Surfaces with $p_g = 0$, $K^2 = 2$

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Dedicated to the victims in the struggle for freedom and justice in the Soviet Union and the Republic of South Africa

0 Introduction

The main object of study of this paper is a Galois etale cover $\psi \colon Y \to X$, where X is a minimal surface of general type; write $G = \mathrm{Gal}(Y/X)$ and $n = |G|$. Recall that the *algebraic fundamental group* is defined as the limit $\pi_1^{\mathrm{alg}}(X) = \varprojlim G$, the limit begin taken over all such covers Y. For brevity I will write $\pi_1 X$ for $\pi_1^{\mathrm{alg}}(X)$; this paper will only incidentally contain information concerning the topological fundamental group.

Theorem 0.1 *Suppose that X has $p_g = 0$, $K^2 = 2$; then any Galois etale cover $Y \to X$ has order $n \le 9$.*

Furthermore,

(i) if $n = 8$ then the canonical map $\varphi_{K_Y} \colon Y \to \overline{Y} \subset \mathbb{P}^6$ is a birational morphism, and the image $\overline{Y}$ is a complete intersection of 4 quadrics.

(ii) if $n = 9$ then $\varphi_{K_Y} \colon Y \to \overline{Y} \subset \mathbb{P}^7$ is a birational morphism and the image $\overline{Y}$ is the section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ with a cubic hypersurface.

Corollary 0.2 *Any cover $Y \to X$ has $q(Y) = 0$; and $H^1(X, \sigma) = 0$ for every torsion element $\sigma \in \mathrm{Pic}\, X$.*

A detailed treatment of the cases $|\pi_1 X| = 9$ and $|\pi_1 X| = 8$ is given in Section 2; for $|\pi_1 X| = 8$, each of the possibilities $(\mathbb{Z}/2)^3$, $\mathbb{Z}/4 \oplus \mathbb{Z}/2$, $\mathbb{Z}_8$ and Q_8 (the quaternion group) occurs and leads to an irreducible (unirational) moduli space; whereas $\pi_1 X = D_8$ (the dihedral group) is impossible.

It should be possible to obtained complete information on the cases of $\pi_1 X$ of order 7, 6, or 5, and I hope to return to this subject in a subsequent paper.

The idea behind Theorem 0.1 is that for n large the canonical map of Y must be "special", so that Y acquires some special properties. On the other hand, any idiosyncracy of the canonical system is intrinsically attached to Y and is therefore compatible with the action of G (see Section 4). In principle this idea can be applied to any surface with small K^2 and large π_1. The following more general results are proved in Section 3.

Theorem 0.3 *Let X be a minimal surface o general type, and suppose that $K_X^2 < \frac{1}{3}c_2(X)$ (equivalently, $K_X^2 < 3\chi(\mathcal{O}_X)$; then either*

(i) $\pi_1 X$ is finite; or

(ii) there exists an etale Galois cover $Y_0 \to X$, Y_0 having a morphism $f\colon Y_0 \to C_0$ to a curve of genus $p > 0$ inducing an isomorphism $f_*\colon \pi_1 Y_0 \stackrel{\cong}{\to} \pi_1 C_0$.

Furthermore in (ii), the fibres of $f\colon Y_0 \to C_0$ are hyperelliptic curves of genus $g \le 5$.

Corollary 0.4 *(i) $\pi_1 X$ is an extension of $\pi_1 C_0$ by a finite group G_0;*

(ii) for every etale cover $Y \to X$ with $q(Y) \neq 0$, the Albanese map $\alpha\colon Y \to \operatorname{Alb} Y$ maps onto a curve;

(iii) if $f\colon X \to C$ is a nonconstant morphism of X with connected fibres to a curve C of genus $p > 0$ then $q(X) = p$.

Dividing $Y_0 \to C_0$ by the equivariant action of G_0 gives a restatement of (ii):

(ii′) $\pi_1 X$ is infinite, and contains a normal subgroup A of order ≤ 4; there exists a morphism $X \to B$ to a curve B; and every etale cover $Y \to X$, which corresponds to a finite quotient of $\pi_1 X/A$, is obtained by making a normalised pullback diagram

$$\begin{array}{ccc} Y = \widetilde{X \times_B C} & \longrightarrow & X \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \end{array}$$

over a cover $C \to B$ ramified only at points of B corresponding to multiple fibres of $X \to B$.

Theorem 0.5 *Let X be a minimal surface with $K_X^2 < 3p_g(X) - 7$, $K_X^2 < 3\chi(\mathcal{O}_X)$, and $p_g \ge 8$; then one of the following 4 cases hold:*

(i) $|K_X|$ *is composed of a rational pencil* $X \dashrightarrow \mathbb{P}^1$*, and* $q(X) = 0$*;*

(ii) $|K_X|$ *is composed of a irrational pencil* $X \to C$*, with* C *a curve of genus* $p > 0$*, and* $q(X) = p$*;*

(iii) $\varphi_{K_X}\colon X \dashrightarrow F \subset \mathbb{P}^{p_g(X)-1}$ *is generically* 2*-to-*1 *onto a rational surface* F*, and* $q(X) = 0$*; or*

(iv) $\varphi_{K_X}\colon X \dashrightarrow F \subset \mathbb{P}^{p_g(X)-1}$ *is generically* 2*-to-*1 *onto a ruled surface of genus* $p > 0$*, and* $q(X) = p$*.*

It is quite likely that (i) and (ii) cannot actually occur; I also do not know if fibres of genus 4 or 5 can occur in Theorem 0.3 – if this happens then $|K_Y|$ must have a fixed part having large intersection number with the fibres.

There is an element of ineffectivity in Theorem 0.3, there being no bound for the finite $\pi_1 X$ which can occur; the problem is to bound the 2-torsion in $\operatorname{Pic} X$ (see Problem 3.2).

A positive answer to Conjecture **??** would be one step in the direction of the following:

Conjecture 0.6 *The hypothesis in Theorem 0.3 can be weakened to* $K_X^2 < \frac{1}{2}c_2(X)$ *(equivalently,* $K_X^2 < 4\chi(\mathcal{O}_X)$*).*

The conjecture can be weakened to ask only that surfaces with $K_X^2 < \frac{1}{2}c_2(X)$ satisfy Corollary 0.4, (ii) and (iii); perhaps the natural approach to this conjecture would be through differential geometric methods, which could also shed light on the problem of the topological fundamental group for surfaces in this range.

F. Sakai points out that the hypothesis of Theorem 0.3 cannot be weakened to $K_X^2 < c_2(X)$, since for every surface of general type S, there exists a cyclic branched cover $T \to S$ with $K_T^2 < c_2(T)$.

Acknowledgements Most of the information in Section 2 was known to Godeaux (see also [2], p. 219); the main part of the proof of Theorem 0.1 (up to the assertion $|\pi_1 X| \le 10$) has been established independently by Beauville (see [1]). Section 7 owes a lot the influence of Andrei Tyurin, especially to [6]. W.D. Geyer has provided me with a simplification in the proof of Miyaoka's theorem (Section 1, Step 4). I would also like to thank Igor Dolgachev and Chris Peters for their interest, and for a very useful correspondence.

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Contents

Characteristic All varieties, morphisms, etc., in this paper are defined over an algebraically closed field k of characteristic 0.

According to [15], p. **??**, a surface of general type has $p_g \geq q$, so that $p_g = 0$ implies also $q = 0$. Actually any surface with $K^2 > 0$ which is not rational (for example, not simply connected) is necessarily of general type.

The assumption of characteristic 0 is used several times in using the formula for the plurigenera $P_n = \chi(\mathcal{O}_X) + \binom{n}{2}K^2$, especially the $n = 2$ case, $P_2 = \chi(\mathcal{O}_X) + K^2$; it is used again in Theorem **??**, a classical result which is to be found (with one and the same proof) in dozens of places in the literature. It would seem highly desirable to have a proof of this result in characteristic p.

A linear system $|L|$ is said to be *free* if it is free of fixed components and base points. I write $H^i \stackrel{\mathrm{d}}{=} H^{n-i}$ to denote duality of vector spaces.

Part I
The main theorem

1 Proof of Theorem 0.1

Let X be as in Theorem 0.1, and $\psi\colon Y \to X$ an etale Galois cover, with G and n as in Section 0; Y has the following invariants:

$$K_Y^2 = nK_X^2 = 2n \quad \text{and} \quad (p_a+1)(Y) = n(p_a+1)(X),$$

that is, $p_g(Y) = n - 1 + q(Y)$. Theorem 0.1 is proved by considering the canonical map

$$\varphi_{K_Y}\colon Y \dashrightarrow \overline{Y} \subset \mathbb{P}^{n-2+q(Y)};$$

the proof proceeds in 8 steps:

Step 1 *For $n \geq 5$, $|K_Y|$ is not composed of a pencil.*

Proof Suppose that

$$|K_Y| = |E^{(r)}| + F,$$

with $E^{(r)} = E_1 + \cdots + E_r$, the E_i being fibres of a rational map $f\colon Y \dashrightarrow C$. Write E for the numerical equivalence class of the E_i. Then

$$2n = K_Y^2 = rK_YE + K_YF \geq rK_YE,$$

and

$$K_YE = rE^2 + EF \geq rE^2;$$

furthermore, since $|E^{(r)}|$ has projective dimension $p_g(Y) - 1$. I have

$$n - 2 \leq p_g(Y) - 1 \leq r.$$

If C has genus $p > 0$ then f is a morphism, and $E^2 = 0$; on the other hand if C is rational then the fibres of f form a rational pencil $|E|$ equivariant under the action of G, so that by Lemma 4.1, (i), $n \mid E^2$, and $2n \geq r^2E^2$ implies at once that $E^2 = 0$. Then

$$K_YE \leq \frac{2n}{r} \leq \frac{2n}{n-2}$$

implies that $K_YE = 2$, so that f is a pencil of curves of genus 2. Some element of G must act on C with nontrivial stabiliser, since otherwise C/G would have genus $p > 0$, with morphism $X = Y/G \to C/G$. Lemma 4.2, (i) now provides a contradiction.

