Chapter 2

Explicit construction of G-Hilb $\mathbb{C}^2$ for binary dihedral groups.

2.1 Representation theory

Throughout this chapter $G = \mathrm{BD}_{4n} \subset \mathrm{SL}(2,\mathbb{C})$ is the binary dihedral group of order $4n$ generated by

$$g = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for ε a primitive $2n$th root of unity. The group has the presentation

$$\left\langle g, \tau \;\middle|\; g^{2n} = \tau^4 = 1,\ g^n = \tau^2,\ \tau g \tau^{-1} = g^{-1} \right\rangle.$$

It is well known that the ring of invariants is generated by

$$u = xy\left(x^{2n} - y^{2n}\right), \quad v = x^{2n} + y^{2n} \quad \text{and} \quad w = (xy)^2$$

where x, y are the coordinates on $\mathbb{C}^2$. See [Rei02] for example. The quotient $X = \mathbb{C}^2/G$ is the hypersurface singularity

$$X\colon (u^2 - v^2w + 4w^{n+1} = 0) \subset \mathbb{C}^3.$$

This has a unique minimal resolution $Y \to X$ with exceptional locus given by a chain of $\mathbb{P}^1$s shown by the Dynkin diagram D_{n+2}.

Let $\operatorname{Irr} G$ be the set of distinct irreducible representations of G. These are given as follows. If ρ is a 1 dimensional representation of the group then $\rho(g)^2 = 1$ and $\rho(\tau)^2 = \rho(g)^n$. Thus there are four 1 dimensional representations given by

$$L_\pm(g) = 1, \quad L_\pm(\tau) = \pm 1 \quad \text{and} \quad M_\pm(g) = -1, \quad M_\pm(\tau) = \pm i^n. \qquad (2.1)$$

Clearly L_+ is the trivial representation.

For $j = 1, \ldots, n-1$ define a 2 dimensional representation $V_j \cong \mathbb{C}^2$ as follows.

$$V_j(g) = \begin{pmatrix} \varepsilon^j & 0 \\ 0 & \varepsilon^{-j} \end{pmatrix} \quad \text{and} \quad V_j(\tau) = \begin{pmatrix} 0 & 1 \\ (-1)^j & 0 \end{pmatrix} \qquad (2.2)$$

Each of these representations is irreducible and $V_i \ncong V_j$ for $i \neq j$. Since $4n = |G| = \sum_{\rho \in \operatorname{Irr} G} (\deg \rho)^2 = 4 \cdot 1 + (n-1) \cdot 4$ these are all the irreducible representations and the regular representation of BD_{4n} is

$$\mathbb{C}[G] \;\cong\; L_+ \oplus L_- \oplus M_+ \oplus M_- \oplus \bigoplus_{j=1}^{n-1} V_j^{\oplus 2}.$$

2.2 G-graphs

Definition 2.1 A polynomial or pair of polynomials is said to belong to one of the irreducible representations of G if one of the following conditions hold.

- A polynomial $f \in \mathbb{C}[x, y]$ *belongs to* $L_\pm$ if $g(f) = f$ and $\tau(f) = \pm f$.

- A polynomial $f \in \mathbb{C}[x,y]$ *belongs to* $M_\pm$ if $g(f) = -f$ and $\tau(f) = \pm i^n f$.

- For a polynomial $f \in \mathbb{C}[x,y]$, the pair of polynomials $(f, \tau(f))$ *belong to* V_j if $g\,(f, \tau(f)) = (\varepsilon^j f, \varepsilon^{-j} \tau\,(f))$.

In the next section we will be interested in the following polynomials that belong to these representations.

L_+	$1,\ x^2y^2, x^{2n} + y^{2n}$
L_-	$xy,\ x^{2n} - y^{2n}$
M_+	$x^n - i^n y^n,\ xy\,(x^n + i^n y^n)$
M_-	$x^n + i^n y^n,\ xy\,(x^n - i^n y^n)$
V_j	$(x^j, y^j)\,,\ (x^{j+1}y, -xy^{j+1})\,,$ $(y^{2n-j}, (-1)^j x^{2n-j})$

We may sometimes abuse this notation and say $(f,g) \in V_j$ when we mean $(f,-g) \in V_j$. For example we say $(x^{j+1}y, xy^{j+1}) \in V_j$ where really $(x^{j+1}y, -xy^{j+1}) \in V_j$ or $(y^{2n-j}, x^{2n-j}) \in V_j$ where really $(y^{2n-j}, (-1)^j x^{2n-j}) \in V_j$. Notice that $(y\,(x^n \pm i^n y^n)\,, x\,(x^n \pm i^n y^n)) \in V_{n-1}$.

