of a rectangle with vertices at $0, -K, -K - iK', -iK'$.]

[Diagram showing the mapping $f(z)$ from $\mathbb{H}$ to $\Delta$.]
More precisely, the internal angles are all equal to $\frac{\pi}{d}$. It we consider the real axis as the boundary of $\mathbb{H}$ then we see that the interval $[0, 1]$ is mapped to the side $[-K, 0]$ where

$$K = \int_0^1 \frac{dw}{(w(1 - w)(\rho - w))^{1/2}}.$$ 

For $1 < w < \rho$ there is a single imaginary root and thus the integral from $[1, \rho]$ is purely imaginary (with negative imaginary part) and the image of $[1, \rho]$ is $[-K, -K - iK']$ where

$$K' = \int_1^\rho \frac{1}{(w(w - 1)(\rho - w))^{1/2}} dw.$$ 

Similarly, we see that the image of $[\rho, \infty)$ is $[-K - iK', -iK')$ and the image of $(-\infty, 0]$ is $(-iK', 0]$.

The upper half plane $\mathbb{H}$ is mapped to the square by

$$z = f(w) = \int \frac{dw}{w(w - 1)(w + 1)}$$

7.4. Generalizations of the Schwarz-Christoffel theorem. More generally one can consider an analogous problem where we introduce holes into the picture. Let $z_1, \ldots, z_d \in \mathbb{D}$ be the centres of subdisks $D_i = B(z_i, r_i) = \{ z \in \mathbb{D} : |z - z_i| < r_i \}$ where $|z_i| + r_i < 1$. We can then associate a “multiply connected region” defined by $\mathbb{M} = \mathbb{D} - \bigcup_{i=1}^d D_i$. When $d = 1$ this simply corresponds to a annulus. Similarly, we can consider sub-polygons $P_1, \ldots, P_d \subset \mathbb{P}$ with sets of vertices $V_1, \ldots, V_d$, respectively. We then let $Q = \mathbb{P} - \bigcup_{i=1}^d P_i$.

Let $C_i = \partial D_i$ and let $\overline{C}_i$ denote the images on reflection in the unit circle $\{ z \in \mathbb{C} : |z| = 1 \} = \partial \mathbb{D}$. In particular, $g_i(z) = z_i + \frac{r_i^2 z}{1 - \bar{z}_i z}$ satisfies $g_i(\overline{C}_i) = C_i$. Let $\Gamma = \langle g_1, \ldots, g_d \rangle$ be the associated Schottky group. By a result of Crowdy et al, there is an explicit formula for a holomorphic bijection $\psi : \mathbb{D} \rightarrow \mathbb{P}$ given by

$$\psi(z) = \int_{z_0}^{z} C(\cdot) \prod_{i=1}^k w(z, w_i)^{\alpha_i} \prod_{i=1}^d \prod_{j=1}^{|V_i|} w(z, w_j^{(i)})^{\alpha_j^{(i)}} dz.$$ (2)

where $w(\cdot, \cdot)$ is the Schottky-Klein prime function and

(1) the exponents $\alpha_1, \ldots, \alpha_k$ correspond to the internal angles of $\mathbb{P}$ and exponents $\alpha_j^{(i)} \ldots, \alpha_n^{(i)}$ correspond to the internal angles of $P^{(i)}$;

(2) the points $W = \{ w_1, \ldots, w_k \} \in \partial D$ on the unit circle correspond to preimages of the vertices $V$ and the points $W^{(i)} = \{ w_1^{(i)}, \ldots, w_{|V_i|}^{(i)} \} \in \partial D_i$ on the circle correspond to preimages of the vertices $V_i$;

(3) $C(\cdot)$ is an explicit correction function.

7.5. Application: Fluid dynamics. The streamlines around a circular region of diameter $2a$ can be described by transforming the horizontal lines past a slit of width $4a$ by the associated conformal map $f(z) = z + a^2/z$. More generally, given a finite number of disjoint circular regions the stream lines described by transforming the horizontal lines past a slits. ??

8. Behaviour of analytic functions

There are very useful tools in complex analysis: The Maximum Principle, and the Schwarz and Pick Lemmas.
8.1. The Maximum Modulus Principle. The next theorem shows that the supremum of the modulus of an analytic function isn’t achieved inside $U$.

**Theorem 8.1 (Maximum Modulus Principle).** Let $f : U \to \mathbb{C}$ be analytic. Let $U$ be connected. If $z_0 \in U$ satisfies $|f(z_0)| \geq |f(z)|$ for all $z \in U$ then $f(z)$ is necessarily a constant function.

**Proof.** By multiplying by a constant, if necessary, we can assume without loss of generality that $f(z_0) = 1$. Let

$$S = \{z \in U : |f(z)| = |f(z_0)|\}.$$

Since $f$ is continuous we see that $S \subset U$ is a closed set.

We claim that $S \subset U$ is also open. Given $z \in S \subset U$, since $U$ is open we can choose $r > 0$ sufficiently small that $B(z,r) \subset U$. Then for any $0 < r' < r$ we have by Cauchy’s Theorem that

$$f(z_0) = |f(z)| = \left| \frac{1}{2\pi i} \int_{C(0,r')} \frac{f(\xi)}{\xi - z} d\xi \right|$$

$$= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + r'e^{i\theta})}{r'e^{i\theta}} (ie^{i\theta}) d\theta \right|$$

$$= \left| \frac{1}{2\pi i} \int_0^{2\pi} f(z + r'e^{i\theta}) d\theta \right|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + r'e^{i\theta})| d\theta \leq f(z_0) = |f(z)|,$$

where $|f(z + r'e^{i\theta})| \leq |f(z)| = f(z_0)$, since $z \in S$. Thus all the inequalities must be equalities and so

$$|f(z + r'e^{i\theta})| = f(z_0)$$

for all $0 \leq \theta < 2\pi$ and all $0 < r' < r$. In particular, $S$ contains the open ball $B(z,r) \subset S$. But since $S$ is both open and closed and $U$ is connected we deduce that $S = U$ and thus $f$ takes the constant value $f(z_0)$.

8.2. The Schwarz Lemma. The next result describes analytic maps of the disk $D = \{z \in \mathbb{C} : |z| < 1\}$ to itself.

**Theorem 8.2 (Schwarz Lemma).** Let $f : D \to D$ be an analytic map.

1. Let $f(0) = 0$ then:
   a. $|f(z_0)| \leq |z_0|$, $\forall z_0 \in D$; and
   b. $|f'(0)| \leq 1$.

2. Moreover, if either $|f(z)| = |z|$ for some $z \in D - \{0\}$ or $|f'(0)| = 1$ then $f$ is a “rotation” of the form $f(z) = e^{i\theta}z$, for some $0 \leq \theta < 2\pi$.

**Proof.** (1) The function $g : D \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} 
   \frac{f(z)}{z} & \text{if } z \neq 0 \\
   f'(0) & \text{if } z = 0
\end{cases}$$

is analytic on $D$. Fix $\epsilon > 0$. We can apply the Maximum Modulus Principle to the restriction $g : B(0,1-\epsilon) \to \mathbb{C}$ to deduce that for $|z| \leq 1 - \epsilon$ we have that

$$|g(z)| \leq \frac{1}{1-\epsilon}.$$

Letting $\epsilon \to 0$ we deduce that $|g(z)| \leq 1$ whenever $|z| \leq 1$. From the definition of $g$ this completes the proof of the first part.

(2) For the second part, if $|f(z_0)| = |z_0|$ then $|g(z_0)| = 1$. The maximum modulus principle applied to $g$ now tells us that $g$ has a constant value, $e^{i\theta}$, say. This implies that $f(z)$ is a rotation.
Similarly, if \( |f'(0)| = |g(0)| \) then the maximum principle principle applied to \( g \) again tells us that \( g \) has a constant value, \( e^{i\theta} \), say. \( \square \)

**Application 8.3.** We denote a Möbius transformations \( \phi_a : \mathbb{D} \to \mathbb{D} \) by

\[
\phi_a(z) = \frac{z - a}{1 - \bar{a}z}
\]

where \( a \in \mathbb{D} \). We can also denote rotations \( \rho_\theta : \mathbb{D} \to \mathbb{D} \) by \( \rho_\theta(z) = e^{i\theta}z \), for \( 0 \leq \theta < 2\pi \).

**Proposition 8.4.** Any analytic function \( f : \mathbb{D} \to \mathbb{D} \) which is a bijection is of the form \( f = \phi_a \circ \rho_\theta \), for some \( a \in \mathbb{D} \) and \( 0 \leq \theta < 2\pi \).

**Proof.** We first observe that \( \phi_a(\partial \mathbb{D}) = \partial \mathbb{D} \). If \( |z| = 1 \) then

\[
|\phi_a(z)| = \left| \frac{z - a}{1 - \bar{a}z} \right| = \frac{|z - a|}{|z - \bar{a}|} = 1.
\]

By the Maximum Modulus Principle, since \( \phi_a \) is not a constant function if \( |z| < 1 \) then \( |\phi_a(z)| < |z| < 1 \). In particular, \( \phi_a(\mathbb{D}) \subset \mathbb{D} \). Observe that \( \phi_a^{-1} = \phi_{-a} \), showing that \( \phi_a : \mathbb{D} \to \mathbb{D} \) is a bijection.

Let us denote \( f(0) = b \), say. Consider \( g = \phi_b \circ f \), which is now an analytic function \( g : \mathbb{D} \to \mathbb{D} \) for which \( g(0) = 0 \). By the Schwarz Lemma, \( |g'(0)| \leq 1 \). Similarly, we can apply the Schwarz lemma to \( g^{-1} : \mathbb{D} \to \mathbb{D} \) to deduce that \( |(g^{-1})'(0)| = \frac{1}{|g'(0)|} \leq 1 \). We therefore have \( |g'(0)| = 1 \). By the uniqueness part of the Schwarz Lemma we have that \( g(z) = \rho_\theta \) for some \( 0 \leq \theta < 2\pi \). But then this means that \( f(z) = \phi_{-b} \circ \rho_\theta \). \( \square \)

**8.3. Pick’s Lemma.** There is a generalization of Schwarz’s lemma in which we don’t require \( f(0) = 0 \)

**Theorem 8.5 (Pick’s Lemma).** Let \( f : \mathbb{D} \to \mathbb{D} \) be analytic.

1. For \( z_1, z_2 \in \mathbb{D} \),

\[
\frac{|f(z_1) - f(z_2)|}{1 - |f(z_1)f(z_2)|} \leq \frac{|z_1 - z_2|}{1 - |z_1z_2|}
\]

and

\[
|f'(z_1)| \leq \frac{1 - |f(z_1)|^2}{1 - |z_1|^2}
\]

2. If there is either an equality in (*) for \( z_1 \neq z_2 \), or an equality in (**) for some \( z_1 \), then \( f \) must be an analytic bijection.

**Proof.** Let

\[
\phi(z) = \frac{z_1 - z}{1 - \bar{z}_1z} \text{ and } \psi(z) = \frac{f(z_1) - z}{1 - f(z_1)z}.
\]

Then \( \psi \circ f \circ \phi \) satisfies the hypothesis of the Schwarz Lemma. Thus for any \( z \in \mathbb{D} \),

\[
|\psi \circ f \circ \phi(z)| \leq |z|.
\]

Setting \( z = \phi^{-1} \) gives the inequality (1).

The Schwarz Lemma also gives that

\[
|(\psi \circ f \circ \phi)'(z)| \leq 1.
\]

Using the chain rule gives the inequality (2). \( \square \)
9. Harmonic functions

We want to consider the real valued functions.

**Definition 9.1.** We say that any function \( u : U \to \mathbb{R} \) is harmonic if \( \Delta u = 0 \) where
\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

**Remark 9.2.** Any scaler multiple of linear combinations of harmonic functions are harmonic.

In particular, if \( f : U \to \mathbb{C} \) is analytic and \( f(x + iy) = u(x, y) + iv(x, y) \) satisfy the Cauchy-Riemann equations then differentiating
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}
\]
with respect to \( x \) implies that
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x}
\]
and differentiating
\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]
with respect to \( y \) implies that
\[
\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.
\]

Thus we have shown:

**Theorem 9.3.** We have that \( u = \text{Re}(f) \) is harmonic. Similarly, we see that \( v = \text{Im}(f) \) is harmonic.

**Example 9.4.** The function \( f : \mathbb{C} \to \mathbb{C} \) given by \( f(z) = z^2 \) is analytic and thus
\[
f(z) = (u(x, y) + iv(x, y)) = (x + iy)^2 = x^2 - y^2 + 2ixy.
\]
where \( u(x, y) = x^2 - y^2 \) and \( v(x, y) = xy \) are harmonic.

**Example 9.5.** Let \( f : U \to \mathbb{C} \) be analytic and for \( z = x + iy \) and write \( f(z) = u(z) + iv(z) \). Let \( w(z) := \log |f(z)| = \frac{1}{2} \log (u(z)^2 + v(z)^2) \) then
\[
\frac{\partial w}{\partial x} = \frac{1}{u(z)^2 + v(z)^2} \left( u(z) \frac{\partial u}{\partial x} + v(z) \frac{\partial v}{\partial x} \right)
\]
\[
\frac{\partial w}{\partial y} = \frac{1}{u(z)^2 + v(z)^2} \left( u(z) \frac{\partial u}{\partial y} + v(z) \frac{\partial v}{\partial y} \right)
\]
and
\[
\frac{\partial^2 w}{\partial x^2} = \frac{-2}{(u(z)^2 + v(z)^2)^2} \left( u(z) \frac{\partial u}{\partial x} + v(z) \frac{\partial v}{\partial x} \right)^2 + \frac{1}{u(z)^2 + v(z)^2} \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + u(z) \frac{\partial^2 u}{\partial x^2} + v(z) \frac{\partial^2 v}{\partial x^2}
\]
\[
\frac{\partial^2 w}{\partial y^2} = \frac{-2}{(u(z)^2 + v(z)^2)^2} \left( u(z) \frac{\partial u}{\partial y} + v(z) \frac{\partial v}{\partial y} \right)^2 + \frac{1}{u(z)^2 + v(z)^2} \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + u(z) \frac{\partial^2 u}{\partial y^2} + v(z) \frac{\partial^2 v}{\partial y^2}
\]
Adding these expressions, the second derivative terms vanish since \( u \) and \( v \) are harmonic. The first order terms vanish by the Cauchy-Riemann equations.

