The Du Val singularities A_n, D_n, E_6, E_7, E_8

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Introduction

Du Val singularities have been studied since antiquity, and there are any number of ways of characterising them (compare Durfee [D]). They appear throughout the classification of surfaces, and in many other areas of geometry, algebraic geometry, singularity theory and group theory. For my present purpose, they make a wonderful introduction to many of the techniques and calculations of surface theory: blowups and birational geometry, intersection numbers, canonical class, first ideas in coherent cohomology, etc. This chapter proposes a number of activities with Du Val singularities, and leaves many of them as enjoyable exercises for the reader. Have fun!

The ideas in this section have many applications and generalisations, including the more complicated classes of surface singularities discussed in [Part I], Chapter 4, and also the terminal and flip singularities of Mori theory of 3-folds; see [YPG] for some of these ideas.

Summary

1. Examples: the ordinary double point and how it arises, the remaining Du Val singularities A_n, D_n, E_6, E_7, E_8 and their resolutions.

2. Quotient singularities $X = \mathbb{C}^2/G$ and covers $\pi\colon Y \to X$.

3. Some characterisations: Du Val singularities "do not affect adjunction"

4. Canonical class: a Du Val singularity $P \in X$ has a resolution $f\colon Y \to X$ with $K_Y = f^*K_X$.

5. Numerical cycle and multiplicity; there is a unique -2-cycle $0 \neq Z_{\mathrm{num}}$, called the *numerical cycle* of $P \in X$, characterised as the minimal nonzero divisor with $Z_{\mathrm{num}}\Gamma \leq 0$ for every exceptional curve Γ. In simple cases, properties of $P \in X$ can be expressed in terms of Z_{num}.

6. Milnor fibres and the amazing concidence

$$\text{Milnor fibre} \overset{\text{diffeo}}{\sim} \text{resolution!}$$

7. The ordinary double point and Atiyah's flop; the Brieskorn–Tyurina theory of simultaneous resolution.
8. Rational and Du Val singularities; how these relate to canonical models of surfaces, projective models of K3s, elliptic pencils

1 Examples

At the introductory level, I like to think of the Du Val singularities as given by the list of Table 1, rather than defined by conditions. Several characterisations are given in Theorem 2.1, and any of these could be taken as the definition.[1] This section introduces the list in terms of examples.

1.1 The ordinary double point A_1

The first surface singularity you ever meet is the ordinary quadratic cone in 3-space, that is, the hypersurface given by

$$P = (0,0) \in X : (xz = y^2) \subset \mathbb{A}^3.$$

This singularity occurs throughout the theory of algebraic surfaces, and can be used to illustrate a whole catalogue of arguments.

Because X is a cone with vertex P and base the plane conic $(xz = y^2) \subset \mathbb{P}^2$, it has the standard "cylinder" resolution $Y \to X$: the cone is a union of generating lines ℓ through P, and Y is the *disjoint* union of these lines. In other words, Y is the correspondence between the cone and its generating lines ℓ:

$$Y = \left\{ (Q,\ell) \text{ with } Q \in X, \ell \text{ a generating line} \;\middle|\; Q \in \ell \right\}.$$

This construction has already appeared in [Part I], Chapter 2, where the surface scroll $Y = \mathbb{F}_2 = \mathbb{F}(0,2)$ has a morphism to $X = \overline{\mathbb{F}}(0,2) \subset \mathbb{P}^3$ contracting the negative section. The exceptional curve $f^{-1}P = \Gamma$ of the resolution is a -2-*curve*, with $\Gamma \cong \mathbb{P}^1$ and $\Gamma^2 = -2$.

In coordinate terms, Y is obtained from X by making the ratio $(x : y : z)$ defined at every point. That is, Y is the *blowup* of $P \in X$.

[1] For the reader who wants to go on to study 3-folds: the right definition for the purposes of surface theory is Du Val singularity = surface canonical singularity. Compare [YPG].

1.2 A_1 as a quotient singularity

The singularity A_1 also appears in a different context: consider the group $G = \mathbb{Z}/2$ acting on $\mathbb{A}^2$ by $u, v \mapsto -u, -v$. The quadratic monomials u^2, uv, v^2 are G-invariant functions on $\mathbb{A}^2$, so that the map $\mathbb{A}^2 \to \mathbb{A}^3$ given by $x = u^2, y = uv, z = v^2$ identifies the quotient space $\mathbb{A}^2/G$ with $X : (xz = y^2) \subset \mathbb{A}^3$.

The blowup $Y \to X$ can alternatively be obtained by first blowing up $P \in \mathbb{A}^2$, then dividing out by the action of $\mathbb{Z}/2$ (see Ex. 1). In general, there may be no very obvious connection between resolving the singularities of a quotient variety V/G and blowups of the cover V and the action of G.