Definition 2.2 A subset $\Gamma \subset \mathbb{C}[x,y]$ is called a *G-graph* if it contains precisely one polynomial belonging to each 1 dimensional representation $L_\pm$ and $M_\pm$ and two pairs of polynomials belonging to each of the irreducible 2 dimensional representations V_i.

An ideal I in $\mathbb{C}[x,y]$ defines a G-cluster if and only if $\mathbb{C}[x,y]/I \cong \mathbb{C}[G]$. Therefore we can choose a basis for the complex vector space $\mathbb{C}[x,y]/I$ which is a G-graph. We note that there will never be a unique choice of G-graph.

Definition 2.3 Let I be an ideal that defines a G-cluster. $\Gamma(I)$ is defined to be the set of G-graphs which form a basis for the complex vector space $\mathbb{C}[x,y]/I$.

Recall that $\mathrm{Soc}(I)$ is defined as the socle of $\mathbb{C}[x,y]/I$ viewed as a module over $\mathbb{C}[x,y]$ localised at the origin.

Theorem 2.4 *Let $G = \mathrm{BD}_{4n}$.*

1. *If an ideal I defines a G-cluster then one of the G-graphs given in the second list in Theorem 2.10 belongs to $\Gamma(I)$.*

2. $\mathrm{Soc}(I)$ *contains at most two irreducible representations of G.*

3. G-Hilb $\mathbb{C}^2$ *is a smooth rational variety.*

To show that G-Hilb $\mathbb{C}^2$ is a smooth rational variety I find a cover of G-Hilb $\mathbb{C}^2$ by smooth affines. In general these affine sets are smooth surfaces in some $\mathbb{C}^n$. However we find that G-Hilb $\mathbb{C}^2$ contains an open set isomorphic to an open subset of $\mathbb{C}^2$ therefore G-Hilb $\mathbb{C}^2$ is rational.

2.3 Preliminaries on G-clusters and G-graphs

Let I be an ideal that defines a G-cluster. Any 1 dimensional representation has multiplicity 1 in $\mathbb{C}[G]$. Thus for any polynomials g and h that belong to the same 1 dimensional representation there exists a polynomial

$$\alpha g - \beta h$$

in I for $\alpha, \beta \in \mathbb{C}$. If $g \in \Gamma$ then $\beta \neq 0$.

Let h be a polynomial belonging to a 1 dimensional representation. Provided that h is not zero in $\mathbb{C}[x,y]/I$ it can be chosen to be basic. In other words the only reason we cannot choose h in Γ is if $h \in I$. In particular if $h = 1$ then 1 can always be chosen to be in Γ. This means that an ideal that defines a G-cluster always contains polynomials of the form

$$A := x^{2n} + y^{2n} - a \quad Q := x^2y^2 - q$$

for some $a, q \in \mathbb{C}$. For any $r \in \mathbb{Z}$ let

$$q_r := \left(x^2y^2\right)^{r-1} + q\left(x^2y^2\right)^{r-2} + \cdots + q^{r-2}\left(x^2y^2\right) + q^{r-1}. \tag{2.3}$$

Then

$$q_rQ = \left(x^2y^2\right)^r - q^r.$$

Lemma 2.5 *The submodule of $\mathbb{C}[x,y]$ consisting of polynomials belonging to L_- is generated by xy and $x^{2n} - y^{2n}$ as a $\mathbb{C}[x,y]^G$-module.*

Proof. First note that this submodule is contained in $\mathbb{C}[x,y]^A$ where $A \cong \mathbb{Z}_{2n}$ is the normal subgroup generated by g. The ring $\mathbb{C}[x,y]^A$ is generated by x^{2n}, xy, y^{2n}. If $f(x,y) \in L_-$ and $xy \mid f(x,y)$ then $f(x,y) = xyf'(x,y)$ for some $f'(x,y) \in \mathbb{C}[x,y]^G$.