**Remark 9.6.** The equation \( \Delta u = 0 \) is called Laplace’s equation.

**Definition 9.7.** If \( u, v \) are harmonic and satisfy the Cauchy-Riemann equations in \( u \) then \( v \) is called the harmonic conjugate of \( u \).

Conversely, if \( u \) is harmonic then it is the real part of an analytic function.

**Theorem 9.8.** If \( u : U \to \mathbb{R} \) is a \( C^2 \) function there there exists a harmonic conjugate, i.e., there exists a \( C^2 \) function \( v : U \to \mathbb{R} \) such that the function \( f : U \to \mathbb{C} \) defined by \( f = u + iv \) is analytic.

**Proof.** Consider the case \( U = \{ z = x + iy : x \in (a - \delta, a + \delta) \text{ and } y \in (b - \epsilon, b + \epsilon) \} \), and let \( f, g : U \to \mathbb{R} \) be \( C^1 \) functions defined by
\[
 f(z) = -\frac{\partial u(z)}{\partial y} \quad \text{and} \quad g(z) = \frac{\partial u(z)}{\partial x}.
\]
Then since \( \Delta u = 0 \) we have that
\[
 \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{by (1)}.
\]
If we define
\[
 v(x, y) = \int_a^x f(t, b) dt + \int_b^y g(x, s) ds
\]
then \( v : U \to \mathbb{R} \) is a \( C^2 \) function such that \( \frac{\partial v}{\partial y} = g \). The derivative with respect to \( x \) is a little more complicated. We can write
\[
 \frac{\partial}{\partial x} \int_a^x f(t, b) dt = f(x, b).
\]
For the second term in (2) we have that
\[
 \frac{\partial}{\partial x} \int_b^y g(x, s) ds = \int_b^y \frac{\partial g}{\partial x}(x, s) ds = \int_b^y \frac{\partial f}{\partial y}(x, s) ds = f(x, y) - f(x, b).
\]
Comparing (2), (3) and (4) gives that
\[
 \frac{\partial v}{\partial x} = f = f(x, b) + (f(x, y) - f(x, b)) = f(x, y).
\]
Finally, \( \frac{\partial u}{\partial x} = g = \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -f = \frac{\partial u}{\partial x} \).

**Exercise 9.9.** Given \( u(x, y) = xy \) find the harmonic conjugate of \( u \). In particular,
\[
 \frac{\partial u}{\partial x} = y = \frac{\partial v}{\partial y} \quad \Rightarrow \quad v(x, y) = \frac{1}{2} y^2 + f(x)
\]
\[
 \frac{\partial v}{\partial x} = f'(x) = -\frac{\partial u}{\partial y} = x \quad \Rightarrow \quad f(x) = -\frac{1}{2} x^2 + C
\]
Therefore \( v(x, y) = \frac{1}{2} (-x^2 + y^2) + C \) and
\[
 f(z) = u + iv
\]
\[
 = xy + i\left(\frac{-x^2 + y^2}{2}\right) + C
\]
\[
 = -\frac{i}{2}(x + iy)^2 + C
\]
\[
 = -\frac{i}{2} z^2 + C.
\]
which is analytic.

One can always find a harmonic conjugate on a disk, but not on any general region.

Example 9.10 (Non-example). Consider the function $u(x, y) = \log(x^2 + y^2)^{1/2}$ on the domain $U = \mathbb{C} - \{0\}$. Then

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u^2}{\partial x^2} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and by symmetry

$$\frac{\partial u^2}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Thus $\Delta u = 0$, i.e., $u$ is harmonic.

Assume for a contradiction we can find $v$ such that $f(z) = u + iv$ is analytic. Then

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} - \frac{i y}{x^2 + y^2} = \frac{1}{x + iy} = \frac{1}{z}$$

Thus $f(z) = \log z + C$. But being analytic the function must be continuous contradicting the jump between branches.

9.2. Poisson Integral formula. A harmonic function on the unit disk can be written in terms of its values on the unit circle.

Theorem 9.11 (Poisson integral formula). Let $z \in \mathbb{D}$ and assume that $u : U \rightarrow \mathbb{C}$ be harmonic and $U \supset \mathbb{D}$. We can write

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)u(e^{it})dt$$

where

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$ 

Proof. Let $f(z) = u(z) + iv(z)$ be the associated analytic function, where $v(z)$ is the harmonic conjugate of $u(z)$.

Given $z \in \mathbb{D}$, let $z_1 = \frac{z}{|z|} \in \mathbb{C} - \mathbb{D}$ be the reflection in the unit circle. By Cauchy’s theorem (applied twice)

$$f(z) = \frac{1}{2\pi i} \int_{C(0,1)} \frac{f(\xi)}{z - \xi}d\xi - \frac{1}{2\pi i} \int_{C(0,1)} \frac{f(\xi)}{z_1 - \xi}d\xi$$

where

$$\zeta = \frac{z}{|z|} \in \mathbb{C} - \mathbb{D}$$

and

$$\zeta_1 = \frac{z_1}{|z_1|} = \frac{z}{|z|}.$$
Let us denote $\xi = e^{it}$ and $z = r e^{i\theta}$ and $z_1 = \frac{1}{r} e^{-i\theta}$ and then with this parameterization we can write

$$f(z) = \frac{1}{2\pi i} \int_0^{2\pi} f(e^{it}) \left( \frac{1}{re^{i\theta} - e^{it}} - \frac{1}{\frac{1}{r} e^{i\theta} - e^{it}} \right) i e^{it} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \left( \frac{1}{re^{i(\theta-t)} - 1} - \frac{1}{\frac{1}{r} e^{i(\theta-t)} - 1} \right) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \left( \frac{1 - re^{-i(\theta-t)}}{1 + r^2 - 2r \cos(\theta - t)} - \frac{1 - \frac{1}{r} e^{-i(\theta)}}{r^2 + 1 - 2r \cos(\theta - t)} \right) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \left( \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - t)} \right) e^{it} dt$$

where in the middle we multiply both expressions by the complex conjugates of their respective denominators. Taking real and imaginary components, the result follows.

**Remark 9.12.** The Poisson kernel can be written in different ways. For example, we can write for $z = r e^{i\theta}$

$$P_r(\theta - t) = \frac{1 - |z|^2}{|1 - rz^{-1}|^2}.$$

**Remark 9.13.** When $z = 0$ we have the mean value property:

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt.$$

**Application 9.14.** The Laplace equation appears in the heat equation, one physical interpretation of this problem is as follows: fix the temperature on the boundary of the domain according to the given specification of the boundary condition. Allow heat to flow until a stationary state is reached in which the temperature at each point on the domain doesn’t change anymore. The temperature distribution in the interior will then be given by the solution to the corresponding Laplace problem.

Harmonic functions have various restrictions on their values.

**Corollary 9.15 (Harnack).** Assume that $u : U \to \mathbb{R}$ is harmonic and $U \supset \mathbb{D}$. Then for any $z \in \mathbb{D}$ we have that

$$\frac{1 - |z|}{1 + |z|} u(0) \leq u(z) \leq \frac{1 + |z|}{1 - |z|} u(0).$$

\[\text{Diagram}\]
9. HARMONIC FUNCTIONS

Proof. We observe that for any $0 \leq t < 2\pi$ and $z \in \mathbb{D}$:

$$
\frac{1 - |z|^2}{|e^{it} - z|^2} \leq \frac{1 - |z|^2}{(1 - |z|)^2} = \frac{1 + |z|}{1 - |z|}.
$$

Thus by the Poisson Integral Theorem

$$
u(z) \leq \frac{1 + |z|}{1 - |z|} \int_0^{2\pi} u(e^{it})dt = \frac{1 + |z|}{1 - |z|} u(0).
$$

The other inequality is proved similarly. □

The following is the analogue of Montel’s theorem for analytic functions.

Corollary 9.16 (Harnack’s Principle). Assume that $u_j : U \to \mathbb{R}, j \geq 1$, are harmonic functions with

$$
u_1 \leq \nu_2 \leq \nu_3 \leq \cdots
$$

Then either $\nu_j \to \infty$ uniformly on compact sets or there is a harmonic function $\nu : U \to \mathbb{R}$ such that $\nu_j \to \nu$ uniformly on compact sets.

This is an exercise.

Corollary 9.17 (Jensen). Let $f : U \to \mathbb{R}$ be analytic and let $U \supset \mathbb{D}$. Let $a_1, \cdots, a_N \in \mathbb{D}$ be the zeros of $f(z)$ with $|a_j| < 1$, for $j = 1, \cdots, N$, and assume there are no zeros on $C(0, 1)$. Then

$$
\log |f(0)| = \sum_{j=1}^N \log |a_j| + \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})|dt.
$$

Proof. If $f(z)$ has no zeros then $u(z) := \log |f(z)|$ is harmonic and by the Poisson Integral Formula (at $z = 0$) we have that

$$
\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})|dt.
$$

More generally, if $f(z)$ has zeros $a_1, \cdots, a_N \in \mathbb{D}$ then we can define

$$
F(z) := \left( \prod_{j=1}^N \frac{1 - z\overline{a_j}}{z - a_j} \right) f(z)
$$

which is now a zero free analytic function. We can apply (1) to $F(z)$ to deduce that

$$
\log |F(0)| = \log |f(0)| + \sum_{i=1}^N \log \left| \frac{1}{a_i} \right|
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{it})|dt
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \left( \log |f(e^{it})| + \sum_{i=1}^N \log \left| \frac{1 - \overline{a_i}e^{it}}{e^{it} - a_i} \right| \right) dt
$$

$$
= \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})|dt
$$

since the maps $z \mapsto \frac{1 - z\overline{a_j}}{z - a_j}$ preserve the unit circles. □

Application 9.18 (The Dirichlet problem). We can construct harmonic functions on $\mathbb{D}$ from a continuous function on the unit circle.
Theorem 9.19. Let \( U : C(0, 1) \rightarrow \mathbb{R} \) be continuous on the boundary of the unit disk \( \mathbb{D} \). Then the function \( u : \mathbb{D} \rightarrow \mathbb{R} \) defined by

\[
u(z) = \begin{cases} U(z) & \text{if } z \in C(0, 1) \\ \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(e^{it}) dt & \text{if } z = re^{it} \in \mathbb{D} \end{cases}
\]

is harmonic (i.e., \( \Delta u = 0 \)).

Proof. We can rewrite

\[
P_r(\theta - t) = \frac{1 - |z|^2}{|z - e^{it}|^2} = \frac{e^{it}}{e^{it} - z} + \frac{e^{-it}}{e^{-it} - \overline{z}} - 1
\]

where \( z = re^{i\theta} \). Then

\[
u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) U(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{e^{it}}{e^{it} - z} + \frac{e^{-it}}{e^{-it} - \overline{z}} - 1 \right) U(e^{it}) dt.
\]

For the first integral we have observe that

\[
\frac{\partial^2}{\partial x^2} \int_0^{2\pi} \left( \frac{e^{it}}{e^{it} - (x + iy)} \right) U(e^{it}) dt = \int_0^{2\pi} \left( \frac{-2e^{it}}{(e^{it} - (x + iy))^3} \right) U(e^{it}) dt
\]

and

\[
\frac{\partial^2}{\partial y^2} \int_0^{2\pi} \left( \frac{e^{it}}{e^{it} - (x + iy)} \right) U(e^{it}) dt = \int_0^{2\pi} \left( \frac{2e^{it}}{(e^{it} - (x + iy))^3} \right) U(e^{it}) dt
\]

and thus adding these two expressions gives

\[
\Delta \int \left( \frac{e^{it}}{e^{it} - (x + iy)} \right) U(e^{it}) dt = 0.
\]

The second integral is similarly. The final term is a constant. Thus three terms are harmonic, and thus so is the sum.

To see that \( u(z) \) is continuous (on the boundary) we can write

\[
\frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) U(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1 - |z|^2}{|z - e^{it}|^2} \right) U(e^{it}) dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} U \left( \frac{e^{it} + z}{1 + \overline{ze}^{it}} \right) dt
\]

using the change of variables \( \phi\prime_{-z}(w) = \frac{w + z}{1 + wz} \) (with \( |\phi\prime_{-z}(w)| = \frac{1 - |z|^2}{|1 + wz|^2} \)) for each \( z \in \mathbb{D} \). Moreover, since for each \( \xi \in C(0, 1) \) we have

\[
\lim_{z \to \xi} \left( \frac{e^{it} + z}{1 + \overline{ze}^{it}} \right) = \xi
\]

if \( \xi \neq -e^{it} \). The result follows from bounded convergence of integrals.
More generally, we see that given a simple closed curve $\gamma$ and a continuous function $U : \gamma \to \mathbb{C}$ we can find a harmonic function $u$ inside $\gamma$ with agrees with $U$ on $\gamma$. It suffices to solve the problem for the unit circle, as above, and then use the Riemann mapping theorem.

However, for the circle we can use the Poisson integral formula.

9.3. Mean Value Property and Maximum Principle for harmonic functions. The following is a characteristic property of harmonic functions.

**Theorem 9.20 (Mean Value Property).** Let $u : U \to \mathbb{R}$ be harmonic. For any $B(z, r) \subset U$ we have that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})d\theta,$$

for any sufficiently small $r > 0$.

**Proof.** Let $r_1 > r > 0$ such that $B(z, r_1) \subset U$. Let $v : B(z, r_1) \to \mathbb{R}$ be the harmonic conjugate for $u : B(z, r_1) \to \mathbb{R}$ and denote the analytic function $f : B(z, r_1) \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$.

By the Cauchy theorem

$$f(z) = u(z) + iv(z) = \frac{1}{2\pi i} \int_{C(z, r)} \frac{f(\xi)}{\xi - z}d\xi$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})d\xi + i \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{i\theta})d\xi$$

Taking real parts gives the result. \[\square\]

This is also the analogue of the maximum principle for analytic functions.