I fill in some background: the *quotient variety* V/G is defined for any finite group G of algebraic automorphisms of an affine variety V (in all the examples here, $V = \mathbb{A}^2$ with coordinates u, v, and $G \subset \mathrm{GL}(2)$). If $k[V]$ is the coordinate ring of V (in our case, $k[V] = k[u, v]$) then G acts on $k[V]$ by k-algebra automorphisms, and invariant polynomials form a subring $k[V]^G \subset k[V]$. Then $k[V]^G$ is a finitely generated k-algebra, and $X = V/G$ is the corresponding variety. It is easy to see that set-theoretically X is the space of orbits of G on V, and is a normal variety.

Remark 1.1 For any finite $G \subset \mathrm{GL}(n)$ acting on $V = \mathbb{A}^n$, a famous theorem of Chevalley and Sheppard–Todd says that the quotient V/G is nonsingular if and only if G is generated by *quasi-reflections* or *unitary reflections* (matrixes that diagonalise to $\mathrm{diag}(\varepsilon, 1, \dots, 1)$, and so generate the cyclic codimension 1 ramification of $x_1 \mapsto x_1^r$); the standard example is the symmetric group S_n acting on $\mathbb{A}^n$ by permuting the coordinates. For any G, the quasi-reflections generate a normal subgroup $G_0 \subset G$, and passing to the quotient G/G_0 acting on $\mathbb{A}^n/G_0 \cong \mathbb{A}^n$ reduces most questions to the case of no quasi-reflections. A quasi-reflection $\mathrm{diag}(\varepsilon, 1, \dots, 1)$ has determinant ε, so a subgroup of $\mathrm{SL}(n)$ satisfies this automatically.

1.3 D_4 and its resolution

Consider the singularity

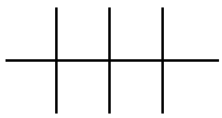
$$P = (0,0,0) \in X : (f = x^2 + y^3 + z^3 = 0) \subset \mathbb{A}^3.$$

The blowup $X_1 \to X$ is covered by 3 affine pieces, of which I only write down one: consider $B_1 = \mathbb{A}^3$ with coordinates x_1, y_1, z, and the morphism $\sigma \colon B_1 \to \mathbb{A}^3$ defined by $x = x_1 z, y = y_1 z, z = z$. The inverse image of X under σ is defined by

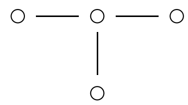
$$f(x_1 z, y_1 z, z) = x_1^2 z^2 + y_1^3 z^3 + z^3 = z^2 f_1, \quad \text{where} \quad f_1 = x_1^2 + (y_1^3 + 1)z.$$

Here the factor z^2 vanishes on the exceptional (x_1, y_1)-plane $\mathbb{A}^2 = \sigma^{-1}P : (z = 0) \subset B_1$, and the residual component $X_1 : (f_1 = 0) \subset B_1$ is the birational transform of X. Now clearly the inverse image of P under $\sigma\colon X_1 \to X$ is the y_1-axis, and $X_1 : x_1^2 + (y_1^3 + 1)z = 0$ has ordinary double points at the 3 points where $x_1 = z = 0$ and $y_1^3 + 1 = 0$. Please check for yourselves that the other affine pieces of the blowup have no further singular points. The resolution $Y \to X_1 \to X$ is obtained on blowing up these three points.

I claim that f^{-1} consists of four -2-curves $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ meeting as follows:



that is,



the configuration is the Dynkin diagram D_4. (The picture on the left is what you draw on paper or on the blackboard, but it's somewhat tedious to typeset, so you usually see the "dual" graph on the right in books; the two pictures contain the same information.) To prove this, it is clear that $\Gamma_1, \Gamma_2, \Gamma_3$ are -2-curves, since they arise from the blowup $Y \to X_1$ of ordinary double points. Also the fact that $\Gamma_0 \cong \mathbb{P}^1$, and Γ_0 meets each of Γ_i transversally in 1 point, can be verified directly from the coordinate description of Y. Finally, to see that $\Gamma_0^2 = -2$, note that y is a regular function on Y whose divisor is $\operatorname{div} y = 2\Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3 + C$, where C is the curve $y_1 = 0$ in Y, which also meets Γ_0 transversally in 1 point. Thus by the rules given in [Part I], Chapter A,

$$0 = (\operatorname{div} y)\Gamma_0 = 2\Gamma_0^2 + \Gamma_0(\Gamma_1 + \Gamma_2 + \Gamma_3 + C) = 2\Gamma_0^2 + 4,$$

so that $\Gamma_0^2 = -2$.