Step 2 *If $n \geq 5$, then $|K_Y|$ is free.*

Proof Write $|K_Y| = |C| + F$, with C an irreducible linear system, and F the fixed part. Then

$$\begin{aligned} K_Y^2 &= K_Y C + K_Y F &\geq K_Y C, \\ K_Y C &= C^2 + CF &\geq C^2; \end{aligned}$$

writing Z for the scheme theoretic base locus of C, I also have

$$C^2 = \deg Z + \deg \varphi_C \cdot \deg \overline{Y},$$

where $\varphi_C = \varphi_{K_Y} \colon Y \dashrightarrow \overline{Y} \subset \mathbb{P}^{n-2+q(Y)}$ is the rational map defined by $|C|$. Then

$$2n = K_Y^2 = \deg \varphi_C \cdot \deg \overline{Y} + KF + CF + \deg Z.$$

According to Lemma 4.1, each of KF, CF and $\deg Z$ is divisible by n, so that if any one of them is nonzero, I get

$$\deg \varphi_C \cdot \deg \overline{Y} = n;$$

and $\deg \overline{Y} \geq n - 3$, since $\overline{Y}$ spans $\mathbb{P}^{n-2+q(Y)}$.

In case $n = 5$ or $n \geq 7$ if follows immediately that $\deg \varphi_C = 1$, that is, φ_C is birational; for $n = 6$ special considerations also show that φ_C must be birational. But for $n = 5$, $\overline{Y}$ would be a quintic in $\mathbb{P}^3$, birational to Y, which contradicts the invariants $p_g = 4$, $K_Y^2 = 10$; for $n \geq 6$ a surface of degree n in $\mathbb{P}^{n-2+q(Y)}$ is not of general type according to Theorem 6.2. This contradiction proves this step.

Step 3 *For $n \geq 6$, either φ_{K_Y} is birational, or $G = (\mathbb{Z}/2)^a$.*

Remark 1.1 For $n = 5$ it can happen that $Y \to \overline{Y} \subset \mathbb{P}^3$ is a double cover of a quintic, ramified in just 20 nodes of $\overline{Y}$; a specific example can be constructed easily out of the Hilbert modular surface (see [3], §4, Theorem 3). This provides the only surface with $p_g = 0$, $K^2 = 2$ and $\pi_1 = \mathbb{Z}/5$ which I know of at time of writing.

Proof $2n = K_Y^2 = \deg \varphi_{K_Y} \cdot \deg \overline{Y}$, with $\deg \overline{Y} \geq n - 3$. Hence

$$\deg \varphi_{K_Y} \leq \frac{2n}{n-3}, \quad \text{and} \quad \deg \varphi_{K_Y} \mid 2n.$$

Thus if φ_{K_Y} is not birational then either $\deg \varphi_{K_Y} = 2$, or I am in one of the following cases:

$$\begin{aligned} & n = 6, \quad \deg \varphi_{K_Y} = 4; \\ & n = 6, \quad \deg \varphi_{K_Y} = 3; \\ \text{or} \quad & n = 9, \quad \deg \varphi_{K_Y} = 3. \end{aligned}$$

The cases with $n=6$ require separate (rather easy) treatment, and are omitted here. The case $n=9$, $\deg_{K_Y}=3$ leads rapidly to a contradiction as follows: $\varphi_{K_Y}\colon Y\to\overline{Y}\subset\mathbb{P}^7$, with $\overline{Y}$ of degree 6; then $\overline{Y}$ must be a rational normal scroll $\mathbb{F}_{n;r}$ with $n+2r=6$. If $r\neq 0$ then $\overline{Y}$ is nonsingular, and the ruling of $\overline{Y}$ leads to a pencil $|E|$ of curves on Y with $Y^2=0$, $K_YE=3$, which is absurd. In the case $r=0$, $\overline{Y}$ is the cone $\overline{\mathbb{F}}_6$, and the generators of this cone again give an irreducible pencil $|E|$ on Y such that $K_Y\geq 6E$; it follows at once that $E^2=0$, $K_YE=3$, which is the same contradiction.

Suppose then that $\deg\varphi_{K_Y}=2$; the image $\overline{Y}=F$ is then a surface of degree n spanning $\mathbb{P}^{n-2+q(Y)}$. By Theorem 6.2, F is either a K3 surface (and $n=6$), or a rational surface, or a ruled surface of genus $p>0$.

(a) Suppose that F is rational; then the double cover $\varphi_{K_Y}\colon Y\to F$ defines a biregular involution i of Y such that the quotient Y/i is rational. The covering $Y\to Y/i$ is equivariant under an action of G, and every element $\sigma\in G$ has a fixed point on Y/i. Lemma 4.1, (ii) then shows that $\sigma^2=1$ for all $\sigma\in G$, so that $G=(\mathbb{Z}/2)^a$.

(a′) If $n=6$ and F is birationally a K3 surface then the same argument applied to an element of G of order 3 (which necessarily has a fixed point on Y/i) gives a contradiction.

(b) Suppose that F is ruled of genus $p\geq 1$; F is of degree n in $\mathbb{P}^{n-2+q(Y)}$, with $q(Y)\geq p>0$. By Corollary 6.5, if $n\geq 5$ then F is ruled by lines. It follows that Y has a canonical pencil $Y\to C$ of curves of genus 2, with C of genus $p>0$. As before, G cannot act freely on C, for otherwise the quotient C/G would also have genus >0, and there would be a morphism $X\to C/G$. Lemma 4.2, (i) then provides the usual contradiction, completing the proof of this step.

Step 4 (Miyaoka) *$G=(\mathbb{Z}/2)^a$ implies that $a\leq 3$.*

According to Theorem 5.4, a Galois cover with group $G=(\mathbb{Z}/2)^a$ corresponds to a subgroup $(\mathbb{Z}/2)^a\subset\operatorname{Pic}X$.

Theorem 1.2 ([5], §4) *Let X be a minimal surface with $p_g=0$, $K^2=2$, and suppose that $(\mathbb{Z}/2)^3\subset\operatorname{Pic}X$; let $\psi\colon Y\to X$ be the corresponding cover. Then $\varphi_{K_Y}\colon Y\to\overline{Y}\subset\mathbb{P}^6$ is birational onto a complete intersection of 4 quadrics in $\mathbb{P}^6$.*

Since $\overline{Y}$ has at worst rational double points as singularities it follows that Y is simply connected.

Proof For each of the 7 nonzero elements $\sigma \in (\mathbb{Z}/2)^3 \subset \operatorname{Pic} X$, I have $H^2(X, \mathcal{O}_X(K_X+\sigma)) = 0$, so that by Riemann–Roch,

$$h^0(X, \mathcal{O}_X(K_X+\sigma)) = 1 + h^1 \geq 1;$$

thus I can choose nonzero sections $x_\sigma \in H^0(K_X+\sigma)$.

The 7 elements all belong to $H^0(2K_X)$, which is a vector space of dimension 3; there are thus (at least) 4 linearly independent relations

$$L_i(x_\sigma^2) = \sum \lambda_{i\sigma} x_\sigma^2 = 0. \tag{1.1}$$

Now for each σ, let x_σ continue to denote $\psi^* x_\sigma \in H^0(K_Y)$; these 7 elements are linearly independent since they belong to different eigenspaces of the group action. Consider the map $\varphi\colon Y \dashrightarrow \mathbb{P}^6$ defined by these 7 sections; exactly as in Step 1, the image $\overline{Y} = \varphi(Y)$ is a surface, which spans $\mathbb{P}^6$ since the $x_\sigma \in H^0(K_Y)$ are linearly independent. On the other hand $\overline{Y}$ is contained in 4 linearly independent *diagonal* quadrics (1.1).

The following simple result completes the proof of Theorem 1.2:

Proposition 1.3 *Let $x_0, \dots, x_n$ be homogeneous coordinates in $\mathbb{P}^n$, and let*

$$Q_i(\underline{x}) = \sum \lambda_{ij} x_j^2 \quad \textit{for } i = 1, \dots, r$$

be r linearly independent diagonal quadrics. Let $W \subset \mathbb{P}^n$ be the locus of common zeros of the Q_i. Then

(i) W is pure of dimension $n-r$;

(ii) either W is irreducible, or every component of W is contained in a hyperplane.

Proof Choose a component U of W; renumbering the x_i if necessary, I can assume that U is not contained in the hyperplane $x_0 = 0$, and that the relations are of the form

$$x_i^2 = L_i(x_0^2, \dots, x_{n-r}^2) \quad \text{for } i = n-r+1, \dots, n,$$

with the L_i linear forms.

For (i), note that by the theorem on the dimension of intersection, U has dimension $\geq n-r$; but on the other hand, the function field $k(U)$ of U is generated by $x_1/x_0, \dots, x_n/x_0$, of which the final r are algebraically dependent on the first $n-r$.