**Theorem 9.21.** Let $u : U \to \mathbb{R}$ be harmonic. If there exists $z_0 \in U$ such that $u(z_0) = \sup_{z \in U} |u(z)|$ then the function $u$ is constant.

**Proof.** We can give two proofs:

**First proof (using harmonic conjugates):** Let

$$\mathcal{M} = \{w \in U : u(w) = \sup_{z \in U} |u(z)|\}.$$ 

Clearly, $z_0 \in \mathcal{M} \neq \emptyset$. It suffices to show that $\mathcal{M} \subset U$ is open, since $U$ being connected would imply $\mathcal{M} = U$.

Let $z \in \mathcal{M}$. For small $r > 0$ we can consider $B(z, r) \subset U$. Let $v : B(z, r) \to \mathbb{R}$ be the harmonic conjugate for $u : B(z, r) \to \mathbb{R}$ and denote the analytic function $f : B(z, r) \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$. We define $g(z) = e^{f(z)}$ then
$|g(z)| = \sup_{\xi \in B(z, r)} |g(\xi)|$. We can apply the maximum modulus principle for analytic functions to deduce that $g(z)$ is constant on $B(z, r)$, and thus $u(z) = \log |g(z)|$ is constant on $B(z, r)$. Thus $\mathcal{M}$ is open.

**Second proof (using mean value property):** Let $L := \sup_{z \in U} |u(z)|$ and let

$$\mathcal{M} = \{ w \in U : u(w) = L \}.$$  

By hypothesis $z_0 \in \mathcal{M} \neq \emptyset$. Since $g$ is continuous we see that $\mathcal{M}$ is closed. It suffices to show it is also open.

Let $z \in \mathcal{M}$ and choose $r > 0$ such that $B(z, r) \subset U$ and then

$$L = g(z) = \frac{1}{2\pi} \int_0^{2\pi} g(z + re^{i\theta})d\theta \leq L$$

and thus $g(z + re^{i\theta}) = L$ for $0 \leq \theta < 2\pi$, for all sufficiently small $r > 0$. We deduce that $\mathcal{M}$ is open.

We can use this to get the converse to the first result.

**Theorem 9.22.** If $h : U \to \mathbb{R}$ satisfies the mean value theorem then it is harmonic.

**Proof.** Assume that $\overline{B(z, r)} \subset U$. By using the Poisson Integral Theorem we have a harmonic function $u : \overline{B(z, r)} \to \mathbb{R}$ which coincides with $h$ on the boundary $C(z_0, r)$. It suffices to show that $u$ and $h$ therefore coincide on all of $\overline{B(z, r)}$.

Let us write $g(z) = u(z) - h(z)$, which vanishes on $C(z_0, r)$. Moreover, it satisfies the mean value theorem since this holds for both $u$ (since it is harmonic) and $h$ (by hypothesis). By the previous lemma we see that $g \leq 0$ (since otherwise the supremum would occur in $B(z, r)$ and this would make the function constant, giving a contradiction). Replacing $w$ by $-w$ we deduce that $w = 0$, i.e., $u(z) = h(z)$ on $B(z, r)$. Since the disk is arbitrary we deduce that $h(z)$ is analytic on $U$. □

### 9.4. Schwarz reflection principle for harmonic functions

We can now formulate an analogue of the Schwarz reflection principle for functions.

**Theorem 9.23.** Given a domain $U$ and suppose that $U \cap \mathbb{R} = (a, b)$. Let $U^+ = \{ z \in U : \text{Im}(z) > 0 \}$. Assume $v : U^+ \to \mathbb{R}$ is harmonic and for each $z_0 \in (a, b)$ we have that $\lim_{z \to z_0} v(z) = 0$. Let $U^- = \{ \overline{z} : z \in U \}$ and define

$$w(z) = \begin{cases} 
   v(z) & \text{if } z \in U^+ \\
   0 & \text{if } z \in (a, b) \\
   -v(\overline{z}) & \text{if } z \in U^-
\end{cases}$$

Then $w : U^+ \cup (a, b) \cup U^- \to \mathbb{R}$ is harmonic.

**Proof.** We claim that the Mean Value Theorem holds on a sufficiently small disk about any point. If $z \in U^+$ then it follows form $u$ being harmonic. Similarly, it holds for $z \in U^-$. If $z \in (a, b)$ then for sufficiently small $r > 0$ we have that

$$\frac{1}{2\pi} \int_0^{2\pi} w(z + re^{i\theta})d\theta = \frac{1}{2\pi} \int_0^{2\pi} w(z + re^{i\theta})d\theta + \frac{1}{2\pi} \int_0^{2\pi} w(z + re^{i\theta + i\pi})d\theta = \frac{1}{2\pi} \int_0^{2\pi} w(z + re^{i\theta})d\theta + \frac{1}{2\pi} \int_0^{2\pi} w(z - re^{i\theta})d\theta = \frac{1}{2\pi} \int_0^{2\pi} w(z + re^{i\theta})d\theta - \frac{1}{2\pi} \int_0^{2\pi} w(z + re^{i\theta})d\theta = 0.$$
since \( w(x + iy) = -w(x - iy) \). Since \( w(z_0) = 0 \) the Mean Value Property holds.

9.5. Application: Continuous extensions to the boundary of analytic bijections. We can use the Schwarz reflection principle for analytic functions to give another (and more generalizable) proof that if \( f : U \to \mathbb{D} \) is an analytic bijection where \( U \subset \mathbb{C} \) is a domain whose boundary consists of straight line segments, or, more generally, circular arcs, then it must extend continuously to the boundary (i.e., the Osgood Carathéodory theorem).

We can consider the case that \( U = P \) is a polygon, the other cases being similar.

**Theorem 9.24.** Let \( f : P \to \mathbb{D} \) be an analytic bijection then it extends continuously to the boundary.

**Proof.** It is easy to see that if \( z_n \to z_0 \in \partial P \) then \( |f(z_n)| \to 1 \) as \( n \to +\infty \). (Otherwise we could find a subsequence with \( f(z_{n_k}) \) accumulating in the interior of \( \mathbb{D} \) and then this would pull back to a closed set containing \( z_{n_k} \), necessarily bounded away from \( \partial P \) and giving a contradiction). Let \( B(0, \frac{1}{2}) \) be the closed ball about 0. Thus \( K = f^{-1}(B(0, \frac{1}{2})) \subset P \) and we can assume that \( z_n \notin K \) for sufficiently large \( n \).

Let us assume that \( z_0 \) is not a vertex of \( P \) then we can consider a small neighbourhood \( D(z_0, \epsilon) := B(z_0, \epsilon) \cap P \) which is a half-disk, provided \( \epsilon > 0 \) is sufficiently small. Moreover, provided \( \epsilon > 0 \) is small enough it is disjoint \( K \) and we have that \( f : D(z_0, \epsilon) \to \mathbb{C} \) is non zero. In particular, then write

\[
  u(z) + iv(z) = i \log |f(z)|.
\]

However, since \( |f(z_n)| \to 1 \) we have that \( v(z) = \text{Im}(\log |f(z_n)|) \to 0 \). We can now apply the Schwarz reflection principle for harmonic functions to \( v(z) \) \( \square \)

**Application 9.25 (Annuli and eccentricity).** Two annuli are conformally equivalent if the ratio of the outer radius and the inner radius is the same for the two. Thus each is conformally equivalent to a unique "standard" annulus \( r_1 < |z| < r_2 \) with \( 0 < r < 1 \).

**Theorem 9.26 (Riemann Mapping Theorem for Annuli).** There is an analytic bijection between the annuli \( A_1 = \{ z \in \mathbb{C} : 1 < |z| < r_1 \} \) and \( A_2 = \{ z \in \mathbb{C} : 1 < |z| < r_2 \} \) if and only if \( r_1 = r_2 \).

**Proof.** Assume we have an analytic map \( f : A_1 \to A_2 \) and thus that \( f \) extends continuously to the closed annuli by the previous result.

By using the Schwarz reflection principle we can show that \( f \) extends to an analytic map between \( A_1^{(1)} = \{ z \in \mathbb{C} : \frac{1}{r_1} < |z| < r_1 \} \) and \( A_2^{(1)} = \{ z \in \mathbb{C} : \frac{1}{r_2} < |z| < r_2 \} \), i.e., the annuli obtained by reflecting \( A_1 \) and \( A_2 \) in the unit circle.
We can reflect again to show that \( f \) extends to an analytic map between \( A^{(2)}_1 = \{ z \in \mathbb{C} : \frac{1}{r_1} < |z| < r_2 \} \) and \( A^{(2)}_2 = \{ z \in \mathbb{C} : \frac{1}{r_2} < |z| < r_2 \} \). Continuing iteratively, we have an analytic map \( f : \mathbb{C} - \{0\} \rightarrow \mathbb{C} - \{0\} \) (since \( \mathbb{C} - \{0\} = \bigcup_{n=1}^{\infty} A^{(n)}_1 \)).

Moreover, we can assume that \( \lim_{z \to 0} f(z) = 0 \) and \( \lim_{z \to \infty} f(z) = \infty \) (the other cases being similar). In particular, we can deduce that \( f(z) \) is a Möbius map and thus a rotation.

\[ \square \]

10. Singularities

We have previously considered analytic functions \( f : U \rightarrow \mathbb{C} \) and their zeros. What about the possibilities of singularities (where the function isn’t analytic)?

10.1. Removable singularities. The first thing to consider is whether we really have a singularity at all, or if we could just extend the analytic function to the missing point.

**Definition 10.1.** We say that \( f : U \rightarrow \mathbb{C} \) has an isolated singularity at \( z_0 \) if \( f \) is defined and analytic in a punctured neighbourhood \( P(z_0, \epsilon) = B(z_0, \epsilon) - \{z_0\} \), but not defined at \( z_0 \).

**Example 10.2.** Let \( f(z) = 1/z \) then \( z_0 = 0 \) is a singularity.

The first result show that at singularities the function blows up, in some sense.

**Theorem 10.3 (Removable Singularities Theorem).** Assume \( f : U \rightarrow \mathbb{C} \) is analytic expect possibly at \( z_0 \in U \). However, if \( U \) is bounded on some punctured ball \( P(z_0, \epsilon) = \{ z \in \mathbb{C} : 0 < |z - z_0| < \epsilon \} \) then \( f(z) \) is analytic at \( z = z_0 \) as well.

**Proof.** Since \( f \) is analytic and bounded on \( P(z_0, \epsilon) \) we see

\[
| (z - z_0)^2 f'(z) | = \left| \frac{(z - z_0)^2}{2\pi i} \int_{|\xi - z_0| = 1/2} \frac{f(\xi)}{(z - \xi)^2} d\xi \right| \leq 4 \left( \sup_{\xi \in P(z_0, \epsilon)} |f(\xi)| \right)
\] (1)
Thus \((z - z_0)^2 f'(z)\) is bounded on \(P(z_0, \epsilon)\)

We can then define 

\[
g(z) = \begin{cases} 
(z - z_0)^2 f(z) & \text{if } z \neq z_0 \\
0 & \text{if } z = z_0
\end{cases}
\]

then by hypothesis we have \(\lim_{z \to z_0} g(z) = 0\) and by (1) we have that

\[
\lim_{z \to z_0} g'(z) = \lim_{z \to z_0} ((z - z_0)^2 f'(z) + 2(z - z_0) f(z)) = 0
\]

In particular, \(g\) is \(C^1\), and by Moreira's theorem we can deduce that \(g(z)\) is analytic.

We can consider the power series expansion \(g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n\). Since \(g(0) = 0\) and \(g'(0) = 0\) we deduce that

\[
g(z) = \sum_{n=2}^{\infty} a_n (z - z_0)^n = (z - z_0)^2 \left( \sum_{n=2}^{\infty} a_n (z - z_0)^{n-2} \right) f(z)
\]

and thus \(f(z) = g(z)/(z - z_0)^2\) is analytic, as required. \(\Box\)

**Example 10.4 (Non-example).** Let \(f(z) = \sin(z)/z\) then this has a singularity at \(z_0 = 0\), but this is a removable singularity.

**10.2. Analogues of Cauchy’s theorem.** We next look for analogues of Cauchy’s theorem in a neighbourhood of the singularity (and in a neighbourhood of \(\infty\)). Assume that \(f\) is defined on the annulus

\[
A(a, r, R) = \{ z \in \mathbb{C} : r < |z - a| < R \}
\]

where \(r\) may be zero and \(R\) may be \(+\infty\).
Lemma 10.5 (Analogue of Cauchy’s integral formula). Let \( f : A(a, r, R) \to \mathbb{C} \) be analytic. Let \( r < r_2 < |z_0 - a| < r_1 < R \) then

\[
f(z_0) = \frac{1}{2\pi i} \int_{C(a, r_1)} \frac{f(z)}{z - z_0} \, dz - \frac{1}{2\pi i} \int_{C(a, r_2)} \frac{f(z)}{z - z_0} \, dz
\]

Proof. Define

\[
F(z) = \begin{cases}  
\frac{f(z) - f(z_0)}{z - z_0} & \text{if } z \in A - \{z_0\} \\
\frac{f'(z_0)}{1} & \text{if } z = z_0
\end{cases}
\]

then since \( f \) is analytic (i.e., complex differentiable) at \( z_0 \) we deduce that \( F \) is continuous on \( A \), and analytic on \( A - \{0\} \). In particular, by the removable singularities theorem we see that \( F \) is analytic. Hence

\[
\int_{C(a, r_1)} F(z) \, dz = \int_{C(a, r_1)} F(z) \, dz
\]

(by joining \( C(a, r_1) \) and \( C(a, r_2) \) by a line, along which the integrals in different directions cancel, and applying Cauchy’s Theorem).