1.4 D_4 as a quotient singularity

The singularity D_4 also appears as a quotient singularity, although the calculation is not quite so trivial: take $\mathbb{A}^2$ with coordinates u, v, and the group of order 16 generated by

$$\alpha\colon u, v \mapsto iu, -iv \quad \text{and} \quad \beta\colon u, v \mapsto v, -u.$$

Thus $\alpha^2 = \beta^2 = (\alpha\beta)^2 = -1$. This is the *binary dihedral group* BD_{16}; see Ex. 9 below. It is not hard to see that

$$u, v \mapsto \begin{cases} x = (u^4 - v^4)uv, \\ y = u^4 + v^4, \\ z = u^2v^2 \end{cases}$$

Name	Equation	Group	Resolution graph
A_n	$x^2 + y^2 + z^{n+1}$	cyclic $\mathbb{Z}/(n+1)$	$\circ - \circ \cdots \circ$
D_n	$x^2 + y^2 z + z^{n-1}$	binary dihedral $\mathrm{BD}_{4(n-2)}$	$\circ - \circ - \circ \cdots \circ$ $\quad\ \vert$ (under 2nd node) $\quad\ \circ$
E_6	$x^2 + y^3 + z^4$	binary tetrahedral	$\circ - \circ - \circ - \circ - \circ$ $\qquad\ \ \vert$ (under 3rd node) $\qquad\ \ \circ$
E_7	$x^2 + y^3 + yz^3$	binary octahedral	$\circ - \circ - \circ - \circ - \circ - \circ$ $\qquad\ \ \vert$ (under 3rd node) $\qquad\ \ \circ$
E_8	$x^2 + y^3 + z^5$	binary icosahedral	$\circ - \circ - \circ - \circ - \circ - \circ - \circ$ $\qquad\ \ \vert$ (under 3rd node) $\qquad\ \ \circ$

Table 1: The Du Val singularities

defines a G-invariant map $\mathbb{A}^2 \to \mathbb{A}^3$, and that the image is the singular surface $X \subset \mathbb{A}^3$ defined by $x^2 - y^2 z + 4z^3$, which is D_4 up to a change of coordinates. See Ex. 9.

1.5 Lists of Du Val singularities

The two examples A_1 and D_4 worked out above are in many ways typical of all the Du Val singularities. These all occurs as quotient singularities, and have resolutions by a bunch of -2-curves. For convenience I tabulate them in Table 1

I explain the terminology. The equation is a polynomial f in x, y, z, with $0 \in X : (f(x, y, z) = 0) \subset \mathbb{A}^3$ an isolated singularity of an embedded surface. The subscript on A_n, D_n, E_6, E_7, E_8 equals the number of -2-curves in the resolution. The configuration of -2-curves on the resolution is given by the corresponding Dynkin diagram. To think of all these cases in families, you may find it convenient to think "$D_3 = A_3$ and $E_5 = D_5$" (and, of course, "A_0 = nonsingular point").

The groups in the middle column are all the possible finite subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ (in suitable coordinates). That is, Γ acts linearly on $\mathbb{A}^2$ or $\mathbb{C}^2$ by matrixes with trivial determinant. The cyclic group $\mathbb{Z}/n$ is generated by $u, v \mapsto \varepsilon u, \varepsilon^{-1} v$, where ε is a primitive nth root of 1 (for example, $\varepsilon = \exp(2\pi i/n)$ if $k = \mathbb{C}$). The other groups can be thought of as a given list.

They are described by generators in Ex. 12; each group is "binary" in the sense that it contains -1 as its centre, and is a double cover of the rotation group of a regular solid in Euclidean 3-space under the standard "spin" double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ (compare Ex. 11). For example, the binary dihedral group BD_{4n} is generated by

$$\alpha = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where ε is a primitive $2n$th root of 1, so that $\beta^2 = \alpha^n = -1$ and $\alpha\beta = \beta\alpha^{-1}$. See Ex. 9. All of this is worked out in the exercises.

2 Various characterisations

So far, a list, but no proper definition. I now give a number of equivalent characterisations, any of which could be taken as the definition.

Theorem 2.1 *The Du Val singularities are characterised by any of the following* 3 *conditions*

(1) Absolutely isolated double point: *$P \in X$ is an isolated double point, and has a resolution $X_n \to X_{n-1} \to \cdots \to X_1 \to X$ where each step $X_i \to X_{i-1}$ is the blowup of an isolated double point over $P \in X$.*

(2) Canonical class: *There exists a resolution of singularities $\varphi\colon Y \to X$ such that $K_Y = \varphi^* K_X$. In other words, K_Y is trivial on a neighbourhood of the exceptional locus. The resolution φ is called a* crepant resolution.

(3) Newton polygon: *In any analytic coordinate system, f has monomials of weight < 1 with respect to each of the weights $\frac{1}{2}(1,1,0)$, $\frac{1}{3}(1,1,1)$, $\frac{1}{4}(2,1,1)$, $\frac{1}{6}(3,2,1)$* (see below for a discussion).

Proof I first sketch the proofs of the implications List 1.5 $\Rightarrow$ (1) and (3) $\Rightarrow$ List 1.5, leaving some of the computational details to the reader. The proofs of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are also easy, but need some basic material on the canonical class and canonical differentials, which I discuss below.