For (ii), note first of all that $k(U)$ contains the purely transcendental extension $k(T_1, \dots, T_{n-r})$ of k with $T_i = x_i^2/x_0^2$. The extension $k(U)/k(\underline{T})$

is generated by x_i/x_0 for $i = 1, \dots, n$, each element of which satisfies an equation

$$x_i^2/x_0^2 = L_i(T_1, \dots, T_{n-r}),$$

with the L_i linear polynomials in the T_i (not necessarily homogeneous).

Now I claim that if no 2 of the L_i are proportional, then $[k(U) : k(\underline{T})] = 2^n$; from this it follows easily that U has degree 2^r in $\mathbb{P}^n$, so that $U = W$ and W is irreducible. On the other hand, if say $L_i = \alpha^2 L_j$ then $x_i^2 = \alpha^2 x_j^2$, so that U is contained in one of the hyperplanes $x_i = \pm\alpha x_j$.

To prove the claim, let $K_i = k(\underline{T})(x_1/x_0, \dots, x_i/x_0)$; then each K_i is either a quadratic extension of K_{i-1}, or $K_i = K_{i-1}$. If the latter happens, consider the valuation of $k(\underline{T})$ at L_i; this is certainly ramified in K_i, so that it must also be ramified in K_{i-1}. However, this implies that the valuation at L_i coincides with the valuation at L_j for some $j < i$, and then the linear forms L_i and L_j are proportional

This completes the proof of the proposition and of Step 4.

Step 5 $n \leq 10$.

Proof By Steps 3 and 4, for $n \geq 6$ the morphism $\varphi_{K_Y} \colon Y \to \overline{Y} \subset P^{n-2+q(Y)}$ must be birational. However, by Theorem **??**, (ii), this implies that $K_Y^2 \geq 3p_g - 7$; substituting the values given at the beginning of this section gives

$$2n \geq 3(n-1) - 7 + 3q(Y),$$

and $n \leq 10$.

Note that this proves Corollary 0.4, since if Y has $q \neq 0$ then X has etale Galois covers of arbitrarily large orders.

Step 6 *For $n = 9$ or 10, or for $n = 8$ if $\overline{Y}$ is not a complete intersection of 4 quadrics, then $\overline{Y} \subset W \subset \mathbb{P}^{n-2}$ with W a component of the intersection of all quadrics through $\overline{Y}$.*

Proof In cases $n = 10$ and $n = 9$, I have $K_Y^2 = 3p_g(Y) - 7$ and $K_Y^2 = 3p_g(Y) - 6$ respectively. The assertion thus follows from Theorem **??**, (iii).

In case $n = 8$, $\overline{Y} \subset \mathbb{P}^6$ has degree 16, and is contained in 4 quadrics. Thus if $\bigcap_{Q \supset \overline{Y}} Q \neq \overline{Y}$, it must contain some component W of dimension ≥ 3; the case of 4-dimensional W is easy to exclude by means of Theorem **??**, (i) and Proposition **??**. But then I can choose 3 quadrics Q_1, Q_2, Q_3 in the web such that $\bigcap_{i=1}^3 Q_i = \bigcup W_j$, with each W_j irreducible and 3-dimensional; there must be at least two W_j, since if $\bigcap_{i=1}^3 Q_i$ is irreducible then for some 4th quadric Q_4 the intersection $\bigcap_{i=1}^4 Q_i$ is purely 2-dimensional, and therefore coincides with $\overline{Y}$. Therefore $\overline{Y} \subset W$ $(= W_1$, say), with $\deg W < 8$; if Q_4 is some quadric containing $\overline{Y}$ by not containing W then $W \cap Q_4$ is purely 2-dimensional, of degree < 16, and this contradiction implies that W is a component of $\bigcap_{Q \supset \overline{Y}} Q$.

Note that since $W \subset \mathbb{P}^{n-2}$ is contained in at least $\binom{n}{2} - 3n$ quadrics, it follows from Proposition **??** that in any of the 3 cases $\deg W \leq 6$. Furthermore W spans $\mathbb{P}^{n-2}$, so that $\deg W \geq n-4$.

The notation of Step 6 is preserved in the next 3 sections.

Step 8 deals with the essential case $n = 8$, $\deg W = 6$.

Step 7 *The following cases are impossible:* $n = 10$; $n = 9$ *and* $\deg W \leq 5$; $n = 8$ *and* $\deg W \leq 5$.

Proof If $\mathbb{F} \subset \mathbb{P}^{m+d-1}$ is a rational normal m-fold scroll of degree d then one calculates easily (compare Section 7)

$$h^0(\mathbb{F}, \mathcal{O}_{\mathbb{F}}(k)) = \binom{k+m-1}{m} d + \binom{k+m-1}{m-1}.$$

Case $n = 10$**,** $\deg W = 6$ Here W is a rational normal 3-fold scroll; then

$$h^0(W, \mathcal{O}_W(4)) = \binom{6}{3} 6 + \binom{6}{2} = 135,$$

$$\text{whereas} \quad h^0(\overline{Y}, \mathcal{O}_{\overline{Y}}(4)) \leq h^0(Y, 4K_Y) = 10 + 20\binom{4}{2} = 130.$$

Thus $\overline{Y}$ is contained in a quartic hypersurface of $\mathbb{P}^8$ not containing W. The ruling of W must then cut out a pencil of curves of genus ≤ 3 on $\overline{Y}$; this gives the usual contradiction to Lemma 4.2, (i).

Case $n = 9$**,** $\deg W = 5$ We have

$$h^0(W, \mathcal{O}_W(5)) = \binom{7}{3} 5 + \binom{7}{2} = 196,$$

$$\text{whereas} \quad h^0(\overline{Y}, \mathcal{O}_{\overline{Y}}(5)) \leq h^0(Y, 5K_Y) = 9 + 18\binom{5}{2} = 189.$$

Thus $\overline{Y}$ is contained in the intersection of W with a quintic. This gives an equivariant pencil $|E|$ on Y with $K_Y E = 4$ or 5; on the other hand, Y has an automorphism of order 3 without fixed points. Applying Lemma 4.1 gives $3 \mid E^2$ and $3 \mid K_Y E + E^2$, and the usual contradiction.

A 3-fold of degree 4 spanning $\mathbb{P}^6$ is either a rational normal scroll or the cone on the Veronese surface.

Case $n = 8$, $\deg W = 4$ and W the Veronese cone The vertex of W is intrinsically determined by Y, and so is fixed by the action of G. Hence $\overline{Y}$ cannot pass through the vertex of W, since otherwise it would have to pass 8-fold through it, giving a contradiction as in Step 2. Thus $\overline{Y}$ is a Cartier divisor on W. Because $\operatorname{Pic} W = \mathbb{Z}$, $\overline{Y}$ is the intersection of W with a quartic hypersurface; the adjunction formula then gives $K_{\overline{Y}} \sim 3h$, where h is the pullback of a line of $\mathbb{P}^2$ under the projection of $\overline{Y}$ to the base of the cone W. If now $\Delta \subset Y$ is the divisor defined by the conductor of $\mathcal{O}_Y$ in $\mathcal{O}_{\overline{Y}}$, we have

$$\Delta \sim \varphi^* K_{\overline{Y}} - K_Y \sim \varphi^* h, \quad \text{and} \quad 2\Delta \sim K_Y.$$

This is a divisor which is invariant under the whole group G, and of arithmetic genus given by $2p_a - 2 = \Delta(K_Y + \Delta) = \frac{3}{4}K_Y^2 = 12$. This contradicts Lemma 4.1, (ii).

Case $n = 8$, $\deg W = 4$ and W a rational normal scroll Then

$$h^0(W, \mathcal{O}_W(6)) = \binom{8}{3}4 + \binom{8}{2} = 252,$$

$$\text{whereas} \quad h^0(\overline{Y}, \mathcal{O}_{\overline{Y}}(6)) \leq h^0(Y, 6K_Y) = 8 + 16\binom{6}{2} = 248.$$

As in the above cases, Y then acquires an equivariant pencil $|E|$ with $K_Y E = 4, 5$ or 6; this is impossible to accommodate with a fixed point free automorphism of order 4. For in cases $K_Y E = 5$ or 6 one obtains a contradiction as before. In case $K_Y E = 4$, $|E|$ is a pencil of curves of genus 3, giving a contradiction by Lemma 4.2, (i).

A 3-fold W of degree 5 in $\mathbb{P}^6$ is either nonsingular, or a cone over a del Pezzo surface of degree 5 (compare [18]).

Case $n = 8$, $\deg W = 5$ and W nonsingular In this case W is the unique Fano 3-fold of index 2 and degree 5 ([8], Theorem 4.2, (iii)); since this is a hyperplane section of a Grassmannian, it has $\operatorname{Pic} W = \mathbb{Z}$. This implies that $5 \mid \deg \overline{Y}$, a contradiction.

Case $n = 8$, $\deg W = 5$ and W a cone on a del Pezzo surface If W is a cone with vertex O, then $\overline{Y}$ cannot pass through O by the argument used above. If H_1 and H_2 are two general hyperplane sections through O then $\overline{Y} \cap H_1 \cap H_2$ consists of $5d$ transversal intersections,where d is the degree of the projection morphism from $\overline{Y}$ to the base of the cone; this again implies that $5 \mid \deg \overline{Y}$, a contradiction.