Observe that that

\[
\int_{C(a, r_1)} \frac{1}{z - z_0} \, dz = 2\pi i \quad \text{and} \quad \int_{C(a, r_2)} \frac{1}{z - z_0} \, dz = 0.
\]

Then comparing (1) and (2) gives:

\[
\int_{C(a, r_1)} \frac{f(z) - f(z_0)}{z - z_0} \, dz = \int_{C(a, r_1)} \frac{f(z)}{z - z_0} \, dz - f(z_0) \int_{C(a, r_1)} \frac{1}{z - z_0} \, dz
\]

\[
= \int_{C(a, r_1)} \frac{f(z)}{z - z_0} \, dz - 2\pi i f(z_0)
\]

\[
= \int_{C(a, r_2)} \frac{f(z) - f(z_0)}{z - z_0} \, dz = \int_{C(a, r_2)} \frac{f(z) - f(z_0)}{z - z_0} \, dz
\]

\[
= \int_{C(a, r_2)} \frac{f(z)}{z - z_0} \, dz.
\]
Rearranging these equalities gives that
\[
f(z_0) = \frac{1}{2\pi i} \int_{C(a,r_1)} \frac{f(z)}{z - z_0} \, dz - \frac{1}{2\pi i} \int_{C(a,r_2)} \frac{f(z)}{z - z_0} \, dz
\]
as required.  

10.3. Laurent’s Theorem. As a consequence of the last theorem we next have the analogue of power series expansions for analytic functions.

Theorem 10.6 (Laurent’s Theorem). Suppose that \( f : A(a,r,R) \to \mathbb{C} \) is analytic. Then there exists \((a_n)_{n \in \mathbb{Z}}\) such that
\[
f(z_0) = \sum_{n=-\infty}^{\infty} a_n (z_0 - a)^n
\]
for all \( z_0 \in A(a,r,R) \). Moreover, for any \( r < r_1 < R \) we can write
\[
a_n = \frac{1}{2\pi i} \int_{C(a,r_1)} \frac{f(z)}{(z - a)^{n+1}} \, dz, \text{ for } n \in \mathbb{Z}.
\]
Moreover, this expansion is unique.

Proof. We present the proof in two parts: First the existence of the expansion and then the uniqueness.

Existence of expansion. Let \( r < r_2 < |z_0 - a| < r_1 < R \) then (by joining \( C(a,r_1) \) and \( C(a,r_2) \) by a line, along which the integrals in different directions cancel, and applying Cauchy’s Theorem) we have
\[
f(z_0) = \frac{1}{2\pi i} \int_{C(a,r_1)} \frac{f(z)}{(z - z_0)} \, dz - \frac{1}{2\pi i} \int_{C(a,r_2)} \frac{f(z)}{(z - z_0)} \, dz
\]
where
\[
I_1 = \sum_{n=0}^{\infty} a_n (z_0 - a)^n \text{ where } a_n = \frac{1}{2\pi i} \int_{C(a,r_1)} \frac{f(z)}{(z - a)^{n+1}} \, dz \text{ for } n \geq 0
\]
by analogy with the proof of the power series expansion for analytic functions.

The evaluation of \( I_2 \) is similar, in as much as we can write
\[
\frac{1}{z - z_0} = -\frac{1}{(z_0 - a)} \left( \frac{1}{1 - \frac{z-a}{z_0-a}} \right)
\]
\[
= -\frac{1}{(z_0 - a)} \left( 1 + \frac{z - a}{z_0 - a} + \left( \frac{z - a}{z_0 - a} \right)^2 + \cdots \right)
\]
for \(|z - a| = r_2 < |z_0 - a|\). Thus

\[
I_2 = \frac{1}{2\pi i} \int_{C(a, r_2)} \frac{f(z)}{(z - z_0)} \, dz = \sum_{n=1}^{\infty} a_{-n}(z_0 - a)^{-n}
\]

where

\[
a_{-n} = \frac{1}{2\pi i} \int_{C(a, r_2)} f(z)(z - a)^{n-1} \, dz = \frac{1}{2\pi i} \int_{C(a, r_1)} f(z)(z - a)^{n-1} \, dz
\]

since \((z - a)^{n-1}f(z)\) is analytic, which is justified by the uniform convergence of the geometric series and integration. Hence

\[
f(z_0) = I_1 - I_2 = \sum_{n=-\infty}^{\infty} a_n(z_0 - a)^n
\]

where \(a_n\) is as given. \(^1\)

**Uniqueness of expansion.** To see the uniqueness of the coefficients, suppose \(f(z) = \sum_{n=-\infty}^{\infty} b_n(z - a)^n\) then dividing by \((z - a)^{k+1}\) and integrating around a circle \(C\) with radius between \(r\) and \(R\), i.e.,

\[
\frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z - a)^{k+1}} \, dz = \sum_{n=-\infty}^{\infty} b_n \left( \frac{1}{2\pi i} \int_{C} (z - a)^{n-k-1} \, dz \right) = b_k
\]

showing uniqueness.

**Example 10.7.** We can consider the function

\[
f(z) = \frac{1}{(z - 1)(z - 2)} = \frac{1}{(z - 2)} - \frac{1}{(z - 1)}
\]
on various domains. Observe that we can expand

\[
\frac{1}{z - 1} = \begin{cases} 
-\sum_{n=0}^{\infty} z^n & \text{if } |z| < 1 \\
\sum_{n=1}^{\infty} z^{-n} & \text{if } |z| > 1
\end{cases}
\]

and

\[
\frac{1}{z - 2} = \begin{cases} 
-\sum_{n=0}^{\infty} 2^{-n-1}z^n & \text{if } |z| < 2 \\
\sum_{n=1}^{\infty} 2^{n-1}z^{-n} & \text{if } |z| > 2.
\end{cases}
\]

**Case I:** For \(z \in D(0, 1)\) we have an expansion

\[
f(z) = -\sum_{n=0}^{\infty} (1 - 2^{-n-1})z^n.
\]

This is a power series which is analytic (and with no negative powers involved),

**Case II:** For \(z \in A(0, 1, 2)\) we have a Laurent series expansion

\[
f(z) = \sum_{n=0}^{\infty} z^{-n} - \sum_{n=0}^{\infty} 2^{-n-1}z^n.
\]

**Case III:** For \(z \in A(0, 2, \infty)\) (i.e., \(|z| > 2\)) we have a Laurent series expansion

\[
f(z) = \sum_{n=0}^{\infty} z^{-n} - \sum_{n=0}^{\infty} (2^{n-1} - 1)z^n
\]

This brings us to the following estimate.

\(^1\)The integrals around \(C(a, r_1)\) and \(C(a, r_2)\) can now be replaced by integrals around any circle of radius between \(r\) and \(R\) (by connecting the circles \(C(a, r_1)\) and \(C(a, r_2)\) by line segments and integrating in both directions (so that these integrals will cancel).
Corollary 10.8. Let \( f : A(0, r, R) \to \mathbb{C} \) be analytic and have a Laurent series
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n
\]
then for any \( r < r_1 < R \) we can bound \( |a_n| \leq \frac{M(r_1)}{r_1^n} \) where \( M(r) = \sup_{|z-a|=r} |f(z)| \).

Proof. Since
\[
a_n = \frac{1}{2\pi i} \int_{C(a,r_1)} \frac{f(z)}{(z-a)^{n+1}} dz
\]
we have that
\[
|a_n| \leq \frac{2\pi r M(r)}{2\pi r_1^{n+1}} = \frac{M(r_1)}{r_1^n}.
\]
\[\square\]

10.4. Classification of singularities. We can now classify the different types of singularities. Consider \( f : P(a, R) \to \mathbb{C} \) be analytic in a punctured disk \( P(a, R) \) centred at a singularity \( a \) and let
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n
\]
be its Laurent series on a punctured disk \( P(a, R) = A(a,0,R) \) (where \( R > 0 \)).

Definition 10.9. If \( a_n \neq 0 \) for infinitely many positive \( n \) then \( a \) is called an essential singularity. If \( a_n = 0 \) for only many positive \( n \) then \( a \) and the least integer \( -n \) such that \( a_n \neq 0 \) is negative then \( a \) is called a pole of order \( n \). In the particular case that \( a_{-1} \neq 0 \) and \( a_k = 0 \) for \( k \leq -2 \) we call \( a \) a simple pole.

Remark 10.10. If the least integer \( n \) such that \( a_n \neq 0 \) is non-negative is non-negative then \( a \) is called a removable singularity for in this case
\[
f(z) = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \cdots
\]
for \( 0 < |z-a| < R \) and \( n \geq 0 \). Therefore we can define (or extend) \( f \) to \( a \) by
\[
f(a) = \lim_{z \to a} f(z) = \begin{cases} 0 & \text{if } n > 0 \\ a_n & \text{if } n = 0 \end{cases}
\]
And with this value we can define \( f : D(a, R) \to \mathbb{C} \) and it is analytic. We shall always adopt the convention of removing singularities, in the future.

Example 10.11. Let \( f(z) = \sin(z)/z = 1 - z^2/3! + z^4/5! - \cdots \) then we can remove the singularity at \( z = 0 \) by setting \( f(0) = 1 \).

Definition 10.12. If the least \( n \) such that that \( a_n \) is strictly positive then \( f(z) \) is said to have a zero at \( a \) of order \( n \).

The order of a function at \( a \) is defined by
\[
\text{Ord}(f,a) = \begin{cases} -\infty & \text{if } a \text{ is an essential singularity} \\ -n > 0 & \text{if } a \text{ is a pole of order } n \\ 0 & \text{if } a \text{ is a removable singularity and } f(a) \neq 0 \\ n & \text{if } a \text{ is a zero of order } n \end{cases}
\]

Definition 10.13. A function is said to be meromorphic if it is analytic in a domain \( U \) except at a set of points which are poles (with no essential singularity).
10.5. Behaviour at singularities. We can consider the behaviour of a function near a zero or pole.

**Theorem 10.14.** \( \text{Ord}(f, a) = n \) where \( n \) is finite if and only if \( \exists \alpha, \beta, \delta > 0 \) such that
\[
\alpha |z - a|^n \leq |f(z)| \leq \beta |z - a|^n
\]
whenever \( 0 < |z - a| < \delta \)

**Proof.** Assume that \( \text{Ord}(f, a) = n \), then we can write
\[
f(z) = a_n(z - a)^n + a_{n+1}(z - a)^{n+1} + \cdots = (a_n + g(z))(z - a)^n
\]
where \( a_n \neq 0 \). Moreover, \( g(z) \to 0 \) as \( z \to a \) and thus given \( 0 < \epsilon < 1 \) we can write \( |g(z)| < \epsilon |a_n| \) provided \( |z - a| < \delta \), provided \( \delta \) is small enough. In particular, we can write
\[
f(z)(z - a)^{-n} = a_n + g(z).
\]
and thus
\[
|a_n|(1 - \epsilon) \leq |f(z)(z - a)^{-n}| \leq |a_n|(1 + \epsilon)
\]
if \( \delta \) is small enough, giving the required bound.

Conversely, suppose
\[
\alpha |z - a|^n \leq |f(z)| \leq \beta |z - a|^n
\]
when \( |z - a| \) is small enough then \( f(z)(z - a)^{-n} \) is bounded in a punctured neighbourhood of \( a \) and hence \( a \) is a removable singularity. In particular,
\[
f(z)(z - a)^{-n} = a_n + a_{n+1}(z - a) + a_{n+2}(z - a)^2 + \cdots
\]
in a punctured neighbourhood \( P(a, \delta) \) of \( a \) (since \( f(z)(z - a)^{-n} \) is analytic at \( a \) by the removable singularity). We want to show that \( a_n \neq 0 \). But from the left hand inequality \( |f(z)(z - a)^{-n}| \) is bounded below since \( \alpha > 0 \), which would not be the case where \( a_n \) is equal to zero. Hence \( a_n \neq 0 \) and \( \text{Ord}(f, a) = n \).

The previous result gives a rough idea of the behaviour of \( f \) near a pole (when \( n \) is negative). The next result describes the behaviour near an essential singularity (i.e., \( f(P(a, \delta)) \subset C \) is dense).

**Theorem 10.15 (Casorati-Weierstrass).** Let \( f \) have an essential singularity at \( a \). Then for any \( b \in \mathbb{C} \) and every \( \epsilon, \delta > 0 \) there exists \( z \) such that \( 0 < |z - a| < \delta \) such that \( |f(z) - b| < \epsilon \).

**Proof.** Assume for a contradiction that there exist \( b, \epsilon, \delta \) such that for all \( z \) such that \( 0 < |z - a| < \delta \) and \( |f(z) - b| \geq \epsilon \). Consider the function \( g : P(a, \delta) \to \mathbb{C} \) defined by
\[
g(z) = \frac{1}{f(z) - b},
\]
which, by assumption, is bounded by \( 1/\epsilon \) in \( P(a, \delta) \). Then \( g \) has a removable singularity at \( a \) (and thus is bounded). In particular, we can write
\[
g(z) = a_n(z - a)^n + a_{n+1}(z - a)^{n+1} + \cdots
\]
where \( a_n \neq 0 \) and thus
\[
g(z) = (z - a)^n h(z)
\]
for an analytic function \( h : B(a, \delta) \to \mathbb{C} \) with \( h(a) \neq 0 \). Thus comparing (1) and (2) gives
\[
f(z) - b = \frac{1}{(z - a)^n h(z)}. \tag{3}
\]
Moreover, \( 1/h(z) \) is analytic in a neighbourhood of \( a \) (since \( h \) does not vanish in a neighbourhood of \( a \) by continuity and the fact \( h(a) \neq 0 \)). But (3) shows that \( f \) has a pole at \( a \) of order \( n \), contradicting the hypothesis of an essential singularity.
Remark 10.16 (Picard’s Big Theorem). The following is an improvement on the Weierstrass-Casorati theorem.

Theorem 10.17 (Picard’s Big Theorem). If an analytic function $f(z)$ has an essential singularity at $a$, then on any punctured disk $B(a, \delta)$ the function $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.