2.1 List 1.5 implies (1)

Let $P \in X$ be any of the singularities in List 1.5, and $X_1 \to X$ the blowup of P. Then, as you can calculate (Ex. 3), X_1 is either nonsingular or has one or more singularities in List 1.5 with smaller subscript. An obvious induction proves (1). QED

2.2 (3) implies List 1.5

(3) is a number of conditions on the Newton polygon of f with respect to any analytic coordinate system x, y, z. If $f = \sum a_{ijk} x^i y^j z^k$, then (3) just says that

$$a_{ijk} \neq 0 \quad \begin{cases} \text{for some monomial } x^i y^j z^k \text{ with } i + j < 2, \\ \text{for some monomial } x^i y^j z^k \text{ with } i + j + k < 3, \\ \text{for some monomial } x^i y^j z^k \text{ with } 2i + j + k < 4, \text{ and} \\ \text{for some monomial } x^i y^j z^k \text{ with } 3i + 2j + k < 6. \end{cases}$$

Condition (3) with $\frac{1}{3}(1,1,1)$ implies that the 2-jet $J_2 f$ is nonzero, and therefore by a linear coordinate change, I can assume that

$$J_2 = x^2 + y^2 + z^2 \quad \text{or} \quad x^2 + y^2 \quad \text{or} \quad x^2.$$

If $J_2 = x^2 + y^2$ then (3) with $\frac{1}{2}(1,1,0)$ implies that at least one term of the form z^m or xz^m or yz^m appears in f, and then an analytic coordinate change can be used to make $f = x^2 + y^2 + z^{n+1}$, and $P \in X$ is of type A_n.

A proper discussion of the analytic coordinate changes would take me too far afield into singularity theory, but fairly typically, if $f = x^2 + y^2 + x^3 + xz^m$ with $m \geq 2$, then $\xi = \sqrt{(x + \frac{1}{2}z^m)^2 + x^3}$ is an analytic function on $\mathbb{C}^3$, and ξ, y, z are new analytic coordinates, with respect to which $f = \xi^2 + y^2 - \frac{1}{4}z^{2m}$. More generally, given that f contains x^2, power series methods allow me to eliminate one by one higher powers of x, or x times monomials in y and z, and the implicit function theorem guarantees that the coordinate change is analytic.[2]

If $J_2 = x^2$ an analytic coordinate change can be used to remove any further appearances of x in f. Then (3) with $\frac{1}{4}(2,1,1)$ implies that the 3-jet is

$$J_3 = x^2 + y^3 + z^3 \quad \text{or} \quad x^2 + y^2 z \quad \text{or} \quad x^2 + y^3.$$

[2] To avoid interrupting the flow of ideas, this chapter uses implicitly material on analytic coordinate changes. It's all standard material in singularity theory, and follows easily from the implicit function theorem, but I should have a section at the end saying explicitly what is involved.

If $J_3 = x^2 + y^2 z$ then (3) with $\frac{1}{2}(1,1,0)$ implies that at least one term of the form z^m or yz^m appears in f, and then an analytic coordinate change can be used to make $f = x^2 + y^2 z + z^{n-1}$, and $P \in X$ is of type D_n.

Finally, if $J_3 = x^2 + y^3$ then (3) with $\frac{1}{6}(3,2,1)$ implies that z^4, yz^3 or z^5 appear. Thus (3) implies that $P \in X$ has one of the equations in List 1.5. QED

2.3 Canonical class and adjunction

I now explain the following statement, which proves (1) $\Rightarrow$ (2):

Lemma 2.2 *Let $P \in X \subset \mathbb{A}^3$ be an isolated double point, and suppose that the blowup $\sigma\colon X_1 \to X$ has only isolated singularities. Then $K_{X_1} = \sigma^* K_X = 0$.*

Formally, this follows easily by the adjunction formula. Whatever the canonical class means, it satisfies the adjunction formula: if $\sigma\colon B \to \mathbb{A}^3$ is the blowup of $P \in \mathbb{A}^3$ (with the exceptional surface $F = \sigma^{-1}P \cong \mathbb{P}^2 \subset B$), then $K_B = \sigma^* K_{\mathbb{A}^3} + 2F$. The blowup X_1 is contained in B, and $X_1 = \sigma^* X - 2F$ (because X has multiplicity 2). Thus by adjunction

$$\begin{aligned} K_{X_1} &= (K_B + X_1)_{|X_1} \\ &= (\sigma^*(K_{\mathbb{A}^3}) + 2F + \sigma^* X - 2F)_{|X_1} = \sigma^*(K_{\mathbb{A}^3} + X)_{|X_1} = \sigma^*(K_X). \end{aligned}$$

This formal argument can be given more substance in terms of rational canonical differentials on X, as follows: let x, y, z be coordinates on $\mathbb{A}^3$, with X defined by $f(x,y,z) = 0$. Since $f = 0$ on X, it follows that