Step 8 *The case $n = 8$, $\deg W = 6$ is impossible.*

Proof Let $V_4 \subset H^0(\mathbb{P}^6, I_W \cdot \mathcal{O}(2))$ be the vector space of quadrics through W. If $V_r \subset H^0(\mathbb{P}^N, \mathcal{O}(2))$ is an r-dimensional subspace, I write

$$I(V_r) = \bigcap_{Q \in V_r} Q$$

for the corresponding (scheme theoretic) intersection. It is easily seen by the arguments of Section 7 that $I(V_4)$ is (set theoretically) no bigger than W itself. By Bertini's theorem it then follows that for general $V_3 \subset V_4$, $I(V_3)$ is reduced and purely 3-dimensional. There are only two possibilities for the components residual to W:

Case (i) $I(V_3) = W \cup \pi_1 \cup \pi_2$, where π_1 and π_2 are two 3-planes with $\dim(\pi_1 \cap \pi_2) \leq 1$.

Case (ii) $I(V_3) = W \cup q$, where $q \subset \mathbb{P}^4$ is a quadric (possibly a pair of 3-planes meeting along a 2-plane).

Proposition **??** disposes of (i); for cutting $W \cup \pi_1 \cup \pi_2$ by a general $\mathbb{P}^4 \subset \mathbb{P}^6$ we get two disjoint lines L_1 and L_2, which intersect $W'' = R$ in a subscheme of degree 3. Then any quadric containing W must also contain π_1 and π_2, and this contradicts the fact that W is contained in 4 linearly independent quadrics.

In case (ii), at least a 2-dimensional space V_2 of quadrics contain the $\mathbb{P}^4 = \langle q \rangle$. Then $I(V_2) = F \cup \mathbb{P}^4$, where F is a cubic 4-fold scroll; since $W \not\subset \mathbb{P}^4$, $W \subset F$, and since W is contained in 4 linearly independent quadrics, $W = F \cap Q$ for some quadric Q not containing F.

The construction so far depends on a particular choice of V_3, and is thus not necessarily invariant under the group action. However, I claim that the rational map $\varphi\colon W \dashrightarrow \mathbb{P}^1$ obtained by projecting from $\mathbb{P}^4$ is uniquely determined. This is so because for any general $V_3' \subset V_4$, and least a $V_2' \subset V_3'$ of quadrics contain F, so that $I(V_2') = W \cup q'$, with $q' \subset \mathbb{P}^{4\prime}$ a quadric; the ruling of F, which is the projection from the original $\mathbb{P}^4$, then coincides with the projection from $\mathbb{P}^{4\prime}$. The fibres of the rational map $W \dashrightarrow \mathbb{P}^1$ constructed above are quadrics; if I show that $\overline{Y}$ is contained in a cubic not containing W then this rational map defines a G-equivariant pencil of curves of geometric genus ≤ 4 on $\overline{Y}$, contradicting the existence of a free automorphism of Y of degree 4. But one sees easily using the above model for W that

$$h^0(W, \mathcal{O}_W(3)) = 58,$$

$$\text{whereas} \quad h^0(\overline{Y}, \mathcal{O}_{\overline{Y}}(3)) \leq h^0(Y, 3K_Y) = 8 + 16\binom{3}{2} = 56.$$

Theorem 0.1 is proved. Amen.

Step 9

2 $|\pi_1 X| = 8$ and 9

2.1 Godeaux type examples

In this section I review Godeaux's construction of surfaces with $p_g = 0$, $K^2 = 2$ and $|G| = 8$; this gives rise to surfaces for which $G = \mathbb{Z}/8$, $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. I then give my own construction of surfaces with $G = Q_8$ (the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$) and $\mathbb{Z}_7$.

Let X be a surface with $p_g = 0$, $K^2 = 2$ and G an Abelian group with $|G| = 8$; the nonzero elements $g \in G^* \subset \operatorname{Pic} X$ give rise to seven sections $x_g \in H^0(K+g)$. It turns out that in all cases there are four linear dependence relations between the quadratic monomials in the x_g. Letting $\pi \colon Y \to X$ be the G-covering of X, the x_g $(= \pi^* x_g)$ form a basis of $H^0(Y, K_Y)$. One gets Godeaux's construction by assuming that $|K_Y|$ is without base points and defines an embedding of Y into $\mathbb{P}^6$ as a complete intersection of 4 quadrics.

$G = \mathbb{Z}/8$**:** for $i = 1, 2, \dots, 7$, I have $x_i \in H^0(K + i)$. The 4 quadrics come from linear relations between

$$\begin{aligned} H^0(2K) &: x_1x_7, x_2x_6, x_3x_5, x_4^2; \\ H^0(2K+2) &: x_3x_7, x_4x_6, x_1^2, x_5^2; \\ H^0(2K+4) &: x_1x_3, x_5x_7, x_2^2, x_6^2; \\ H^0(2K+6) &: x_1x_5, x_2x_4, x_3^2, x_7^2. \end{aligned}$$

$G = \mathbb{Z}/4 \oplus \mathbb{Z}/2$**:** the nonzero x_g are $x_{10}, x_{20}, x_{30}, x_{01}, x_{11}, x_{21}, x_{31}$ with $x_{ij} \in H^0(K + ij)$. The quadrics come from two linear relations between each of

$$\begin{aligned} H^0(2K) &: x_{10}x_{30}, x_{11}x_{31}, x_{20}^2, x_{01}^2, x_{21}^2; \\ H^0(2K+20) &: x_{01}x_{21}, x_{10}^2, x_{30}^2, x_{11}^2, x_{31}^2. \end{aligned}$$

$G = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$**:** there are seven x_g; since $2g = 0$ for all $g \in G$, the seven squares x_g^2 all belong to $H^0(2K)$. There are thus 4 linear dependence relations between these squares.

In all three cases it is trivial to give explicit examples of the 4 quadric relations chosen such that $Y \subset \mathbb{P}^6$ is nonsingular and disjoint from the fixed locuses of the action of G on $\mathbb{P}^6$; this gives a construction for X.

2.2 The quaternion group Q_8

Let X be a surface with $p_g = 0$ and $K^2 = 2$ and $f\colon Y \to X$ an etale Galois cover with Galois group $\mathrm{Gal}(Y/X) = Q_8$. Then by Theorem 0.1, (i), the canonical model of Y is an intersection of 4 quadrics in $\mathbb{P}^6$. Moreover, Corollary 5.3 tells us the representation of Q_8 on $H^0(K_Y)$, and hence on $\mathbb{P}^6$, and the representation on the vector space of quadrics defining Y; introduce the notation $A = kQ_8$ for the regular representation of Q_8 and A^+ for the complement of the trivial representation. Then Corollary 5.3 says that $H^0(K_Y) \cong A^+$, $H^0(2K_Y) = 3A$, and the quadrics are the kernel of

$$S^2H^0(K_Y) \to H^0(2K_Y),$$

that is, of a surjection $S^2A^+ \to 3A$. Write $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, and

$$A = 1 \oplus L_i \oplus L_j \oplus L_k \oplus H,$$

for its regular representation, where L_i, L_j, L_k are the three 1-dimensional representations on which the centre $\{\pm 1\}$ and one of i, j and k act trivially, and $H = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ the obvious (reducible) quaternion representation. Choose a basis $\{x_i, x_j, x_k, y_1, y_i, y_j, y_k\}$ for A^+; one sees easily that the invariant quadratic forms are based by the 7 expressions

$$x_i^2, x_j^2, x_k^2, y_1y_i + y_jy_k, y_1y_j + y_iy_k, y_1y_k + y_iy_j, y_1^2 + y_i^2 + y_j^2 + y_k^2,$$

and in fact $S^2A^+ = 7 \cdot 1 \oplus 3L_i \oplus 3L_j \oplus 3L_k \oplus 3H$. Choosing the four quadrics

$$x_i^2 + y_1y_i + y_jy_k, \quad x_j^2 + y_1y_j + y_iy_k, \quad x_k^2 + y_1y_k + y_iy_j, \quad y_1^2 + y_i^2 + y_j^2 + y_k^2,$$

one sees that these define a nonsingular surface Y on which G_8 acts freely.

2.3 The case $G = \mathbb{Z}/7$

I am now going to construct a surface with $p_g = 0$, $K^2 = 2$ and $G = \mathbb{Z}/7$; let X be such a surface, and for $i = 1, 2, \ldots, 6$, let $x_i \in H^0(K + i)$ be a nonzero section. In each of the $H^0(2K + i)$ (including $i = 0$) there are 3 quadratic monomials, which I will take to be linearly independent.

In each of the $H^0(3K + i)$ (including $i = 0$) there are 8 cubic monomials; I am going to assume that these span $H^0(3K + i)$, so that there is a single nontrivial relation R_i between them.

In each of the $H^0(4K+i)$ (including $i=0$) there are 18 quartic monomials; if we assume that these span $H^0(4K+i)$, then there are 5 linearly independent relations between them; however, there are already 6 relations x_jR_{i-j} (for $j=1,2,\dots,6$), and so there is a 2nd relation (syzygy) $S_i=\sum a_{ij}x_jR_{i-j}\equiv 0$ between these.

There is now a single 3rd relation which comes from considering $H^0(7K)$: this has dimension $P_7=1+2\binom{7}{2}=43$. In $H^0(7K)$ there are

114 septic monomials in the x_i

126 relations of the form (monomial in $H^0(4K+j)\cdot R_{-j}$

56 syzygies of the form (monomial in $H^0(3K+j)\cdot S_{-j}$

It follows that there is a single 3rd relation which can be written $\sum f_{-i}S_i$, with f_i a sum of the monomials in $H^0(3K+j)$.