1. Show that the following series define a meromorphic function on $\mathbb{C}$ and determine the set of poles and their orders
   
   (1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)}$
   
   (2) $\sum_{n=0}^{\infty} \frac{1}{z^n + n^2}$
   
   (3) $\sum_{n=0}^{\infty} \frac{1}{(n+z)^2}$
   
   (4) $\frac{1}{z} + \sum_{n \in \mathbb{Z} - \{0\}} \left( \frac{1}{z-n} + \frac{1}{n} \right)$
   
   (5) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)}$

2. Show that
   
   $$f(z) = \sum_{n=1}^{\infty} \frac{z^2}{n^2 z^2 + 8}$$

   is defined and continuous for the real values of $z$. Determine the region of the complex plane in which this function is analytic. Determine its poles.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that the image of $f$ is dense in $\mathbb{C}$.

4. Show that the order of a pole is invariant under analytic bijections (i.e., if $f : U \to \mathbb{C}$ has a pole at $z_0$ of order $n$ and $\phi : V \to U$ is an analytic bijection with $\phi(w) = z$ then $w$ is a pole for $f \circ \phi : V \to \mathbb{C}$ of order $n$).

5. Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function with $\lim_{|z| \to +\infty} |f(z)| = \infty$. Prove that $f(z)$ is a rational function (i.e., the ratio of two polynomials).

6. Show that the Laurent series of a function on an annulus can be differentiated term by term to give the derivative of the function on the annulus.

7. Find the Laurent series for the following functions:
   
   (1) $\frac{z}{(z+2)}$ for $|z| > 2$,
   
   (2) $\sin(1/z)$ for $z \neq 0$,
   
   (3) $\cos(1/z)$ for $z \neq 0$,
   
   (4) $1/(z-3)$ for $|z| > 3$.

8. Prove the following expansions:
   
   (1) $e^z = 1 + e \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}$,
   
   (2) $1/z = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$ for $|z-1| < 1$
   
   (3) $1/z^2 = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$ for $|z+1| < 1$

9. Let $f(z)$ be analytic on the annulus $0 < r < |z| \leq R$. Prove that there is a function $f_1(z)$ analytic on $|z| \leq R$ and $f_2(z)$ analytic on $|z| \geq r$ and

   $$f = f_1 + f_2$$

   on the annulus.
11. Entire functions, their order and their zeros

11.1. Growth of entire functions. Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function.

**Definition 11.1.** We say that \( f(z) \) is of finite order if there exists \( a > 0 \) and \( r > 0 \) such that
\[
|f(z)| \leq e^{|z|^a} \quad \text{for} \quad |z| > r
\]
The least such \( a \) (i.e., the largest lower bound) is called the order of the entire function.

**Example 11.2.**
1. Any polynomial is of order 0.
2. The function \( f(z) = \sin(z) \) is of order 1.
3. The function \( f(z) = \sin(z^2) \) is of order 2.

We can denote \( M_f(r) = \max_{|z|=r} |f(z)| \). This function conveys a lot of information about the zeros of the entire function.

**Lemma 11.3.** Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. If \( N \in \mathbb{Z}^+ \) and
\[
\liminf_{r \to +\infty} \frac{M_f(r)}{r^N} < +\infty
\]
then \( f(z) \) is a polynomial of degree at most \( N + 1 \).

**Proof.** Assume that \( f(z) \) has a power series expansion (about \( z = 0 \))
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]
and consider the associated polynomial \( P(z) = \sum_{n=0}^{N} a_n z^n \). The function \( g : \mathbb{C} \to \mathbb{C} \) defined by
\[
g(z) := z^{N-1} (f(z) - P(z)) = a_{N+1} + a_{N+2} z + a_{N+3} z^2 + \cdots
\]
is analytic and not identically zero. Assume that \( r_n^{-N} M_f(r_n) \) is bounded for some sequence \( r_n \to +\infty \) then
\[
\frac{M_f(r_n)}{r_n^{N+1}} \to 0 \quad \text{and thus} \quad M_g(r_n) \to 0
\]
as \( n \) tends to infinity. In particular, we can choose \( R > 0 \) such that
\[
\sup\{M_g(r) : 0 \leq r \leq R\} > R.
\]
By the maximum modulus principle applied to \( g(z) \) the disk \( B(0, R) \) we deduce that \( g(z) \) is constant (i.e., \( g(z) = a_{N+1} \)) and that \( f(z) = P(z) + a_{N+1} z^{N+1} \) is a polynomial.

There is also a close connection between \( M_f(r) \) and the number of zeros of \( f(z) \).

**Definition 11.4.** We denote by \( n(r) \) the number of zeros of \( f(z) \) in the ball \( B(0, r) \) (i.e., \( n(z) = \text{Card}\{z \in B(0, r) : f(z) = 0\} \}).

**Lemma 11.5.** Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function with \( f(0) = 1 \). Then
\[
n(r) \leq \frac{M_f(2r)}{\log 2}
\]
for all \( r > 0 \).

**Proof.** We can apply Jensen’s formula to the function \( F(z) := f(z/2r) \), say, on the unit disk \( \mathbb{D} \) to write
\[
0 = \log |f(0)| = - \sum_{k=1}^{n(2r)} \log \left( \frac{2r}{|a_k|} \right) + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(2re^{i\theta})|d\theta
\]
where \( a_1, \ldots, a_{n(2r)} \) are the zeros of \( f(z) \) with \( |a_i| < 2r \) (and \( a_1, \ldots, a_{n(r)} \) are the zeros of \( f(z) \) with \( |a_i| < r \)). In particular,
\[
n(r) \log 2 \leq \sum_{k=1}^{n(r)} \log \left( \frac{2}{|a_i|} \right)_{\leq 1} \leq \sum_{k=1}^{n(2r)} \log \left( \frac{2}{|a_i|} \right)_{\leq 1} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(2re^{i\theta})| \, d\theta \leq \log M_f(2r) \]
\[
\leq \log M_f(2r)
\]

As a corollary, we have the following relationship between all the zeros and the order.

**Lemma 11.6.** Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function of order \( a \) with \( f(0) = 1 \). If \( (a_i)_{i=1}^{\infty} \) are the zeros for \( f(z) \) ordered by modulus, i.e.,
\[
|a_1| \leq |a_2| \leq |a_3| \leq \cdots \leq |a_n| \leq \cdots
\]
then
\[
\sum_{n=1}^{\infty} \frac{1}{|a_n|^a} < +\infty.
\]

**Proof.** Let \( \epsilon > 0 \), then for \( r > 0 \) sufficiently large we have that
\[
M(r) \leq \exp(r^{a+\epsilon}).
\]
Thus by the previous theorem
\[
n(r)r^{-(r-2\epsilon)} \leq \frac{r^{-(a-\epsilon)}}{\log 2} M(2r) \leq \frac{r^{-(a-\epsilon)}}{\log 2} 2r^{a+\epsilon} \to 0 \text{ as } r \to +\infty.
\]
In particular, there exists \( r_0 \) such that providing \( r > r_0 \) have \( n(r) \leq C r^{a+2\epsilon} \). Thus since \( n \leq n(|a_n|) \leq |a_n|^{a+2\epsilon} \) we have that
\[
\frac{1}{|a_n|^{a+1}} \leq \left( \frac{1}{n} \right)^{\frac{a+1}{a+2\epsilon}}.
\]
Providing \( 0 < \epsilon < \frac{1}{2} \) we have that
\[
\sum_{n=1}^{\infty} \frac{1}{|a_n|^{a+1}} \leq \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^{\frac{a+1}{a+2\epsilon}} < +\infty.
\]

**11.2. Factorization of entire functions.** One of the most important applications of entire functions of finite order is their representation as infinite products. This can be thought of as a generalization of the simple representation of a polynomial \( P(z) = a_n z^n + \cdots + a_1 z + a_0 \) in terms of a finite product
\[
P(z) = (a_n a_0) \prod_{i=1}^{n} \left( 1 - \frac{z}{a_i} \right)
\]
where $\alpha_1, \cdots, \alpha_n$ are zeros of $P(z)$.

**Definition 11.7.** We can define the elementary entire functions

$$E_p(z) = (1 - z) \exp \left( z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right)$$

We have the following bound.

**Lemma 11.8.** If $z \in \mathbb{D}$ then $|E_p(z) - 1| \leq |z|^{p+1}$.

**Proof.** We leave the proof as an exercise. It requires showing that $E_p(z)$ has a power series expansion

$$E_p(z) = 1 + \sum_{n=p+1}^{\infty} b_n z^n$$

where $b_n \leq 0$. Then since $0 = E_p(1) = 1 + \sum_{n=p+1}^{\infty} b_n$ we have that $1 = \sum_{n=p+1}^{\infty} |b_n|$ and then

$$|E_p(z) - 1| = \left| \sum_{n=p+1}^{\infty} b_n z^n \right| \leq |z|^{p+1} \sum_{n=p+1}^{\infty} |b_n|.$$ 

This leads to the following.

**Lemma 11.9 (Weierstrass).** If $a_n$ occur with multiplicity $p_n$ then

$$f_0(z) := \prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{|a_n|} \right)$$

is an entire function with zeros $a_n$ having multiplicity $p_n$.

**Proof.** Let $r > 0$ then there exists $N$ such that for $n \geq N$ we have that $|a_n| > r$. We can then use the previous lemma to bound

$$|1 - E_{p_n} \left( \frac{z}{|a_n|} \right)| \leq \frac{1}{|a_n|^{p_n+1}}$$

and thus

$$\sum_{n=N}^{\infty} \left| E_{p_n} \left( \frac{z}{|a_n|} \right) - 1 \right| < +\infty.$$ 

In particular,

$$\prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{|a_n|} \right) = \prod_{n=1}^{\infty} \left( 1 + \left( E_{p_n} \left( \frac{z}{|a_n|} \right) - 1 \right) \right)$$

which is enough to guarantee convergence.

This brings us to the statement of the main result.

**Theorem 11.10 (Hadamard Factorization).** If $f(z)$ is an entire function of order $a$ and with zeros $(a_n)$ then we can write

$$f(z) = e^{Q(z)} \prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{|a_n|} \right)$$

where $Q(z)$ is a polynomial of degree at most $a$. 
The proof of Selberg and Erdős from 1949, but these are even more difficult. The idea is that given \( f(z) \) we can consider its zeros and thus construct the function \( f_0(z) \). Moreover \( e^{g(z)} := f(z)/f_0(z) \) will be an entire function.

Let \( r > 0 \) and let \( a_1, \ldots, a_n \) be the zeros for \( f(z) \) with \( |a_i| < r \). We can apply the Poisson formula to \( \log |f(z)| \) to write

\[
\log |f(z)| = \sum_{i=1}^{n} \log \left| \frac{r^2 - a_i z}{r(z - a_i)} \right| + \frac{1}{2\pi} \int_{0}^{2\pi} \text{Re} \left( \frac{r e^{i\theta} + z}{r e^{i\theta} - z} \right) \log |f(r e^{i\theta})| d\theta
\]

We can apply \( \frac{\partial}{\partial z} - i \frac{\partial}{\partial \theta} \) to (1) to get that

\[
\frac{f'(z)}{f(z)} = - \sum_{i} \frac{1}{z - a_i} + \sum_{i} \frac{\pi}{r^2 - \overline{a_i} z} + \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{2re^{i\theta}}{(re^{i\theta} - z)^2} \right) \log |f(re^{i\theta})| d\theta
\]

We can then differentiate this \( a \) times with respect to \( z \) to get

\[
- \sum_{i} \frac{a!}{(z - a_i)^{a+1}} + \sum_{i} \frac{a! \overline{a_i}^{a+1}}{r^2 - \overline{a_i} z} + \frac{(h + 1)!}{2\pi} \int_{0}^{2\pi} \left( \frac{2re^{i\theta}}{(re^{i\theta} - z)^{a+2}} \right) \log |f(re^{i\theta})| d\theta
\]

We want to let \( r \to +\infty \) and it is possible to show that this becomes

\[
- \sum_{i} \frac{a!}{(z - a_i)^{a+1}}.
\]

In particular, we have that

\[
g^{a+1}(z) = D^a \left( \frac{f'}{f} \right)(z) - D^a \left( \frac{f'}{f_0} \right)(z)
\]

The second term on the Right Hand Side of (2) is precisely (2). We deduce that \( g^{a+1} = 0 \) and thus \( g(z) \) is a polynomial of degree \( a \).

\[\square\]

**Example 11.11.** We can consider \( \sin(z) \) which is entire, of order 1, and zeros at \( \pi n \), where \( n \in \mathbb{Z} \). Thus

\[
\sin(z) = C \prod_{n \in \mathbb{Z}} \left( 1 - \frac{z}{\pi n} \right).
\]

(The function \( g(z) = az + b \) must be constant by the periodicity under \( z \mapsto z + 2\pi \).)

**12. Prime number theorem**

Perhaps one of the most interesting applications of complex analysis to another discipline is the proof of the prime number theorem in number theory.

**Theorem 12.1.** Let \( \pi(x) \) denote the number of prime numbers less than \( x \) then \( \pi(x) \sim \frac{x}{\log x} \) as \( x \to +\infty \), i.e., \( \lim_{x \to +\infty} \frac{\pi(x)}{x/\log x} = 1 \).

**Example 12.2.** We can compute: \( \pi(10) = 4 \), \( \pi(100) = 25 \), \( \pi(1,000) = 168 \), \( \pi(1,000,000) = 78,498 \) and \( \pi(1,000,000,000) = 50,847,534 \).

In 1808 Legendre conjecture that \( \pi(x) \sim \frac{x}{\log(x) - 1.08366} \). The correct form was proved independently by Hadamard and de la Vallée-Poussin in 1896, using complex analysis and the zeta function introduced by Riemann in 1859.

There are proofs which do not use complex analysis, including the “elementary” proof of Selberg and Erdős from 1949, but these are even more difficult.
12.1. Riemann zeta function. Recall that the Riemann zeta function is defined by \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \), provided \( \text{Re}(s) > 1 \).