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z = 0$$

as a 1-form on X. Write $S = \frac{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z}{f} \in \Omega^3_{\mathbb{A}^3}(X)$; this is a basis for 3-forms on $\mathbb{A}^3$ with pole of order 1 along X. It follows at once that the expressions

$$s = \mathrm{Res}_{\mathbb{A}^3|X}\, S = \frac{\mathrm{d}x \wedge \mathrm{d}y}{\partial f/\partial z} = -\frac{\mathrm{d}y \wedge \mathrm{d}z}{\partial f/\partial x} = \frac{\mathrm{d}z \wedge \mathrm{d}x}{\partial f/\partial y}.$$

coincide, and define a rational canonical form on X, the *Poincaré residue* of S. (By the way: the Poincaré residue

$$\mathrm{Res}\colon \Omega^3_{\mathbb{A}^3}(X) \to \mathcal{O}_X(K_X)$$

is an intrinsically defined map that generalises the residue of a meromorphic differential form at a pole in Cauchy's integral formula, and realises the adjunction formula $K_X = (K_{\mathbb{A}^3} + X)_{|X}$.) The rational form s is regular and nonzero at every nonsingular point $Q \in X$. Indeed, one of the partial derivatives is nonzero at Q, so that, say, $\partial f/\partial z$ is an invertible function, and x, y are local coordinates at Q, and therefore s is an invertible multiple of the volume element $\mathrm{d}x \wedge \mathrm{d}y$.

Now s is also a rational canonical form on X_1 (since X_1 is birational to X), and the following calculation shows that it is also a basis for the regular canonical forms at every nonsingular point of X_1, so that $K_{X_1} = \operatorname{div}(s) = 0$. Indeed, the blowup B is covered by 3 affine pieces, one of which has coordinates (say) x_1, y_1, z with $x = x_1 z, y = y_1 z, z = z$, and $X_1 \subset B$ is defined on that affine piece by $f_1 = 0$, where $f_1 = f(x_1 z, y_1 z, z)/z^2$. By the same argument as for X,

$$s_1 = \operatorname{Res}_{B|X_1} \frac{\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}z}{f_1} = \frac{\mathrm{d}x_1 \wedge \mathrm{d}y_1}{\partial f_1/\partial z} = \text{etc.}$$

Now compare S and $S_1 = \frac{\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}z}{f_1}$. First $x = x_1 z$ gives $\mathrm{d}x = z\,\mathrm{d}x_1 +$ a multiple of $\mathrm{d}z$, and similarly for $y = y_1 z$. Thus the numerator of S splits off z^2, as does the denominator, and therefore $S = S_1$. Now $s = \operatorname{Res}_{\mathbb{A}^3|X} S$ and $s_1 = \operatorname{Res}_{B|X_1} S_1$. Outside the exceptional locus φ is an isomorphism, so that $s = s_1$ as a rational differential on X_1. Thus the differential form s lifts to s_1, that is a basis for $\mathcal{O}_{X_1}(K_{X_1})$. Thus $K_{X_1} = \varphi^*(K_X)$. QED

2.4 Sketch proof of (2) implies (3)

I clear denominators, and treat $\alpha = (1,1,0)$ or $(1,1,1)$ or $(2,1,1)$ or $(3,2,1)$ as integral weightings on monomials in x, y, z; write $d = \alpha(xyz) = 2, 3, 4, 6$ respectively.

I can associate a *weighted blowup* $\sigma\colon B \to \mathbb{A}^3$ with any of the weighting $\alpha = (1,1,0)$ or $(1,1,1)$ or $(2,1,1)$ or $(3,2,1)$ (in any analytic coordinate system on $\mathbb{A}^3$). The exceptional divisor $F \subset B$ of σ is an irreducible surface, and any monomial $m = x^a y^b z^c$ vanishes on F to multiplicity $\alpha(m)$. For example, for $\alpha = (3,2,1)$, one affine piece of the $(3,2,1)$ blowup is $B_1 = \mathbb{A}^3(x_1, y_1, z)$ with $x = x_1 z^3$, $y = y_1 z^2$, $z = z$. Then $\sigma^* x = x_1 z^3$ vanishes to order 3 on $F : (z = 0) \subset B$.

Thus if $X : (f = 0) \subset \mathbb{A}^3$, and $S = \frac{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z}{f}$ is a basis of $\Omega^3(X)$ with residue the basis of K_X, on the blowup we have $\sigma^* f = z^{\alpha(f)} f_1$ where f_1

defines $X_1 \subset B$. A basis of $\Omega^3_B(X_1)$ is given by $S_1 = \frac{\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}z}{f_1}$, and

$$\sigma^*(S) = z^{-\alpha(f)+\alpha(xy)} S_1 = z^{-\alpha(f)+\alpha(xyz)-1} S_1.$$

Thus if $\alpha(f) \geq \alpha(xyz)$, the natural basis element s of K_X has poles on X_1. See [YPG], (4.8) Proposition and p. 374 for more details.