The relations can be written compactly as follows: let $\mathfrak{S}$ be the matrix $(a_{ij}x_j)$. Then the relations take the form

$$\mathfrak{S}\begin{pmatrix}R_0\\R_6\\R_5\\R_4\\R_3\\R_2\\R_1\end{pmatrix}=0,$$

and

$$(f_0,f_6,f_5,f_4,f_3,f_2,f_1)\mathfrak{S}=0.$$

There are are no more relations; this can be checked by making use of the binomial coefficient identity

$$\binom{5+m}{5}-\binom{-2+m}{5}=7\binom{1+m}{4}+14\binom{m}{2}+7$$

Now to show how to get relations R_i, $\mathfrak{S}$ and f_i satisfying the above conditions. Let us take $\mathfrak{S}$ to be the 7×7 skewsymmetric matrix

$$\mathfrak{S}=\begin{pmatrix}0 & a_{01}x_1 & a_{02}x_2 & \dots & \\ -a_{01}x_1 & 0 & a_{12}x_3 & \dots & \\ & & & & \\ & \dots & & & \dots \\ & & & & \\ -\ \mathrm{sym} & & \dots & & \end{pmatrix}=0,$$

and

$$\operatorname{Adj}\mathfrak{S}=(A_iA_j)\quad \text{where } A_i=\text{Pfaffian of diagonal } 6\times 6 \text{ minor}$$

No proof of nonsingularity is offered here, but see [13], App. to Round 2.

3 Proof of Theorems 0.3 and 0.5

Suppose that X is as in Theorem 0.3, and that $\pi_1 X$ is infinite. I will prove (ii).

Let $Y \to X$ be an etale Galois cover such that $G = \mathrm{Gal}(Y/X)$ has $|G| = n$; then Y has the invariants

$$K_Y^2 = nK_X^2, \quad \chi(\mathcal{O}_Y) = n\chi(\mathcal{O}_X).$$

Since $K_X^2 \le 3\chi(\mathcal{O}_X) - 1$, I have

$$K_Y^2 \le 3p_g(Y) + 3 - 3q(Y) - n.$$

Exactly as in Section 1, Step 5, if $n \ge 11$ then $K_Y^2 < 3p_g - 7$, so that by Theorem **??**, (ii), φ_{K_Y} cannot be birational; suppose that $|K_Y|$ is not composed of a pencil. Then φ_{K_Y} is generically m-to-1 onto a surface $\overline{Y}$, and the standard argument as in Section 1, Step 2 gives

$$m \deg \overline{Y} \le K_Y^2.$$

Since $\overline{Y}$ is a surface spanning $\mathbb{P}^{p_g(Y)-1}$, for $n \ge 10$ I must have $m = 2$ and $\deg \overline{Y} < \frac{3}{2}(p_g(Y) - 1)$. Thus using Theorem 6.2 and Corollary 6.5, I find the following 3 possibilities for φ_{K_Y}:

(1) $|K_Y|$ is composed of a pencil;

(2) $\varphi_{K_Y} \dashrightarrow Y \to \overline{Y}$ is generically 2-to-1 onto a rational surface $\overline{Y}$;

(3) $\varphi_{K_Y} \dashrightarrow Y \to \overline{Y}$ is generically 2-to-1 onto a surface $\overline{Y}$ having an irrational ruling by lines and conics.

In (ii) there is a biregular involution i of Y such that the quotient $F = Y/i$ is rational. As in Section 1, Step 3, it follows from Lemma 4.2, (ii) that $G = (\mathbb{Z}/2)^a$. By Theorem 5.4 it follows that $(\mathbb{Z}/2)^a \subset \mathrm{Pic}\, X$.

Proposition 3.1 *There is a bound T (depending on X) such that $(\mathbb{Z}/2)^a \subset \mathrm{Pic}\, X$ implies that $a \le T$.*

Proof T can be taken as $2q(X)$ plus the number of generators for the Néron–Severi group (which is finitely generated).

Problem 3.2 Find an effective bound T in terms of $p_g(X)$, $q(X)$ and K_X^2.

Step 4 of Section 1 solves this problem in the simplest case $p_g = q = 0$, $K^2 = 2$; the most hopeful method to attack this problem would be to try to extend the method of Step 4 to quadrics which although not diagonal, consist of "small" diagonal blocks. However, I do not know how to do this even for $p_g = 0$, $K^2 = 3$, although it seems likely that in this case $T = 4$ is a bound.

Thus under the hypothesis that $\pi_1 X$ is infinite, there exist etale Galois covers $Y \to X$ for which Y falls under (i) or (iii) above.

Proposition 3.3 *Suppose that for some $Y \to X$ with $p_g \geq 8$ the canonical system $|K_Y|$ is composed of a pencil; then $\varphi_{K_Y}\colon Y \to C \subset \mathbb{P}^{p_g(Y)-1}$ is a morphism with irreducible generic fibres of genus* 2.

Proof Write

$$|K_Y| = |E^{(r)}| + F,$$

with F the fixed part and $|E^{(r)}| = E_1 + \cdots + E_r$, the E_i being irreducible fibres of a map $Y \dashrightarrow C$. Let E be the numerical class of E_i.

Then as in Section 1, Step 1, $K_Y^2 \geq r^2 E^2$, where furthermore $r \geq p_g(Y) - 1$, so that $E^2 = 0$; also

$$K_Y E \leq \frac{1}{r} K_Y^2 \leq \frac{3p_g(Y) + 3}{p_g(Y) - 1} < 4,$$

so that $K_Y E = 2$.

Thus φ_{K_Y} is obtained by composing the morphism $Y \to C$ with the rational map $C \to \mathbb{P}^{p_g(Y)-1}$ given by a linear system $|d|$ on C of degree r; if this map is not birational, then $r \geq 2p_g(Y) - 2$, again contradicting the numerical conditions. The proposition is proved.

I write $\varphi_{K_Y}\colon Y \dashrightarrow \overline{Y} \subset \mathbb{P}^{p_g(Y)-1}$ for the canonical rational map, regardless of the dimension of $\overline{Y}$.

Proposition 3.4 *Let $Y_2 \xrightarrow{\psi_{2,1}} Y_1 \to X$ be an etale tower, with $p_g(Y_1) \geq 8$; then there exists a rational map $\overline{Y}_2 \dashrightarrow \overline{Y}_1$ making the diagram*

$$\begin{array}{ccccc} Y_2 & \dashrightarrow & \overline{Y}_2 & \subset & \mathbb{P}^{p_g(Y_2)-1} \\ \downarrow & & \downarrow & & \\ Y_1 & \dashrightarrow & \overline{Y}_1 & \subset & \mathbb{P}^{p_g(Y_1)-1} \end{array}$$

commute. Furthermore

(i) if $\overline{Y}_2$ is a curve then so is $\overline{Y}_1$;

(ii) if $\overline{Y}_2$ is a surface ruled by lines then $\overline{Y}_1$ is either a curve, or a surface ruled by lines.

Proof The composite

$$Y_2 \to Y_1 \dashrightarrow \overline{Y}_1 \subset \mathbb{P}^{p_g(Y_1)-1}$$

is defined by the subspace $\psi_{2,1}^* H^0(K_{Y_1}) \subset H^0(K_{Y_2})$; hence the required map is defined by the linear projection from $\mathbb{P}^{p_g(Y_2)-1}$ onto $\mathbb{P}^{p_g(Y_1)-1}$. The final assertion is obvious.

Thus my surface X falls under one of the following 3 cases:

Case 0 There exists some etale Galois cover $Y_0 \to X$ such that $|K_Y|$ is composed of a pencil of curves of genus 2 for every etale cover $Y \to Y_0$.

Case i (for $i = 1$ or 2) There exists an etale Galois cover $Y_0 \to X$ such that for every etale cover $Y \to Y_0$,

$$\left\{ \begin{array}{l} \varphi_{K_Y} \colon Y \dashrightarrow \overline{Y} \subset \mathbb{P}^{p_g(Y)-1} \text{ is a double cover of a surface } \overline{Y} \\ \text{having an irrational pencil of rational curves of degree } i. \end{array} \right. \qquad (3.1)$$

It seems quite likely that Case 0 cannot occur; in any case it is easy to deal with:

Proposition 3.5 *In Case* 0 *there exists a morphism $f \colon X \to B$ inducing an isomorphism $f_* \colon \pi_1 X \xrightarrow{\cong} \pi_1 B$.*

Proof Y_0 has a G-equivariant pencil $Y_0 \to C_0$ of curves of genus 2; by Lemma 4.2, (i), G must act freely on C_0, so that the following diagram is a pullback:

$$\begin{array}{ccc} Y_0 & \longrightarrow & C_0 \\ \downarrow & & \downarrow \\ X = Y_0/G & \longrightarrow & B = C_0/G. \end{array}$$

Now for every etale cover $Y_1 \to Y_0$, $\varphi_{K_{Y_1}}$ defines a pencil $Y_1 \to C_1$, and by Proposition 3.4 there is a morphism $C_1 \to C_0$ compatible with the canonical maps of Y_1 and Y_0. But these maps then form a pullback diagram, and Y_1 can also be obtained as the pullback $X \times_B C_1$; since every etale cover $Y \to X$ fits into a Galois tower under some such Y_1, the proposition is proved.