**Lemma 12.3.** \( \zeta(s) - \frac{1}{s-1} \) is analytic for \( \text{Re}(s) > 0 \).

**Proof.** For \( \text{Re}(s) > 1 \) we can write

\[
\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^\infty \frac{1}{x^s} \, dx = \sum_{n=1}^{\infty} \int_n^{n+1} \left( \frac{1}{n^s} - \frac{1}{x^s} \right) \, dx
\]

and bound

\[
\left| \frac{1}{n^s} - \frac{1}{x^s} \right| = \left| s \int_n^x \frac{1}{u^{s+1}} \, du \right| \leq \frac{|s|}{n^{\text{Re}(s)+1}}.
\]

This leads to the result. \( \square \)

**Remark 12.4.** The zeta function \( \zeta(s) - \frac{1}{s-1} \) actually extends to an entire function. The extension to \( \text{Re}(s) > 0 \) is given at the end.

**Lemma 12.5.** We can write

\[
\zeta(s) = \prod_p \left( 1 - p^{-s} \right)^{-1}
\]

for \( \text{Re}(s) > 1 \) where the product is over all primes \( p \).

**Proof.** This follows from the prime factorization of natural numbers. \( \square \)

12.2. A related complex function. We define

\[
\Phi(s) = \sum_p \frac{\log p}{p^s}
\]

for \( \text{Re}(s) > 1 \) where the sum is over all primes \( p \).

**Lemma 12.6.** \( \Phi(s) - \frac{1}{s-1} \) is meromorphic for \( \text{Re}(s) > 1/2 \). Moreover, the poles occur at the zeros of \( \zeta(s) \) in this region.

**Proof.** For \( \text{Re}(s) > 1 \) we have that

\[
-\frac{\zeta'(s)}{\zeta(s)} = \frac{\partial}{\partial s} \log \zeta(s)
\]

meromorphic for \( \text{Re}(s) > 0 \)

\[
= \frac{\partial}{\partial s} \left( \sum_p \log(1 - p^{-s}) \right)
\]

\[
= \sum_p \frac{\log p}{(p^s - 1)}
\]

\[
= \sum_p \frac{\log p}{p^s} + \sum_p \frac{\log p}{p^s(p^s - 1)} =: \Phi(s)
\]

converges for \( \text{Re}(s) > 1/2 \). \( \square \)

**Lemma 12.7.** \( \zeta(s) \) has no zeros on \( \text{Re}(s) = 1 \). (In particular, \( \Phi(s) \) is analytic on \( \text{Re}(s) = 1 \)).
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Proof. We can write for $s = \sigma + it$

$\zeta(\sigma)^3|\zeta(\sigma + it)|^4|\zeta(\sigma + 2it)|$

$= \exp \left( \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{\sigma}} \left( \frac{3 + 4 \cos(mt \log p) + \cos(2mt \log p)}{m^2} \geq 0 \right) \right) \geq 1(1)$

(since $3 + 4 \cos \theta + \cos(2\theta) = 2(1 + \cos \theta)^2 \geq 0$). Assume for a contradiction that $1 + it_0$ is a zero for $\zeta(s)$. Since $s = 1$ is a simple pole for $\zeta(s)$ we see that

$\lim_{\sigma \to 1} \zeta(\sigma)^3|\zeta(\sigma + it_0)|^4|\zeta(\sigma + 2it_0)| = 0$

giving the contradiction to (1). □

12.3. A related counting function. We denote

$\Theta(x) = \sum_{p \leq x} \log p$ for $x > 0$

Lemma 12.8. We can bound $\Theta(x) \leq (8 \log 2)x$

Proof. Using the binomial expansion we can bound

$2^{2n} = (1 + 1)^{2n} \geq \binom{2n}{n} \geq \prod_{n < p \leq 2n} p = \exp \left( \sum_{n < p \leq 2n} \log p \right) = e^{\Theta(2n) - \Theta(n)}.$

i.e., $\Theta(2n) - \Theta(n) \leq n(2 \log 2)$. If $2^k < x \leq 2^{k+1}$, for some $k > 0$, then we can now bound

$\Theta(x) \leq \Theta(2^{k+1}) \leq \sum_{m=0}^{k} \left( \Theta(2^{m+1}) - \Theta(2^m) \right) \leq 2^m (2 \log 2)

\leq \left( \sum_{m=0}^{2^m} 2^m \right) 2 \log 2

= 2(2^{k+1} - 1)2 \log 2

\leq (8 \log 2)x$

□

12.4. Finiteness of integrals. The core of the proof is a finite integral, using Cauchy’s theorem.

Lemma 12.9. The following integral is finite:

$\lim_{y \to +\infty} \int_1^y \frac{\Theta(x) - x}{x^2} dx = \int_1^\infty \frac{\Theta(x) - x}{x^2} dx < +\infty$

Proof. We break the proof up into simpler steps.

Step 1 (Relating the integral to $\Phi(s)$): For $\Re(s) > 1$ we have

$\Phi(s) = \sum_p \frac{\log p}{p^s} = s \int_1^\infty \frac{\Theta(x)}{x^{s+1}} dx.$

(1)

Moreover, by a change of variables ($x = e^t$) we can write

$\int_1^\infty \frac{\Theta(x)}{x^{s+1}} dx = \int_0^\infty e^{-st} \Theta(e^t) dt$ and $\frac{1}{s} = \int_1^\infty \frac{1}{x^{s+1}} dx = \int_0^\infty e^{-st} dt.$

(2)

Let us denote $f(t) := \Theta(e^t)e^{-t} - 1$ then by combining (1) and (2) we have that

$g(s) := \int_0^\infty f(t)e^{-st} dt = \frac{\Phi(s+1)}{(s+1)} - \frac{1}{s}$
for \( \text{Re}(s) > 0 \). Moreover, this is analytic on a open set \( U \supset \{ z : \text{Re}(z) \geq 0 \} \).

**Step 2 (\( g(z) \) as a limit):** For \( T > 0 \) we can consider the entire functions

\[
g_T(z) = \int_0^T f(t)e^{-zt} \, dt = \left( \int_1^{\log T} \frac{\Theta(x) - x}{x^2} \, dx \right).
\]

We need to show that

\[
\lim_{T \to \infty} \int_0^T f(t) \, dt = \lim_{T \to \infty} g_T(0) = g(0) = \int_0^\infty f(t) \, dt.
\]

Let \( C = C(R, \delta) \) be the boundary of the D-shaped region defined my \( D = \{ z \in \mathbb{C} : |z| \leq R, \text{Re}(z) \geq 0 \} \).

**Step 3 (Integral over \( C_+ \)):** For \( z \in C_+ \) we can bound the integrand of (3) using

\[
|g(z) - g_T(z)| = \left| \int_C (g(z) - g_T(z)) \frac{e^{zT}}{z} \left( 1 + \frac{z^2}{R^2} \right) \, dz \right|.
\]

We can write \( C = C^+ \cup C^- \) where \( C^+ = \{ z \in C : \text{Re}(z) \geq 0 \} \) and \( C^- = \{ z : \text{Re}(z) \leq 0 \} \).

Together (4) and (5) give that the integral over \( C_+ \) has a bound

\[
\left| \frac{1}{2\pi i} \int_{C_+} (g(z) - g_T(z)) \frac{e^{zT}}{z} \left( 1 + \frac{z^2}{R^2} \right) \, dz \right| \leq \frac{B}{R}.
\]
Step 4 (Integral of \( g_T(z) \) over \( C_- \)): By analyticity of \( g_T(z) \) we can write
\[
\int_{C_-} g_T(z) \frac{e^{zT}}{z} \left( 1 + \frac{z^2}{R^2} \right) \, dz = \int_{C'_-} g_T(z) \frac{e^{zT}}{z} \left( 1 + \frac{z^2}{R^2} \right) \, dz
\] (7)
where we replace the integral over \( C_- \) by that over \( C'_- = \{ z \in \mathbb{C} : |z| = R, Re(z) < 0 \} \).

Furthermore, we can again bound
\[
|g_T(z)| = | \int_0^T f(t)e^{-zt} \, dt | \leq B \int_0^T |e^{-zt}| \, dt = \frac{Be^{-Re(z)T}}{|Re(z)|}
\] (4')
for \( Re(z) < 0 \) (compare with (4)). The analogue of (5) is
\[
\left| e^{zT} \left( 1 + \frac{z^2}{R^2} \right) \frac{1}{z} \right| = e^{Re(z)T} \frac{2|Re(z)|}{R^2}.
\] (5')
Together (4') and (5') give us a bound on the contribution of (7) to (3) of
\[
\left| \frac{1}{2\pi i} \int_{C'_-} g_T(z) \frac{e^{zT}}{z} \left( 1 + \frac{z^2}{R^2} \right) \, dz \right| \leq \frac{B}{R}
\] (8)
(compare with (6)).

Step 5 (Integral of \( g(z) \) over \( C_- \)): Finally, we can bound the remaining contribution to (3) coming from the integral over \( C_- \) by
\[
\int_{C_-} \frac{g(z)}{z} \left( 1 + \frac{z^2}{R^2} \right) e^{zT} \, dz \to 0 \text{ as } T \to 0
\] (9)
since \( Re(z) < 0 \) on \( C_- \).

Step 6 (Putting the bounds together): Finally, from the identity (3) and the estimates (6), (8) and (9) we have that
\[
\limsup_{T \to +\infty} |g(0) - g_T(0)| \leq \frac{2B}{R}.
\]
However, since the \( R \) can be arbitrarily large we deduce that \( \lim_{T \to \infty} g_T(0) = g(0) \).
12.5. Proof of the Prime Number Theorem. It is actually easier to first show the following.

**Lemma 12.10.** \( \Theta(x) \sim x \) as \( x \to +\infty \)

**Proof.** Assume that \( \lambda > 1 \) we can find \( x_n \to +\infty \) such that \( \Theta(x_n) \geq \lambda x_n \). Since \( \Theta(x) \) is monotone increasing:

\[
\int_{1}^{\infty} \frac{\Theta(x) - x}{x^2} \, dx \geq \sum_{k=1}^{\infty} \left( \lambda x_k \int_{x_k}^{\infty} \frac{1}{x^2} \, dx - \int_{x_k}^{\infty} \frac{1}{x} \, dx \right)
\]

\[
= \sum_{k=1}^{\infty} \left( \int_{1}^{\lambda} \frac{1}{x^2} \, dx - \int_{1}^{\lambda} \frac{1}{x} \, dx \right)
\]

(By change of variables)

\[
= \sum_{k=1}^{\infty} \left( \frac{1 - \lambda}{\lambda} - \log \lambda \right) = +\infty.
\]

This contradicts the integral converging.

On the other hand, assume that \( \lambda < 1 \) we can find \( x_n \to +\infty \) such that \( \theta(x_n) \leq \lambda x_n \) we again show that the integral doesn’t converge.

Either way, we see that \( \Theta(x) \sim x \) as \( x \to +\infty \). \( \square \)

This leads to the final result.

**Theorem 12.11 (Prime Number Theorem).**

\( \pi(x) \sim \frac{x}{\log x} \) as \( x \to +\infty \).

**Proof.** We can bound

\[
\Theta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x
\]

For any \( \epsilon > 0 \),

\[
\Theta(x) \geq \sum_{x^{1-\epsilon} \leq p \leq x} \log p \geq (1 - \epsilon) \log x \sum_{x^{1-\epsilon} \leq p \leq x} 1 \geq (1 - \epsilon) \log x \left( \pi(x) - \pi(x^{1-\epsilon}) \right)
\]

Since \( \epsilon > 0 \) is arbitrary, together these show that \( \pi(x) \sim \frac{x}{\log x} \). \( \square \)

**Application 12.12 (More on the Riemann zeta function).** Recall that we define the Riemann zeta function by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

It is possible to show that this has an extension to \( \mathbb{C} \). We can write

\[
\Gamma(s) = \int_{0}^{\infty} x^{s-1} e^{-x} \, dx
\]
for \( \sigma = \Re(s) \) and then
\[
n^{-s} \Gamma(s) = \int_0^{\infty} x^{s-1} e^{-nx} \, dx.
\]
Thus summing over \( \sigma \) for \( s \) integral is entire in \( s > 0 \). Moreover, \( \Gamma(1-s) \) is analytic there. However, the integral at \( s = 1 \) must actually cancel with the zeros of the integral since we already know that \( \zeta(s) \) is analytic there. However, the integral at \( s = 1 \) has value 1.

**Corollary 12.14.** \( \zeta(s)(s-1) \) is an entire function (i.e., \( \zeta(s) \) has a simple zero at \( s = 1 \)).

### 13. Further Topics

There are a number of topics which would have fitted nicely into the course with more time.


If \( \omega_1, \omega_2 \in \mathbb{C} \) such that \( \Im(\omega_1/\omega_2) \) then we can consider the set of points
\[
L = \{2n_1\omega_1 + 2n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\} \subset \mathbb{C}
\]

**Definition 13.1.** We can define the Weierstrass elliptic functions are defined by
\[
P(z) = \frac{1}{z^2} + \sum_{w \in L} \frac{1}{z-w^2} - \frac{1}{w^2}
\]

In particular, we have that \( P(z + 2\omega_i) = P(z) \) for \( i = 1, 2 \).

**Theorem 13.2.** This has a Laurent series expansion
\[
P(z) = \frac{1}{z^2} + \frac{1}{20}g_2z^2 + \frac{1}{28}g_3z^4 + O(z^6).
\]
where
\[
g_2(\omega_1, \omega_2) = 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^4} \quad \text{and} \quad g_3(\omega_1, \omega_2) = 60 \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^6}
\]
The coefficients satisfy \( g_2(\lambda \omega_1, \lambda \omega_2) = \lambda^{-4}g_2(\omega_1, \omega_2) \) and \( g_3(\lambda \omega_1, \lambda \omega_2) = \lambda^{-6}g_3(\omega_1, \omega_2) \), for \( \lambda > 0 \).

We can let \( \tau = \omega_1/\omega_2 \) with \( \Im(\tau) > 0 \) be the period ratio and consider \( g_2(\tau) = g_2(1, \omega_2/\omega_1) \) and \( g_3(\tau) = g_3(1, \omega_2/\omega_1) \).
Definition 13.3. The modular discriminant $\Delta$ is defined as

$$\Delta(\tau) = g_2^3 - 27g_3^2$$

This is studied in its own right, as a cusp form, in modular form theory (that is, as a function of the period lattice).