3 Discussion of other properties

Du Val singularities are a focal point for very many different areas of math, and have a great many different characterisations (see [D]). Each of the following topics deserves a few paragraphs of explanation, but I do not have time to write it out properly.

1. **Bunches of -2-curves and the root lattices** The exceptional curves E_i of a crepant resolution is a -2-curve, that is, a curve $\cong \mathbb{P}^1$ with $E^2 = -2$. We have already seen this in some cases. It follows because $E^2 < 0$ (as an exceptional curve) and $K_Y E = 0$ (because $K_Y \sim 0$ on a neighbourhood of E); the adjunction formula $2g(E) - 2 = (K_Y + E) \cdot E$ then only allows one possibility, namely $g - 0$ and $E^2 = -2$.

   See Ex. 5 for an derivation of the possible configurations of -2-curves based on negative definite lattices.

2. **Milnor number** Let f be a regular function on $M = \mathbb{C}^3$ with an isolated critical point at P. The Milnor algebra $J(f) = \mathcal{O}_{\mathbb{C}^3}/(\partial f/\partial x_i$ is the quotient of the local ring of M at P by the ideal of partial derivatives. The Milnor number $\mu(f) = \dim_{\mathbb{C}} J(f)$ is the main invariant of an isolated critical point. See Ex. 8

3. **Milnor fibre** Let f be a regular function on $M = \mathbb{C}^3$ with an isolated critical point at $P \in \mathbb{C}^3$ (and critical value $f(P) = 0 \in \mathbb{C}$). The Milnor fibre of f is the intersection of the neighbouring fibre $f^{-1}(t)$ with a small ball around P, say $B(P, \varepsilon)$. First choose sufficiently small $\varepsilon > 0$, so that f has no other critical points in $B(P, \varepsilon)$, then choose $t \neq 0$ close to

   The Milnor fibre is diffeomorphic to the resolution.

4. **Simultaneous resolution**

5. **McKay correspondence** See [B].

Homework

1. Let $\mathbb{A}^2 \to X$ be the quotient morphism given by $x = u^2, y = uv, z = v^2$ described in 1.1. Show that the blowup $B \to \mathbb{A}^2$ of $0 \in \mathbb{A}^2$ fits into a commutative diagram

$$\begin{array}{ccc} B & \to & \mathbb{A}^2 \\ {}^{\pi}\downarrow & & \downarrow \\ Y & \to & X \end{array}$$

and that $\pi \colon B \to Y$ is a double cover ramified along $\ell \subset B$ and $\Gamma \subset Y$. Verify the compatibility of the intersection numbers $\ell^2 = -1$ in B and $\Gamma^2 = -2$ in Y.

2. Blow up $x^2 = yz(y + z)$ as in §1.3 and find all the singular points of the blowup X_1. [Hint: This is a simple exercise in not missing a singularity "at infinity", which will happen if you only take the obvious coordinate piece of the blowup.] Find a change of coordinates that makes $x^2 = yz(y + z)$ into D_4, and check that your result is compatible with that of 1.3.

3. Do all the resolutions of the Du Val singularities from the equations. [Hint: Successively blow up isolated double points in the spirit of 1.3, e.g., for D_n with $n \geq 5$, make sure you've taken note of the warning in the preceding exercise, then calculate

$$\begin{array}{ccc} D_n & & D_{n-2} \quad A_1 \\ \bullet & \longleftarrow & -\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet- \end{array}.]$$

4. Prove that the exceptional locus of the minimal resolution is a bunch of -2-curves with the configuration of the Dynkin diagram. [Hint: A_1 and D_4 were done in 1.1–1.3.]

5. A connected bunch $\bigcup \Gamma_i$ of -2-curves Γ_i with negative definite intersection matrix $\{\Gamma_i \Gamma_j\}$ base a lattice L. Classify negative definite lattices L based by a connected bunch of vectors $\{e_i\}$ with $e_i^2 = -2$. Check that the configuration is given by one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8. [Hint: Required to prove that the graph is a tree, with no node of valency ≥ 4, at most one node of valency 3, with restrictions on the lengths of the branches coming out of it. Sample arguments: $e_i e_j \geq 2$ is impossible because it leads to $(e_i + e_j)^2 \geq 0$.

A node of the graph with valency ≥ 4, or the bunch of 9 vectors with the graph "$E_9 = \widetilde{E}_8$:"

$$\begin{array}{ccccccc} & & & & \circ & & \\ & & & & | & & \\ \widetilde{D}_4\colon & \circ & — & & \circ & — & \circ \\ & & & & | & & \\ & & & & \circ & & \end{array} \quad \text{or} \quad \begin{array}{cccccccccccccccc} \widetilde{E}_8\colon & \circ & — & \circ & — & \circ & — & \circ & — & \circ & — & \circ & — & \circ & — & \circ \\ & & & & & | & & & & & & & & & & \\ & & & & & \circ & & & & & & & & & & \end{array}$$

are impossible because $v = \sum u_i e_i$ has $v^2 \geq 0$. If you do this exercise properly, all the completed Dynkin diagrams $\widetilde{A}_n, \dots, \widetilde{E}_8$ appear as logical ends of the argument.]