In Cases 1 and 2 the irrational pencil on $\overline{Y}$ defines a pencil $Y \to C$ on Y.

Lemma 3.6 *In Case* 1 *the fibres of $Y \to C$ have genus ≤ 3; in Case* 2 *they have genus ≤ 5.*

Proof Since every fibre of $Y \to C$ goes into a rational curve of degree i, each fibre imposes at most $i+1$ conditions on a divisor in $|K_Y|$, and setting $r = \left[\frac{p_g-2}{i+1}\right]$, it follows that a divisor of $|K_Y|$ can be found containing r fibres of $Y \to C$; thus

$$3p_g(Y) + 3 - 3q(Y) - n \geq K_Y^2 \geq rK_Y E.$$

It then follows that $K_Y E \leq 4$ in Case 1, and $K_Y E \leq 8$ in Case 2, as required.

I can now make the further requirement on Y_0:

$$\left\{\begin{array}{l}\text{as in (3.1), and for some } g \text{, for every etale } Y \to Y_0, \\ \text{the irrational pencil on } \overline{Y} \text{ defines a pencil of curves} \\ \text{of genus } g \text{ on } Y.\end{array}\right. \tag{3.2}$$

Now consider the multiple fibres of $Y_0 \to C_0$; by making a pullback by a cover of C_0 ramified only in the points corresponding to the multiple fibres, I arrive at a cover Y_0 which satisfies in addition

$$\text{as in (3.2), and } Y_0 \to C_0 \text{ has no multiple fibres.} \tag{3.3}$$

Now let $Y_1 \to Y_0$ be an etale Galois cover; by Proposition 3.4, the canonical double covers of Y_0 and Y_1 fit into a commutative diagram

$$\begin{array}{ccc} Y_1 & \dashrightarrow & \overline{Y}_1 \\ \downarrow & & \downarrow \\ Y_0 & \dashrightarrow & \overline{Y}_0 \end{array}$$

on the other hand, both $\overline{Y}_i$ are ruled by lines or conics, so that the map $\overline{Y}_1 \to \overline{Y}_0$, defined by linear projection, induces a map $C_1 \to C_0$ between the curves parametrising the lines or conics of $\overline{Y}_1$ and $\overline{Y}_0$. I thus get a diagram

$$\begin{array}{ccccc} Y_1 & \dashrightarrow & \overline{Y}_1 & \dashrightarrow & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \dashrightarrow & \overline{Y}_1 & \dashrightarrow & C_1 \end{array}$$

Now since both $Y_i \to C_i$ have fibres of the same genus it follows that no element of $\mathrm{Gal}(Y_1/Y_0)$ acts trivially on C_1. Thus $C_1 \to C_0$ is also Galois, with $\mathrm{Gal}(Y_1/Y_0) = \mathrm{Gal}(C_1/C_0)$; under these circumstances it follows that Y_1 is birational to the pullback $Y_0 \times_{C_0} C_1$, and then since the pencil $Y_0 \to C_0$ is without multiple fibres it follows that $C_1 \to C_0$ is also etale. This proves Theorem 0.3, via the following assertion:

Theorem 3.7 *X has a cover Y_0 satisfying* (3.3) *above. For any such Y_0, $f\colon Y_0 \to C_0$ induces an isomorphism $f_*\colon \pi_1 Y_0 \xrightarrow{\cong} \pi_1 C_0$.*

Proof of Corollary 0.4, (ii) Since $Y_0 \to C_0$ induces an isomorphism on π_1, it also induces an isomorphism $f_*\colon \operatorname{Alb} Y_0 \stackrel{\sim}{\to} JC_0$, so that the Albanese map of Y_0 is just the composite of $Y_0 \to C_0$ with the embedding of C_0 into its Jacobian JC_0.

For X itself the Albanese map of X fits into a commutative diagram

$$\begin{array}{ccccc} Y_0 & \to & C_0 & \subset & \operatorname{Alb} Y_0 \\ \downarrow & & & & \downarrow \psi_* \\ \alpha\colon X & & \longrightarrow & & \operatorname{Alb} X, \end{array}$$

so that $\alpha(X) \subset \psi_*(C_0)$; this proves (ii).

Proof of Corollary 0.4, (iii) Given $f\colon X \to C$, consider the diagram

$$\begin{array}{ccc} \alpha\colon X & \longrightarrow & \operatorname{Alb} X, \\ \downarrow & & \downarrow f_* \\ C & \longrightarrow & JC; \end{array}$$

the image $\alpha(X)$ is a curve D which maps onto C under f_*; since f has irreducible fibres it follows that $D = C$, proving (iii).

Proof of Corollary 0.4, (ii) Given $f\colon X \to C$, consider the diagram

$$\begin{array}{ccc} X & \stackrel{\alpha}{\longrightarrow} & \operatorname{Alb} X \\ \downarrow & & \downarrow f_* \\ C & \longrightarrow & JC; \end{array}$$

the image $\alpha(X)$ is a curve D which maps onto C under f_*; since f has irreducible fibres it follows that $D = C$, proving (iii).

Proof of Theorem 0.5 In view of Corollary 0.4, (iii) and Theorem **??**, (ii), the only nontrivial assertion remaining to prove is that if $q(X) \neq 0$ then (i) and (iii) are impossible.

Let $Y_0 \to X$ be the cover as in Theorem 0.3; since $q(X) \neq 0$ the curve $C_0 \subset \operatorname{Alb} Y_0$ has a nontrivial map to the curve $\alpha(X) \subset \operatorname{Alb} X$, and this implies that C_0/G_0 has genus > 0.

The map $\overline{Y}_0 \dashrightarrow \overline{X}$ provided by Proposition 3.4 and the quotient map $X = Y_0/G_0 \to \overline{Y}_0/G_0$ fit into a diagram

$$\begin{array}{ccccc} Y_0 & & \dashrightarrow & & \overline{Y}_0 \subset \mathbb{P}^{p_g(Y_0)-1} \\ \downarrow & & & \swarrow & \downarrow \\ X & \to & \overline{Y}_0/G_0 & \dashrightarrow & \overline{X} \subset \mathbb{P}^{p_g(Y_0)-1} \end{array}$$

Now split into cases: if $\overline{Y}_0$ is a curve then it is birational to C_0; then $\overline{Y}_0/G_0$ is a non-rational curve, birational to $\overline{X}$ according to Proposition 3.3.

On the other hand, if $\overline{Y}_0$ is a surface then it is ruled over C_0, so that $\overline{Y}_0/G_0$ is ruled over C_0/G_0; if $X \dashrightarrow \overline{X}$ is generically 2-to-1 onto a surface then this surface is birational to $\overline{Y}_0/G_0$. If $\overline{X}$ is a curve then by Proposition 3.3 $X \dashrightarrow \overline{X}$ has irreducible fibres, so that the same holds for $\overline{Y}_0/G \dashrightarrow \overline{X}$. The ruled surface $\overline{Y}_0/G$ is birationally a product $\mathbb{P}^1 \times C$, with $C = C_0/G_0$ of genus $p > 0$. There are thus just two possibilities for the rational map $\overline{Y}_0/G \dashrightarrow \overline{X}$, the projection on the two factors. I am home if it is projection onto C. But the other case is impossible: for $\varphi\colon X \dashrightarrow \overline{Y}_0/G$ is a double cover, and φ composed with the first projection is birationally the canonical map of X, and thus by Proposition 3.3 has fibres curves of genus 2. On the other hand, φ composed with the second projection is the map $X \to C_0/G_0$ deduced from $Y_0 \to C_0$ by quotienting by G_0, and thus has fibres of genus ≤ 5. Thus the branch locus of the birational double cover $\varphi\colon X \dashrightarrow \mathbb{P}^1 \times C$ has degree ≤ 12 on the first factor, degree 2 on the second factor, and C has genus 1. It follows easily that then $p_g \leq 7$.

Part II
Technical digressions

4 The geometry of Galois covers

Let $\psi\colon Y \to X$ be an etale Galois cover with group $G = \mathrm{Gal}(Y/X)$ of order $n = |G|$. A linear system $|D|$ on Y is said to be *invariant* under G if

$$\sigma^* D \in |D| \quad \text{for all } D \in |D| \text{ and } \sigma \in G;$$

the obvious examples are the complete canonical system $|D| = |K_Y|$, or any system determined in a unique way from $|K_Y|$.

Lemma 4.1 *Let $|D|$ be G-invariant.*

(i) Suppose that $|D|$ is irreducible, and let Z be the scheme theoretic base locus of $|D|$; then $Z = \psi^* Z_X$ for some subscheme $Z_X \subset X$, and in particular

$$n \mid \deg Z.$$

(ii) Let $|D| = |D_m| + D_f$ be the decomposition of D into fixed a mobile parts; then $D_f = \psi^* E_X$ for some divisor E_X on X, and in particular

$$n \mid D_f^2, \quad n \mid K_Y D_f \quad \text{and} \quad 2n \mid D_f^2 + K_Y D_f.$$

Proof D_f is the divisor theoretic intersection of all $D \in |D|$ (that is, their greatest common divisor), and is hence invariant under G, and thus of the form $\psi^* E_X$; similarly, Z is the scheme theoretic intersection of all $D \in |D|$, and is thus also invariant under G. Q.E.D.