Theorem 13.4. The discriminant is a modular form of weight 12. That is, under the action of the modular group, it transforms as

$$\Delta(\frac{z\tau + b}{c\tau + d}) = (c\tau + d)^{12}\Delta(\tau)$$

with $a, b, c, d$ being integers, with $ad - bc = 1$.

13.2. Picard’s Little theorem. These describe the values of entire functions.

Theorem 13.5 (Picard’s Little Theorem). A non-constant entire function takes every value, except at most one.

13.3. Runge’s theorem. This deals with approximations on compact sets.

Theorem 13.6 (Runge). Let $U$ be a (simply-connected) domain. Then any function analytic in $U$ can be approximated uniformly on compact sets $K \subset U$ by polynomials, i.e., for each $\epsilon > 0$ there is a polynomial $p(z)$ with complex coefficients such that $|f(z) - p(z)| < \epsilon$ for all $z \in K$.

13.4. Conformality: Property of analytic maps. Any complex number $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg(z)$ is defined up to a multiple of $2\pi$.

Let $f : U \to \mathbb{C}$ be analytic and let $\gamma$ be a smooth contour in $U$ represented by a parameterization $z : [a, b] \to \mathbb{C}$ and let $a < t_0 < b$ be such that $z'(t_0) \neq 0$.

Definition 13.7. Let $U$ be an open neighbourhood. We say that $f : U \to \mathbb{C}$ is conformal on $U$ if for each $z_0 \in U$ the function $f$ is complex differentiable at $z_0$ and $f'(z_0) \neq 0$ exists.

This has a more natural interpretation.

Lemma 13.8. $f$ is conformal if and only if it preserves angles, i.e., given two smooth paths $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0)$ we have that

$$\arg(\gamma_1'(0) - \gamma_2'(0)) = \arg((f \circ \gamma_1)'(0) - (f \circ \gamma_2)'(0)).$$

Proof. We can write $\gamma_j(t) = x_j(t) + iy_j(t)$ and denote $z_0 = \gamma_1(0) = \gamma_2(0)$. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Suppose that $\gamma_1'(0) = \gamma_2'(0)$. We can write $(f \circ \gamma_j)'(0) = f'(\gamma_j(0))\gamma_j'(0)$ for $j = 1, 2$. Thus

$$\arg((f \circ \gamma_1)'(0)) = \arg(f'(\gamma_1(0))) + \arg(\gamma_1'(0))$$

up to a multiple of $2\pi$. We can then write

$$\arg((f \circ \gamma_1)'(0)) - \arg((f \circ \gamma_2)'(0)) = \arg(\gamma_1'(0)) - \arg(\gamma_2'(0))$$

up to a multiple of $2\pi$. \qed

Theorem 13.9. Analytic maps are conformal at all points at all points $z_0$ such that $f'(z_0) \neq 0$.

Method I: This can be illustrated by considering two two orthogonal families of contours such as the vertical lines and horizontal lines (or concentric circles and radial lines). Let $f(z) = u(x, y) + iv(x, y)$ with vertical lines $x = \alpha$ and horizontal lines $y = \beta$.

Consider the image of 2 perpendicular curves passing through $z_0$, i.e., $z_0 = \alpha + i\beta$. The image of the line $x = \alpha$ is now $z = u(\alpha, y) + iv(\alpha, y)$ for $y \in \mathbb{R}$. The image of the line $y = \beta$ is now $z = u(x, \beta) + iv(x, \beta)$ for $x \in \mathbb{R}$.
Method II: Consider the so-called level curves of \( f(z) = u(z) + iv(z) \), i.e., \( u(x, y) = \alpha \) and \( v(x, y) = \beta \). In particular, these curves are the ones mapped to the straight lines \( u = \alpha \) and \( \beta = v \).

Provided \( f'(z_0) \neq 0 \) we have that the angle between these preimages is \( \theta = \pi/2 \).

Example 13.10 (Method I). Let \( f(z) = e^z \) then \( f'(z) = e^z \neq 0 \) everywhere. Writing \( z = x + iy \) gives \( f(z) = e^x \cos y + i e^x \sin y \).

If \( x = \alpha \) then \( u = e^\alpha \cos y \) and \( v = e^\alpha \cos y \) with \( y \in \mathbb{R} \) a parameter, i.e., \( u^2 + v^2 = e^{2\alpha} \) which describes a circle.

If \( y = \alpha \) then \( u + iv = e^x \cos (\beta + i \sin \beta) \). Thus \( \frac{u}{v} = \tan(\beta) \), i.e., a straight line through the origin at angle \( \beta \).

Example 13.11 (Method II). Let \( f(z) = z^2 \) then \( f'(z) = 2z \neq 0 \) except at \( z = 0 \). Writing \( f(z) = u + iv \), the preimage of \( u = \alpha \) are those \( x + iy \) such that \( u(x, y) = \alpha \) and the preimage of \( v = \beta \) are those \( x + iy \) such that \( v(x, y) = \beta \).

If \( x = \alpha \) then \( u = e^\alpha \cos y \) and \( v = e^\alpha \cos y \) with \( y \in \mathbb{R} \) a parameter, i.e., \( u^2 + v^2 = e^{2\alpha} \) which describes a circle.

If \( y = \alpha \) then \( u + iv = e^x \cos (\beta + i \sin \beta) \). Thus \( \frac{u}{v} = \tan(\beta) \), i.e., a straight line through the origin at angle \( \beta \).

Thus \( z^2 = (x + iy)^2 = (x^2 - y^2) + i2xy \). The level sets correspond to \( x^2 - y^2 = \alpha \) or \( 2xy = \beta \).

Remark 13.12. In practise, it can be difficult to find the appropriate constants for these mappings. This involves numerical evaluation of elliptic integrals.

13.5. Functions of several complex variables. We can consider analytic functions \( f: U \to \mathbb{C} \), where \( U \subset \mathbb{C}^2 \) is an open set. We can define analyticity in terms of the convergence of power series in two variables (on poly disks \( B(z_0, w_0, \epsilon_1, \epsilon_2) = \{ (z, w) \in \mathbb{C}^2 : |z - z_0| < \epsilon_1, |w - w_0| < \epsilon_2 \} \) upon which we can apply the Cauchy theorem twice to write

\[
f(z, w) = \frac{1}{4\pi^2} \int_{C(z_0, \epsilon_1)} \int_{C(w_0, \epsilon_1)} \frac{f(z, w)}{(z - z_0)(w - w_0)} dzdw
\]

There are two particularly interesting theorems: The Bochner Tube Theorem and Hartog’s Theorem

Theorem 13.13 (Bochner Tube theorem). Let \( \Omega \subset \mathbb{R}^2 \) be a connected open set and let

\( \text{co}(\Omega) = \{ \lambda x + (1 - \lambda)y : x, y \in \Omega, 0 \leq \lambda \leq 1 \} \)

be the convex hull. If \( f: \Omega + i\mathbb{R}^2 \to \mathbb{C} \) is analytic then \( f: \text{co}(\Omega) + i\mathbb{R}^2 \to \mathbb{C} \)

Theorem 13.14 (Hartog Theorem). Assume that \( f: B(z_0, w_0, \epsilon_1/2, \epsilon_2/2) \to \mathbb{C} \) is analytic in two variables and \( f(\cdot, w): B(z_0, r_1) \to \mathbb{C} \) is analytic in one variable for each \( w \in B(w_0, r_2) \). Then \( f: B(z_0, w_0, \epsilon_1/2, \epsilon_2/2) \to \mathbb{C} \) is also analytic.

14. Problems

1. Show that the preimage of a circle or a straight line in \( \mathbb{C} \) is a circle on the Riemann sphere under stereographic projection.

2. Show that the metric on \( \hat{\mathbb{C}} \) coming from the stereographic projection of the usual Euclidean metric is given as follows: If \( z := \pi((x_1, x_2, x_3)) \) and \( w := \pi((x'_1, x'_2, x'_3)) \) then

\[
d(z, w) = \sqrt{\frac{|z - w|}{(1 + |z|^2)(1 + |w|^2)}}
\]
whenever \((x_1, x_2, x_3), (x'_1, x'_2, x'_3) \neq (0, 0, 1)\) (i.e., \(z, w \neq \infty\)) and
\[
d(z, \infty) := d((x_1, x_2, x_3), (0, 0, 1)) = \frac{2}{\sqrt{1 + |z|^2}}
\]
whenever \((x_1, x_2, x_3) \neq (0, 0, 1)\) (i.e., \(z \neq \infty\))

3. Given a complex matrix \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) we can associate a Möbius map \(T_M : \mathbb{C} \to \mathbb{C}\) of the form
\[
T_M(z) = \frac{az + b}{cz + d}.
\]
Given two Möbius maps, relate \(M_{T_1 \circ T_2}\) to \(M_{T_1}\) and \(M_{T_2}\).

4. Find the Möbius transformation that takes:
   (1) The points \(-i, -1, i\) to the points \(-i, 0, i\), respectively;
   (2) The points \(-1 + i, 0, 1 - i\) to the points \(-1, -i, i\), respectively;
   (3) The points \(0, 1 + i, 1 - i\) to the points \(1, -i, i\), respectively.

5. What is the image of the unit disk \(D = \{ z \in \mathbb{C} : |z| < 1 \}\) under the following Möbius maps:
   (1) \(T(z) = \frac{z+i}{z+i+1}\);
   (2) \(T(z) = \frac{z-1}{z+1}\);
   (3) \(T(z) = \frac{1+i(z+i-1)}{z+1+i}\);
   (4) \(T(z) = \frac{2z-1}{z+1}\);
   (5) \(T(z) = \frac{\sqrt{2z-1}+i}{\sqrt{2z+1}+i}\);
   (6) \(T(z) = \frac{(1+i)(z-1-i)}{z+1}\).

6. The cross ratio of four points \(z_0, z_1, z_2, z_3 \in \mathbb{C}\) is given by
\[
(z_0, z_1, z_2, z_3) = \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}.
\]
If \(\lambda = (z_0, z_1, z_2, z_3)\) show that different permutations of the four numbers gives cross-ratios with the values
\[
\lambda, 1 - \lambda, 1 - \frac{1}{\lambda}, \frac{1}{1 - \lambda}, \frac{\lambda}{1 - \lambda}.
\]
Show that these are all possible values.

Apollonius circle packing

1. Show that if \(E\) is a circle of radius \(k\) then:
   (1) if we reflect in \(E\) a circle \(C\) of radius \(r\) whose centres are separated by \(d > r + k\) then the image \(f(C)\) is a circle with radius \(k^2r/(d^2 - r^2)\).
   (2) if we reflect in \(E\) a straight line at a distance \(b > k\) from the centre of \(E\) then the image \(f(C)\) is a circle of radius \(k^2/2b\).

2. If a Möbius map \(f(z) = \frac{az+b}{cz+d}\) takes real numbers to real numbers (i.e., \(t \in \mathbb{R} \implies f(t) \in \mathbb{R}\)) then show that \(a, b, c, d\) are all real numbers.

Classification of Möbius maps

3. Show that any Möbius map other than the identity must fix at least one point and at most two.
4. Since we are assuming $ad - bc = 1$ show that we can write $(a - d)^2 + 4bc = (a - d)^2 + 4ad - 4 = 0$

5. Let $f : \mathbb{C} \to \mathbb{C}$ be a Möbius map. Show that
   (1) If $f(\infty) = \infty$ then $f(z) = \alpha z + \beta$, for some $\alpha, \beta \in \mathbb{C}$.
   (2) If $f(\infty) = \infty$ and $f(0) = 0$ then $f(z) = \alpha z$, for some $\alpha \in \mathbb{C}$.

6. We say that two linear fractional transformations $f_1, f_2$ are conjugate if $f_2 = g^{-1} f_1 g$ for some linear fractional map $g$. We can associate to the Möbius map $f(z) = \frac{az + b}{cz + d}$ the value $tr(f) := a + d$. Show that the value $tr(f)$ is preserved by conjugacy, i.e., if $f_1, f_2$ are conjugate then $tr(f_1) = tr(f_2)$.

7. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. The non-identity Möbius transformations are commonly classified into three types:
   (1) parabolic (conjugate to $z \mapsto z + \beta$ with a single fixed point $\infty$);
   (2) elliptic (conjugate to $z \mapsto \alpha z$ with $|\alpha| = 1$ with two points $0, \infty$); and
   (3) loxodromic (everything else).
   Show that:
   (1) Elliptic Möbius transformations correspond to $tr(f)^2 \in [0, 4]$;
   (2) Parabolic Möbius transformations correspond to $tr(f)^2 = 4$;
   (3) Hyperbolic Möbius transformations (i.e., conjugate to $z \mapsto \lambda z$ with $0 < \lambda < 1$ or $\lambda \in (1, \infty))$ correspond to $tr(f)^2 \in (4, \infty)$.


Here $\Re z$ and $\Im z$ stand for real and imaginary parts of the complex number $z$, respectively.

0. Let $f(z) : \mathbb{C} \to \mathbb{C}$ be a continuous function. Is it true that
   (1) $\Re f(z)$ and $\Im f(z)$
   (2) $|f(z)|$ and $(f(z))$
are continuous functions?

1. Show that if $f : U \to \mathbb{C}$ is analytic and there exist $a \in \mathbb{C}$ and a convergent sequence $z_n \to z$ with $z_n, z \in U$ such that $f(z_n) = a$ then $f|_U = a$. What happens if $z \notin U$?

2. Show that the equivalence of parameterizations of curves is an equivalence relation.

3. Show that $\int_{\gamma} f(w)dw = -\int_{\gamma} f(z)dz$ for any closed curve $\gamma$.

4. Show that if $R = [a, b] \times [c, d]$ is a rectangle then
   (1) We can evaluate $\int_{\partial R} dz = 0$;
   (2) We can evaluate $\int_{\partial R} (z - z_0)dz = 0$.