6. For each of A_n, D_n, E_6, E_7, E_8, show how to write down an effective combination $Z = \sum a_i \Gamma_i$ with $a_i \in \mathbb{Z}$ and every $a_i > 0$ such that $Z^2 = -2$. Show that it is the the biggest divisor with $Z^2 = -2$, and the smallest divisor satisfying $Z\Gamma_i \leq 0$ for every Γ_i.

7. Prove the assertions of 1.3: $x = (u^4 - v^4)uv$, $y = u^4 + v^4$ and $z = u^2v^2$ are BD_8-invariant functions; they define a morphism
$$\mathbb{C}^2 \to X : (x^2 = y^2 z - 4z^3) \subset \mathbb{C}^3;$$
and X is the orbit space $\mathbb{C}^2/\mathrm{BD}_8$. In other words, prove that two point (u_1, v_1) and $(u_2, v_2) \in \mathbb{C}^2$ give rise to the same value of x, y, z if and only if there is an element of BD_8 taking one to the other.

8. Calculate the Milnor number of the Du Val surface singularities; show that A_n and D_n have $\mu = n$, and E_6, E_7, E_8 have $\mu = 6, 7, 8$ respectively. [Hint: for example, if $f = x^2 + y^3 + z^5$ then the calculation is
$$\partial f/\partial x = 2x, \quad \partial f/\partial y = 3y^2, \quad \partial f/\partial z = 5z^4,$$
so $J(f) = \mathbb{C}[x, y, z]/(x, y^2, z^4)$ is a vector space with basis $1, z, z^2, z^3, y, yz, yz^2, yz^3$, and it has dimension 8.]

9. Let ε be a primitive $2n$th root of 1 and consider the two matrixes
$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
Show that they generate a binary dihedral subgroup $\mathrm{BD}_{4n} \subset \mathrm{SL}(2, \mathbb{C})$ of order $4n$. Find the ring of invariants of BD_{4n} acting on $\mathbb{C}^2$ and prove that $\mathbb{C}^2/\mathrm{BD}_{4n}$ is isomorphic to the Du Val singularity D_{n+2}. [Hint: Find first the ring of invariants of $\langle A \rangle \cong \mathbb{Z}/n$, and show that B acts by an involution on it.]

10. **Cyclic group** of order m is generated by

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{where} \quad \varepsilon = \exp \frac{2\pi i}{m}$$

Binary dihedral group BD_{4m} is generated by

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Binary tetrahedral group BT_{24} is generated by

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$$

Check $A^2 = B^2 = (AB)^2 = C^3 = -1$ and $(AC)^3 = (BC)^3 = 1$; check also that A, B, C generate a group G of order 24, and that the quotient $\overline{G} = G/(\pm 1)$ is isomorphic to the alternating group A_4.

Binary octahedral group BO is generated by

$$D = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, \quad \text{where } \varepsilon = \exp \tfrac{2\pi i}{8}$$

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$$

or by BT and the matrix D (note $D^2 = A$).

Binary icosahedral group Write ε for 5th root of 1.

$$A = \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E = \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon + \varepsilon^4 & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix}.$$

Check that $E^2 = -1$ and $(AE)^3 = -1$. Find the other relations. Show that B is in the group generated by A and E. Show that A, E generate a group of order 120.

11. There is an action of the multiplicative group $\mathbb{H}^*$ of nonzero quaternions by rotations of Euclidean 3-space $\mathbb{R}^3$ defined as follows: identify Euclidean 3-space $\mathbb{R}^3$ with the space of imaginary quaternions $x = x_1 i + x_2 j + x_3 k \in \mathbb{H}$, and let $\mathbb{H}^* \ni q \colon x \mapsto qxq^{-1}$. This action makes the unitary group $\mathrm{SU}(2, \mathbb{C})$ into a double cover of $\mathrm{SO}(3)$.

12. Write out matrix generators for the binary tetrahedral groups $G \subset \mathrm{SL}(2,\mathbb{C})$ and calculate its invariants. See the book by Blichfeldt, in English [G. A. Miller, H. F. Blichfeldt and L. E. Dickson, Theory and applications of finite groups, Wiley, 1916].

13. In Ex. 10 you saw that the inclusion $\mathbb{Z}/2n \lhd \mathrm{BD}_n$ defines a double cover $A_{2n-1} \to D_{n+2}$ between Du Val singularities. Show also that $\mathrm{BD}_{4n} \subset \mathrm{BD}_{8n}$ is a normal subgroup of index 2 and that it defines a double cover $D_{n+2} \to D_{2n+2}$ between Du Val singularities.