Lemma 4.2 *(i) Let $Y \to C$ be a G-equivariant morphism of a surface Y onto a normal curve C with connected fibres of genus g; suppose that the G-action on Y is fixed point free. Then for every $c \in C$, the order of the stabiliser group* $\operatorname{Stab}_G(c) = \{g \in G \mid gc = c\}$ *divides $g - 1$:*

$$|\operatorname{Stab}_G(c)| \mid g - 1$$

(ii) Let $\varphi\colon Y \to F$ be a finite morphism of Y onto a surface F, equivariant under a group action G which is fixed point free on Y. Then for every $x \in F$ the order of the stabiliser group of x, $\operatorname{Stab}_G(x)$ *has order not exceeding* $\deg \varphi$*:*

$$|\operatorname{Stab}_G(x)| \leq \deg \varphi.$$

Proof (i) Write $G_0 = \operatorname{Stab}_G(c)$, and $X_0 = Y/G_0$; since $G_0 \subset G$ acts freely on Y, the cover $\psi_0\colon Y \to X_0$ is etale of degree $|G_0|$. From the adjunction formula defining the arithmetic genus of a divisor, and from the fact that $K_Y = \psi_0^*(K_{X_0})$ if follows easily that for a divisor D on X_0, I have

$$p_a(\psi_0^* D) - 1 = |G_0| \cdot (p_a D - 1).$$

But on the other hand the fibre $\varphi^*(c)$ is invariant under G_0, and is thus of the form $\psi_0^* D$ for some divisor D on X_0. This proves (i). Note that if no element of G acts trivially on C then the quotient $Y/G = X$ has a pencil $X \to C/G$ of curves of genus g with a multiple fibre of multiplicity $|\operatorname{Stab}_G(c)|$ at c; this gives a simpler proof of (i) in this case.

(ii) $\operatorname{Stab}_G(x)$ acts freely on the inverse image $\varphi^{-1}(x)$, which is a set of points of cardinality $\leq \deg \varphi$ (see for example [14], p. 169).

Remark 4.3 It seems quite likely that (ii) can be strengthened to

$$|\operatorname{Stab}_G(x)| \mid \deg \varphi.$$

This holds for example if φ is either flat or Galois.

5 The algebra of etale Galois covers

I introduce some notation from elementary group representation theory. Let G be a finite group whose order $n = |G|$ is not divisible by the characteristic of k; as always, k is algebraically closed. Let $\rho_i \colon G \to \mathrm{GL}(W_i)$ for $i = 1, \dots, h$ be a fixed choice of a complete set of nonisomorphic irreducible representations of G; for each i, let $n_i = \deg \rho_i = \dim W_i$. Let kG be the group algebra of G; considered as a representation of G, kG is the *regular representation.* The following result is well known (see [16], I.2.4).

Proposition 5.1 (Schur's lemma) *(i) Let V be any finite dimensional representation of G; then there is a natural isomorphism of kG-modules*

$$V = \bigoplus_{i=1}^{h} V_i \otimes W_i, \quad \textit{where} \quad V_i = \mathrm{Hom}_{kG}(W_i, V). \tag{5.1}$$

In (5.1), W_i is the fixed set of irreducibles, and V_i is considered as a trivial G-module.

(ii) In the case of the regular representation kG, $\dim V_i = n_i$; in particular $n = \sum n_i^2$. □

Now let $\psi \colon Y \to X$ be an etale Galois cover, with group G. The normal basis theorem of Galois theory ([17], §67) tells us that the function field $k(Y)$ is the regular representation of G over $k(X)$, that is, $k(Y) \cong k(X) \otimes_k kG$ (as a kG-module); this is a result for the generic stalk of $\psi_* \mathcal{O}_Y$, which I will extend in Theorem 5.2 to $\psi_* \mathcal{O}_Y$ itself. For the present, note that $\psi_* \mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-algebra of rank $n = \deg \psi$, and that the action of G makes it into a module over the sheaf of algebras $\mathcal{O}_X[G] = \mathcal{O}_X \otimes_k kG$.

If $\mathcal{L} \in \mathrm{Pic}\, X$ then G acts on the cohomology groups

$$H^j(Y, \psi^*\mathcal{L}) = H^j(X, \psi_*\psi^*\mathcal{L}) = H^j(X, \psi_*\mathcal{O}_Y \otimes \mathcal{L}).$$

I can consider the Euler characteristic $\chi(Y, \psi^*\mathcal{L}) = \sum (-1)^j H^j(Y, \psi^*\mathcal{L})$ as a virtual representation of G. The knowledge of its class $\chi(Y, \psi^*\mathcal{L}) \in R_k(G)$ in the representation ring of G is equivalent to knowing, for each irreducible representation W_i, the alternating sum of the number of times W_i appears in $H^j(Y, \psi^*\mathcal{L})$.

Theorem 5.2 (normal basis theorem) *Let X be a scheme of finite type over k, and $\psi \colon Y \to X$ an etale Galois cover. Then*

(i) $\psi_* \mathcal{O}_Y$ is locally free of rank 1 as an $\mathcal{O}_X[G]$-module;

(ii) $\psi_* \mathcal{O}_Y$ *has a natural decomposition*

$$\psi_* \mathcal{O}_Y = \bigoplus_{i=1}^{h} \mathcal{E}_i \otimes_k W_i; \tag{5.2}$$

where $\mathcal{E}_i = \operatorname{Hom}_{\mathcal{O}_X[G]}(\mathcal{O}_X \otimes_k W_i, \psi_* \mathcal{O}_Y)$. *Each* $\mathcal{E}_i$ *is a locally free sheaf of rank* $n_i = \dim W_i$ *on which* G *acts trivially; and each* $\mathcal{E}_i$ *becomes trivial on lifting to* Y, *that is,* $\psi^* \mathcal{E}_i \cong \mathcal{O}_Y^{n_i}$.

(iii) Suppose that X *is complete, so that* $H^i(X, \mathcal{F})$ *is finite dimensional for any coherent* $\mathcal{F}$*; then in* $R_k(G)$ *we have*

$$\chi(Y, \psi^* \mathcal{L}) = \chi(X, \mathcal{L}) \otimes kG \in R_k(G).$$

Corollary 5.3 *(i) Suppose that* $H^i(Y, \psi^* \mathcal{L}) = 0$ *for all* $i > 0$*; then as* G*-modules,* $H^0(Y, \psi^* \mathcal{L}) \cong H^0(X, \mathcal{L}) \otimes kG$. *In particualr if* X *is a minimal surface of general type, and* $\mathcal{L} = \mathcal{O}_X(nK_X)$ *with* $n \geq 2$ *then* $H^0(Y, \mathcal{O}_Y(nK_Y)) \cong H^0(X, \mathcal{O}_X(nK_X)) \otimes kG$.

(ii) If X *is a minimal surface of general type, and* $H^1(Y, \mathcal{O}_Y) = 0$ *then* $H^0(Y, \mathcal{O}_Y(K_Y)) \oplus k \cong H^0(X, \mathcal{O}_X(K_X)) \otimes kG$, *where* G *acts trivially on the factor* k *on the left-hand side.*

The corollary is an immediate consequence of (iii) of the theorem. For (i) one must observe that $\mathcal{O}_Y(K_Y) = \psi^*(\mathcal{O}_X(K_X))$, and that $H^i(\mathcal{O}_Y(nK_Y)) = 0$ for $n \geq 2$ by the Kodaira–Ramanujam vanishing theorem. For (ii),

$$H^1(Y, \mathcal{O}_Y(K_Y)) \overset{\mathrm{d}}{=} H^1(Y, \mathcal{O}_Y) = 0;$$

on the other hand $H^2(Y, \mathcal{O}_Y(K_Y)) \overset{\mathrm{d}}{=} H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X) = k$, on which G acts trivially.

Proof of Theorem 5.2 For $x \in X$, let $\mathcal{O}_{Y,x}$ denote the semilocal ring of rational functions on Y regular at each of the n points of $\psi^{-1}x$: $\mathcal{O}_{Y,x} = \psi_* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_{X,x}$ is the stalk of $\mathcal{O}_Y$ at x. Now $\mathcal{O}_{Y,x}/m_x \mathcal{O}_{Y,x}$ is naturally the ring of k-valued functions on the n points of $\psi^{-1}x$; if $\overline{a} \in \mathcal{O}_{Y,x}$ is the function that takes the value 1 at one of these, and 0 at all the others, then the set $\{g\overline{a}\}$ as g runs through G, forms a basis of $\mathcal{O}_{Y,x}/m_x \mathcal{O}_{Y,x}$. Now if $a \in \mathcal{O}_{Y,x}$ is an element that reduces to $\overline{a}$ modulo m_x, then the G-orbit $\{ga\}$ forms a basis of $\mathcal{O}_{Y,x}$ over $\mathcal{O}_{X,x}$ by Nakayama's lemma. This proves (i).

The first part of (ii), namely the splitting (5.2), is now obvious from Proposition 5.1. Standard considerations in Galois theory provide a splitting of the algebra $k(Y) \otimes_{k(X)} k(Y)$ (with G acting on the second factor) into the Cartesian product of n copies of $k(Y)$ permuted by G (see [17], §67).