5. Write the Cauchy-Riemann equations in polar coordinates.

6. Let $f : \mathbb{D} \to \mathbb{C}$ be a function well-defined in the unit disc, such that partial derivatives exist everywhere and Cauchy-Riemann equations hold true. Is it true that $f$ is complex differentiable everywhere in the unit disc? Hint: consider $f(z) = \exp(-z^{-4})$. 
8. Find all holomorphic functions that map the unit disc to the real line.

9. Find domains of complex differentiability of the following functions
   (1) \( f_1(z) = |z|; \)
   (2) \( f_2(z) = z\bar{z}; \)
   (3) \( f_3(z) = \bar{z}. \)

14.3. Problems 3.

1. Define the Bernoulli numbers by the series \( \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n. \) Prove that
   \[ B_0 n! 0! + B_1 (n - 1)! 1! + \cdots + B_{n-1} 1!(n-1)! = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases} \]

2. Compute \( \int_\gamma z e^{z^2} \, dz \) where:
   (1) \( \gamma = [i, -i + 2]; \) and
   (2) \( \gamma = \{x + ix^2 : 0 \leq x \leq 1\}. \)

3. Compute \( \int_\gamma \sin(z) \, dz \) where \( \gamma = \{x + ix : 0 \leq x \leq 1\}. \)

4. Let \( f : D \to D \) be an analytic map which is injective (i.e., \( f(z_1) = f(z_2) \) implies \( z_1 = z_2 \)) of the form \( f(z) = \sum_{n=1}^{\infty} a_n z^n. \) Show that the image has area:
   \[ \text{Area}(f(D)) = \pi \sum_{n=1}^{\infty} n^2 |a_n|^2. \]

5. Let \( f : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \) be a continuous function such that
   (1) for \( w \in \mathbb{D} \) the map \( D \ni z \mapsto f(z, w) \in \mathbb{C} \) is analytic; and
   (2) for \( z \in \mathbb{D} \) the map \( D \ni zw \mapsto f(z, w) \in \mathbb{C} \) is analytic.
   Show that we can write \( f(z, w) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} z^n w^m, \) which converges absolutely for \( z, w \in \mathbb{D}. \)

6. Use Cauchy’s theorem to evaluate
   \[ \int_{C(0,1)} \left( \frac{z-a}{z-b} \right)^2 \, dz \]
   where \( b < 1 \) and \( C(0,1) = \{z \in \mathbb{C} : |z| = 1\}. \)

7. Let \( f : U \to \mathbb{C} \) be an analytic function. Let \( z_0 \in U \) with \( f'(z_0) \neq 0. \) Let \( C \) be a sufficiently small circle centred at \( z_0 \) then show
   \[ \frac{2\pi i}{f'(z_0)} = \int \frac{dz}{f(z) - f(z_0)} \]

8. If \( f \) is analytic in \( U \) and \( z_0 \in U \) define
   \[ g(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0. \end{cases} \]
   Show that this is analytic.

9. Use Rouche’s theorem to show that:
   (1) \( z^7 - 4z^3 + z - 1 = 0 \) has three zeros inside the unit circle;
   (2) \( z^7 + 5z^3 - z - 2 = 0 \) has three zeros inside the unit circle.
10. Use Rouche’s theorem to show that \( z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1 \)

(1) has 4 zeros inside the unit circle.

(2) has 87 zeros inside the circle \( \{ z \in \mathbb{C} : |z| = 2 \} \).


1. If \( |a| > e \) then show that \( e^z = az^n \) has \( n \) roots in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \).

2. Consider the following polynomials:

   (1) How many zeros of \( z^7 - 2z^5 + 6z^3 - z + 1 \) are in the unit disk \( \mathbb{D} \);

   (2) How many zeros of \( z^4 - 6z + 3 \) are in the annulus \( \{ z \in \mathbb{C} : 1 < |z| < 2 \} \);

   (3) How many zeros of \( z^3 - 4z^2 + 6z^2 - 4z^3 + 3 \) are in the set \( \{ z \in \mathbb{C} : |z - 1| < 1 \} \).

3. Let \( h \) be analytic in the unit disk \( \mathbb{D} \) and assume that

\[
|h(z) - h(w)| < |z - w|
\]

for all \( z, w \in \mathbb{D} \). Show that the function \( f : \mathbb{D} \to \mathbb{C} \) defined by \( f(z) = h(z) + z \) is one-to-one.

4. Let \( a > 0 \). Show that if \( f : \mathbb{C} \to \mathbb{C} \) is analytic and satisfies \( |f(z)| < |z|^a \) then \( f \) is a polynomial of degree at most \( |a| \).

5. Show that if \( f : \mathbb{C} \to \mathbb{C} \) is analytic and satisfies \( |f(z)| > 1 \) whenever \( |z| > 1 \) then \( f \) is a polynomial.

6. Show that if \( f : \mathbb{C} \to \mathbb{C} \) is analytic, non-constant and has no zeros then there must exist a sequence of points \( |z_n| \to \infty \) such that \( f(z_n) \to 0 \).

7. Let \( f : \mathbb{D} \to \mathbb{C} \) be analytic and continuous on the closed unit disk \( \overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \} \). Moreover, assume that:

   (1) \( |f(z)| \leq M \) whenever \( |z| = 1 \) and \( \text{Im}(z) \geq 0 \); and

   (2) \( |f(z)| \leq N \) whenever \( |z| = 1 \) and \( \text{Im}(z) < 0 \).

Then show that \( |f(0)| \leq \sqrt{NM} \).

8. Show that if \( f : \mathbb{C} \to \mathbb{C} \) is analytic and \( \text{Im}(f(z)) \neq 0 \) whenever \( |z| \neq 1 \) then \( f \) is constant.

9. Show that if \( f \) is a polynomial and \( z \in \mathbb{C} \) then either \( f^n(z) \to \infty \) as \( n \to +\infty \) or \( \{ f^k(z) : k \geq 0 \} \) is a bounded set.

### 14.5. Problems 5

1. Consider the family of functions \( f_n : \mathbb{D} \to \mathbb{C} \) defined by \( f_n(z) = z^n \), \( n \geq 1 \). Show that for all \( z \in \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) we have that \( f_n(z) \to 0 \) as \( n \to \infty \) and that the convergence is uniform on compact sets, but that it is not uniform on \( \mathbb{D} \).

2. Give an example of a family of analytic functions \( f_n : \mathbb{D} \to \mathbb{C} \), \( n \geq 1 \), for which \( f_n(z) \) has precisely one zero, but \( f_n \) converges uniformly on \( \mathbb{D} \) to the function \( f = 0 \) (i.e., the function which is identically zero).

3. Give an example of a family of analytic functions \( f_n : \mathbb{C} \to \mathbb{C} \), \( n \geq 1 \) such that \( f_n \to \infty \) uniformly on \( \mathbb{C} \) (i.e., for all compact sets \( K \subset \mathbb{C} \) and \( M > 0 \) we have that \( \inf_{z \in K} |f_n(z)| \geq M \) for sufficiently large \( n \)) but for which the derivatives \( f_n' : \mathbb{C} \to \mathbb{C} \), \( n \geq 1 \), neither tend to \( \infty \) nor form a normal family.
4. Given two domains $U_1, U_2$ and two points $z_1 \in U_1$ and $z_2 \in U_2$ show that there exists an analytic bijection $f : U_1 \rightarrow U_2$ such that $f(z_1) = z_2$. [Hint: Use the Riemann Mapping Theorem].

5. Find the image of the unit disk $\mathbb{D}$ with respect to the map $z \mapsto z + \frac{1}{z}$. Find the image of the upper half plane $\mathbb{H}$ with respect to the map $z \mapsto \sqrt{z^2 - 1}$.

6. Find an analytic bijection between the following domains:
   (1) $\mathbb{R} \times (0, \pi)$ and $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$;
   (2) $\mathbb{H} - [0, i\pi]$ and $\mathbb{H}$;
   (3) $\mathbb{D}$ and $\mathbb{C} - (-\infty, -\frac{i}{2}]$;
   (4) $(\mathbb{R} \times (0, 2)) - ((-\infty, i - \epsilon] \cup [i + \epsilon, i + \infty))$ and $\mathbb{H}$;
   (5) $\mathbb{H}$ and the equilateral triangle.

7. Let $U \subset \mathbb{C}$ be defined by
   \[
   \{ z \in \mathbb{C} : 0 < \text{Re}(z) < 1, 0 < \text{Im}(z) < 1 \} - \bigcup_{n=1}^{\infty} \left[ \frac{1}{2^n}, \frac{1}{2^n} + i \frac{1}{2} \right].
   \]
   (1) Show that $U$ is (path) connected;
   (2) Show that $U$ is simply connected;
   (3) Show that $0 \in \partial U$ in the boundary cannot be reached by a continuous path from any point in $U$;
   (4) Show that any analytic bijection $f : U \rightarrow \mathbb{D}$ cannot extend to a continuous map from the boundary $\partial U$ (of $U$) to the unit circle $\partial \mathbb{D}$.

8. Let $U \subset \mathbb{C}$ be defined by
   \[
   \{ z \in \mathbb{C} : \text{Im}(z) > 0 \text{ and } |z| < 1 \} - \bigcup_{p/q \in \mathbb{Q}} \left[ 0, \frac{1}{q} e^{i\pi p/q} \right].
   \]
   Show that every point in the boundary $\partial U$ can be reached by a continuous path from any point in $U$. However, show that any analytic bijection $f : U \rightarrow \mathbb{D}$ cannot extend to a continuous map from the boundary $\partial U$ of $U$ to the unit circle $\partial \mathbb{D}$.

9. Consider an analytic bijection $f : U \rightarrow \mathbb{D}$ that satisfies that for $z_0 \in U \cap \mathbb{R}$ we have that $f(z_0) = 0$ and $f'(z_0) > 0$. Show that if $U$ is symmetric about the real axis then $f(\overline{z}) = \overline{f(z)}$.

10. Show that there is no analytic bijection between $\mathbb{D}$ and $\mathbb{C}$.

3. Let $f : \mathbb{P} \to \mathbb{D}$ be continuous. Assume that $f$ extends to a bijection $f : \partial \mathbb{P} \to \partial \mathbb{D}$ on the boundary. Show that $f : \mathbb{P} \to \mathbb{D}$ is surjective.

4. Let $g : U \to \mathbb{C}$ be analytic. Let $D \subset U$ be a closed region whose boundary curve $\gamma = \partial D$ is simple. Assume that $g : \gamma \to g(\gamma)$ is one-to-one. Show that $g : D \to g(D)$ is one-to-one. (This is Darboux’s theorem). [Hint: Use the Argument Principle]

5. Do questions 2 – 4 give another proof that the Schwartz-Christoffel formula gives an analytic bijection $f : \mathbb{H} \to \mathbb{P}$ for a polygon $\mathbb{P}$?

6. Consider the semi-infinite strip
   
   $$S = \{x + iy : -a < x < a \text{ and } y > 0\}$$

   for $a > 0$. Use the Schwarz-Christoffel theorem to find the analytic bijection $f : \mathbb{H} \to S$ which extends to $f(\pm 1) = \pm a$ and $f(\infty) = \infty$.

7. Consider the triangle $\Delta$ in $\mathbb{C}$ with vertices at $-1, 0$ and $i$. Use the Schwarz-Christoffel theorem to find the analytic bijection $f : \mathbb{H} \to \Delta$ which extends to $f(-1) = i, f(1) = -1$ and $f(\infty) = 0$.

8. Derive the version of the Schwarz-Christoffel theorem for an analytic bijection $f : \mathbb{D} \to \mathbb{P}$ from the unit disk to a polygon.

9. Describe the image of the unit disk $\mathbb{D}$ under the map $f(z) = \int_0^z (1 - \xi^n)^{-2/n}d\xi$, for $n \geq 3$.

10. Describe the image of the unit disk $\mathbb{D}$ under the map $f(z) = \int_0^z (1 - \xi^4)^{-1}(1 + \xi^4)^{-1/2}d\xi$.


1. Let $f : U \to \mathbb{C}$ be analytic. Assume that $f$ never vanishes. If there is a point $z_0 \in U$ such that $|f(z_0)| \leq |f(z)|$, for all $z \in U$ then $f$ is a constant.

2. Suppose that both $f(z)$ and $g(z)$ are analytic on $\mathbb{D}$ with $|f(z)| = |g(z)|$ for $|z| = 1$. Show that if neither $f(z)$ nor $g(z)$ vanishes in $\mathbb{D}$ then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $f = \lambda g$.

3. Suppose that $f(z)$ is analytic on $\mathbb{D}$ and for $0 \leq r < 1$ define $A(r) = \max_{|z|=r} |Re f(z)|$. Show that unless $f$ is a constant, $A(r)$ is a strictly increasing function of $r$.

4. Does there exist an analytic function $f : \mathbb{D} \to \mathbb{D}$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{3}{4}$?

5. Let $f(z) = (z + 1)^2$ on $U = \{z = x + iy : 0 \leq x \leq 2$ and $0 \leq y \leq 1\}$. Where does $|f(z)|$ have its maximum and minimum values?

6. Show that the function $w(x, y) = e^x \sin(y)$ is harmonic.

7. Let $f : U \to \mathbb{C}$ be analytic and non-zero. Show that $\log |f(z)|$ is harmonic.

8. Show Harnack’s principle: Let $u_1 \leq u_2 \leq \cdots$ be harmonic functions on an open set $U$. Then either $u_j \to \infty$ uniformly on compacts sets or there is a harmonic function $u$ on $U$ such that $u_j \to u$ uniformly on compact sets. [Hint: Use Harnack’s Inequality].

9. Find the harmonic conjugate of $u(x, y) = y^3 - 3x^2y$. 

10. Describe suitable domains for the following harmonic functions, and find their harmonic conjugates of the following functions:

   (i) \(2x(1 - y)\);
   (ii) \(2x - x^3 + 3xy^2\);
   (iii) \(\sinh(x) \sin(y)\);
   (iv) \(y/(x^2 + y^2)\).

11. Show that if \(v_1, v_2\) are both harmonic conjugates of \(u\) then \(v_1 - v_2\) is a constant.