14. Every finite subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$ leaves invariant a Hermitian inner product, and is thus conjugate to a subgroup of the unitary group $\mathrm{SU}(2,\mathbb{C})$.

15. Exercises repeated from [Part I], Chap. 4.

16. Calculate the numerical cycle Z_{num} for each of the Du Val singularities ([Part I], 4.5). Check that $Z_{\mathrm{num}}^2 = -2$. [Hint: Start from $Z_0 = \sum \Gamma_i$, and successively increase the coefficients of Γ_i only if $Z_j\Gamma_i > 0$.]

17. Prove negative semidefiniteness for a fibre $F = \sum a_i\Gamma_i$ of a surface fibration ([Part I], 8.7). [Hint: In more detail, assume that $\bigcup \Gamma_i$ is connected. Then $F\Gamma_i = 0$; if $\mathrm{hcf}\{a_i\} = m$ then $F = mF'$, where $F' = \sum a_i'\Gamma_i$, and the $a_i' = a_i/m$ have no common factor. Now adapt the argument of ([Part I], 8.7) to prove that $E = \sum b_i\Gamma_i$ has $E^2 \le 0$, with equality if and only if E is an integer multiple of F'.]

18. Prove [Part I], Proposition 4.12, (2). [Hint: Several methods are possible. For example, you can prove that $h^0(\mathcal{O}_A(-Z_{\mathrm{num}})) \ne 0$, so that $h^0(\mathcal{O}_{A+Z}) \ge 2$ for every $A > 0$. Or you can calculate $\chi(\mathcal{O}_{A+Z})$ using the numerical games of (i).]

19. Prove [Part I], Proposition 4.12, (3). [Hint: If $Z_{\mathrm{num}} = D_1 + D_2$, write out $p_a(Z_{\mathrm{num}}) = 1$, $p_a(D_1), p_a(D_2) \le 1$ in terms of the adjunction formula ([Part I], A.11).]

20. Consider the codimension 2 singularity $P \in X \subset \mathbb{C}^4$ defined by the two equations $x_1^2 = y_1^3 - y_2^3$ and $x_2^2 = y_1^3 + y_2^3$. Prove that $P \in X$ has a resolution $f\colon Y \to X$ such that $f^{-1}P$ is a nonsingular curve C of genus 2 and $C^2 = -1$, but with $H^0(\mathcal{O}_C(-C)) = 0$. Prove that x_1, x_2 vanish along C with multiplicity 2 and y_1, y_2 with multiplicity 3. Hence $Z_{\mathrm{fib}} = 2C$, although obviously $Z_{\mathrm{num}} = C$.

21. If $(\Gamma_i\Gamma_j)$ is negative definite, prove that $H^0(\mathcal{O}_A(A)) = 0$ for every effective divisor supported on Γ_i. More generally, if $\mathcal{L}$ is a line bundle on A such that $\deg_{\Gamma_i} \mathcal{L} \leq 0$ for all i then $H^0(\mathcal{O}_A(A) \otimes \mathcal{L}) = 0$. [Hint: Since $A^2 < 0$, any section $s \in H^0(\mathcal{O}_A(A))$ must vanish on some components of A. Now use the argument of [Part I], Lemma 3.10.]

22. Suppose that X is projective with ample H, and $f\colon Y \to X$ as usual. Let $Z = \sum a_i\Gamma_i$ be an effective exceptional divisor with $Z\Gamma_i < 0$. Prove that $n(f^*H) - Z$ is ample on Y for $n \gg 0$. [Hint: Some section on nH on X vanishes on Z, so that $n(f^*H) - Z$ is effective for some n. Now prove that by taking a larger n if necessary, $(n(f^*H) - Z)C > 0$ for every curve $C \subset Y$. The result follows by the Nakai–Moishezon ampleness criterion (which is easy, see [H], Chap. V, 1.10).]

References

[D] Alan Durfee, Fifteen characterizations of rational double points and simple critical points, Enseign. Math. **25** (1979) 131–163

[H] Robin Hartshorne, Algebraic geometry, Graduate Texts in Mathematics 52, Springer 1977

[M] J. Milnor, Singular points of complex hypersurfaces. Annals of Math Studies **61** Princeton, 1968

[Part I] M. Reid, Chapters on algebraic surfaces, in Complex algebraic geometry (Park City, UT, 1993), 3–159, IAS/Park City Math. Ser. **3**, AMS 1997

[YPG] M. Reid, Young person's guide to canonical singularities, in Algebraic geometry (Bowdoin, 1985), Proc. Sympos. Pure Math. **46** Part 1, AMS 1987, pp. 345–414

[B] M. Reid, La correspondance de McKay, Séminaire Bourbaki, Vol. 1999/2000, Astérisque **276** (2002) pp. 53–72