Hence the Galois cover $Y \times_X Y \to Y$ splits as n isomorphic copies of Y. It follows that $\psi^*\psi_*\mathcal{O}_Y \cong \mathcal{O}_Y \otimes_k kG$; the subsheaf $\psi^*(\mathcal{E}_i \otimes W_i) \subset \psi^*\psi_*\mathcal{O}_Y$ must then coincide with the piece $\mathcal{O}_Y \otimes_k V_i \otimes W_i$ of $\mathcal{O}_Y \otimes_k kG$ belonging to the representation W_i, and hence $\psi^*\mathcal{E}_i \cong \mathcal{O}_Y \otimes_k V_i$. This proves (ii).

(iii) is very easy from (ii) and the Riemann–Roch theorem: by (ii), I have $\chi(\psi_*\mathcal{O}_Y \otimes \mathcal{L}) = \sum_i \chi(\mathcal{E}_i \otimes \mathcal{L}) \otimes W_i$; and by Riemann–Roch

$$\chi(\mathcal{E}_i \otimes \mathcal{L}) = \varkappa_2\big(\mathrm{ch}(\mathcal{E}_i \otimes \mathcal{L}) \cdot \mathrm{Td}(X)\big)[X].$$

However, since $\psi^*\mathcal{E}_i$ is trivial, the Chern classes of $\mathcal{E}_i$ all vanish, so that the right-hand side has the same value as if $\mathcal{E}_i$ were the trivial sheaf of rank n_i, that is, $n_i\chi(\mathcal{L})$. Thus

$$\chi(\psi_*\mathcal{O}_Y \otimes \mathcal{L}) = \sum n_i W_i \otimes \chi(\mathcal{L}). \quad \text{Q.E.D.}$$

In the Abelian case, all the $\mathcal{E}_i$ have rank 1, and one can easily see that the multiplicative structure of $\psi_*\mathcal{O}_Y$ is given by isomorphisms $\mathcal{E}_i \otimes \mathcal{E}_j \xrightarrow{\cong} \mathcal{E}_{i+j}$; the subscript $i+j$ makes sense, since the $\mathcal{E}_i$ correspond to characters of G, that is, to elements of the dual group $G^* = \mathrm{Hom}(G, k^*)$. This formalism also works in any characteristic:

Theorem 5.4 *Let X be a scheme of finite type over k, and let G be a commutative group scheme over k; then there is a natural bijection*

$$\big\{G\text{-torsors } Y \to X\big\} \leftrightarrow \big\{\text{homomorphisms } G^* \to \mathrm{Pic}\, X.\big\}$$

The straightforward proof is omitted.

Problem 5.5 Let G be a group of transformations of a nonsingular minimal surface of general type Y, and suppose that the action has finitely many elliptic points $P \in Y$ where $\mathrm{Stab}\, P = G_P \subset \mathrm{SL}(T_P Y)$. Then $Y/G = X$ is a surface with only Du Val singularities, and $\psi^* K_X = K_Y$. The problem is to determine the representations of G on $H^0(nK_Y)$ in terms of $p_g(X)$, K_X^2 and the subgroups $G_P \subset \mathrm{SL}(2, k)$.

Example 5.6 ($G = \mathbb{Z}/2$) Suppose that the involution $\sigma \in G$ has μ fixed points, so that the quotient $X = Y/G$ has μ nodes. Then $\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{L}$, where $\mathcal{L}$ is a divisorial sheaf (invertible outside the nodes), and $\mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_X$ is an isomorphism outside the nodes. Then

$$H^0(nK_Y) = H^0(nK_X) \oplus H^0(\mathcal{L} \otimes \mathcal{O}_X(nK_X)),$$

and

$$\dim H^0(nK_X) = \chi(\mathcal{O}_X) + K_X^2\binom{n}{2},$$
$$\dim H^0(nK_X \otimes \mathcal{L}) = \chi(\mathcal{O}_X) + K_X^2\binom{n}{2} - \frac{\mu}{4}.$$

Note that $K_X^2 = \frac{1}{2}K_Y^2$; but $p_g(X)$ is not in general determined by Y only. If $q(Y) = 0$ then

$$p_g = 1 + 2p_g(X) - \frac{\mu}{4},$$
$$c_2(Y) = 2c_2(\widetilde{X}) - 3\mu$$

(where $\widetilde{X}$ denotes the resolution).

6 Nagata's method for surfaces of low degree

Let me start by giving a convenient statement of a classical result on linear system on curves.

Theorem 6.1 (Clifford's theorem + Riemann–Roch) *Let C be a curve of genus g, and let $|d|$ be a linear system on C of degree d and projective dimension n; then either*

(i) $|d|$ is special, that is, $h^1(\mathcal{O}_C(d)) \neq 0$; then

$$d \geq 2n \quad \textit{and} \quad g - 1 \geq n.$$

or

(ii) $|d|$ is nonspecial, that is, $h^1(\mathcal{O}_C(d)) = 0$; then

$$n + 1 \leq h^0(\mathcal{O}_C(d)) = 1 - g + d$$

Theorem 6.2 *Let $F \subset \mathbb{P}^n$ be an irreducible surface spanning $\mathbb{P}^n$, and of degree $d \leq 2n - 2$; then F is birational either to a ruled surface, or to a K3 surface, and in the latter case $d = 2n - 2$.*

Proof Let $F_1 \to F \subset \mathbb{P}^n$ be a resolution of the singularities of F, and $|H|$ the linear system on F_1 defining the map to $\mathbb{P}^n$; let $C \in |H|$ be an irreducible nonsingular curve, and g its genus.

Consider the characteristic system $|C|_C$; according to the above, either

(i) $d \geq 2n-2$ and $g-1 \geq n-1$; or

(ii) $n \leq 1-g+d$.

Now if F_1 is not ruled, I can find a divisor $D \geq 0$ such that $D \sim rK_{F_1}$, with $r > 0$; the adjunction formula for C gives

$$2g-2 = C^2 + CK_{F_1} = d + \frac{1}{r}CD \geq d.$$

(ii) is now impossible, since $2n \leq 2-2g+2d \leq d$ contradicts the hypothesis $d \leq 2n-2$.

(i) gives $d = 2n-2$ and $g = n$, so that the hyperplane sections of F are canonically embedded curves. If the resolution $F_1 \to F$ is chosen to be relatively minimal (that is, no fibre contains exceptional curves of the first kind) then it is easy to see that $D = 0$, and F is a K3 surface.

Remark 6.3 I have proved on the way that if $d \leq 2n-3$ then

$$g \leq 1+d-n. \tag{6.1}$$

Now suppose that F_1 is a ruled surface of genus $p > 0$, so that F has a unique ruling by irreducible rational curves.

Theorem 6.4 (Nagata [18]) *Let $F \subset \mathbb{P}^n$ be a ruled surface of genus $p > 0$; suppose that F is ruled by rational curves of degree σ. Let g denote the geometric genus of the general hyperplane section of F. Then*

$$2g-2 \geq \sigma(2p-2) + \left(1 - \frac{1}{\sigma}\right) d. \tag{6.2}$$

Proof A nonsingular minimal model of F is a ruled surface $F_2 = \mathbb{P}_B(E)$, with $\pi\colon \mathbb{P}_B(E) \to B$ a $\mathbb{P}^1$-fibre bundle over a curve B of genus p. The birational map $F_2 \dashrightarrow F \subset \mathbb{P}^n$ is given by an irreducible system L on F_2, with base points O_i of multiplicity m_i.

The Néron–Severi group of F_2 has generators H (a divisor corresponding to the tautological line bundle $\mathcal{O}_{\mathbb{P}_B(E)}(1)$) and A (a fibre of $\mathbb{P}_B(E) \to B$), with intersection numbers

$$H^2 = c_1(E), \quad HA = 1, \quad A^2 = 0.$$

Furthermore, the canonical divisor class of $\mathbb{P}_B(E)$ is given by

$$K_{\mathbb{P}_B(E)} \sim -2H + \pi^*(\det E + K_B).$$

By hypothesis, L is contained in a complete linear system $L \subset |D|$, with D numerically equivalent to $\sigma H + rA$. Then

$$D^2 = \sigma^2 c_1(E) + 2\sigma r,$$

and

$$2p_a D - 2 = KD + D^2 = (\sigma^2 - \sigma)c_1(E) + 2(\sigma - 1)r + \sigma(2p - 2).$$

Taking account of the effect of the fixed points on the degree and genus, I get

$$d = \deg L = \sigma^2 c_2(E) + 2\sigma r - \sum_i m_i^2 \tag{i}$$

and

$$2g - 2 = (\sigma^2 - \sigma)c_1(E) + 2(\sigma - 1)r + \sigma(2p - 2) - 2\sum \binom{m_i}{2}. \tag{ii}$$

Subtracting $(1 - 1/\sigma)$ times (ii) from (i) then gives

$$2g - 2 - \left(1 - \frac{1}{\sigma}\right) d = \sigma(2p - 2) + \sum m_i \left(1 - \frac{m_i}{\sigma}\right).$$

Since $0 \le m_i \le \sigma$ for each i, each term under the summation is positive, and the theorem follows.

(6.1) and (6.2) impose rather strong conditions on a nonrational surface F of degree $d \le 2n - 3$.

Corollary 6.5 *If F is as in Theorem 6.4 with degree $d \le 2n - 3$ then*

$$d \ge 2n \left(1 - \frac{1}{1 + \sigma}\right).$$

In particular, if F is not *ruled by lines then $d \ge \frac{4}{3}n$, and if F is not ruled by lines or conics then $d \ge \frac{3}{2}n$.*

7 Counting quadrics

(For this section and the next, see my L'Aquila paper.)

8 Castelnuovo's inequalities for surfaces of general type

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