Therefore we may assume that $f(x,y)$ has no monomial term which is divisible by xy and so $f(x,y) = P_1(x^{2n}) + P_2(y^{2n})$ where the P_i are polynomials in one variable. Therefore $\tau(f) = P_1(y^{2n}) + P_2(x^{2n})$. Since $f \in L_-$ it must be that $\tau(f) = -f$ so $P_1 = -P_2$ and $f = P_1(x^{2n}) - P_1(y^{2n})$. This implies that $(x^{2n} - y^{2n}) \mid f$ and so $f = (x^{2n} - y^{2n}) f'(x,y)$ for some $f'(x,y) \in \mathbb{C}[x,y]^G$. $\square$

Corollary 2.6 *If neither xy nor $x^{2n} - y^{2n}$ can be chosen to be basic then $\mathbb{C}[x,y]/I$ contains no copy of L_-.*

Proof. If neither xy nor $x^{2n} - y^{2n}$ can be chosen to be basic then

$$xy, x^{2n} - y^{2n} \in I.$$

Let f be any polynomial belonging to L_-. By Lemma 2.5

$$f = xyf' + \left(x^{2n} - y^{2n}\right) f''$$

for $f', f'' \in \mathbb{C}[x,y]^G$. Therefore $f \in I$. □

Lemma 2.7 *Let I be an ideal containing the polynomials A and Q. If neither $x^n + i^n y^n$ nor $xy(x^n - i^n y^n)$ is basic then $\mathbb{C}[x,y]/I$ contains no copy of M_-. If neither $x^n - i^n y^n$ nor $xy(x^n + i^n y^n)$ is basic then $\mathbb{C}[x,y]/I$ contains no copy of M_+.*

Proof. We prove the first assertion but the second is analogous. If $x^n + i^n y^n$ cannot be basic then $x^n + i^n y^n \in I$ and so

$$x^2(x^n + i^n y^n) - i^n y^{n-2} Q = x^{n+2} - i^n q y^{n-2} \in I.$$

Therefore only polynomials that have monomial terms of the form $x^\alpha y^\beta$ can be basic where $\alpha < n+2$ and $\beta < 2$ or $\beta < n+2$ and $\alpha < 2$. The only polynomials of this form that belong to M_- are $x^n + i^n y^n$ and $xy(x^n - i^n y^n)$. These polynomials are both contained in I so $\mathbb{C}[x,y]/I$ contains no copy of M_-. □

Suppose (f_1, f_2), (g_1, g_2) and (h_1, h_2) are pairs of polynomials that belong to some irreducible 2 dimensional representation V_i. Since V_i has multiplicity 2 in $\mathbb{C}[G]$ these three pairs cannot be linearly independent in $\mathbb{C}[x,y]/I \cong \mathbb{C}[G]$. Therefore there exist polynomials of the form

$$\alpha f_1 + \beta g_1 + \gamma h_1, \quad \alpha f_2 + \beta g_2 + \gamma h_2$$

in I for $\alpha, \beta, \gamma \in \mathbb{C}$. If (f_1, f_2) and (g_1, g_2) are chosen in a G-graph for I then γ is nonzero.

Lemma 2.8 *Let I be an ideal that defines a G-cluster. Then there exists a G-graph $\Gamma \in \Gamma(I)$ containing the set $\{1, (x, y), \ldots, (x^{n-1}, y^{n-1})\}$.*

Proof. The G-graph Γ must contain $4n$ elements. If one of the pairs (x^i, y^i) for $0 < i < n$ cannot be chosen to be in Γ then both x^i and y^i belong to I. Therefore $x^j, y^j \in I$ for all $j \geq i$ and so Γ is contained in the set

$$\left\{1, x, \ldots, x^{i-1}, y, \ldots, y^{i-1}, xy, \ldots, x^{i-1}y, xy^2, \ldots, xy^{i-1}\right\}.$$

This has only $4(i-1) < 4n$ elements. Therefore the pairs (x^i, y^i) can always be chosen to be in Γ for $i = 1, \ldots, n-1$. $\square$

We therefore consider only G-graphs containing these monomials.

Suppose $\{1, (x, y), \ldots, (x^{n-1}, y^{n-1})\} \subset \Gamma$ and $(h_1, h_2) \in V_i$. The only reason that this pair cannot be chosen to be in Γ is if there exist polynomials of the form

$$h_1 - \alpha x^i, \quad h_2 - \alpha y^i$$

in I.

Lemma 2.9 *Let I be an ideal containing the polynomials A and Q. Suppose neither of the pairs $\left(x^{k+1}y, xy^{k+1}\right)$ nor $\left(y^{2n-k}, x^{2n-k}\right)$ can be chosen as part of the basis of $\mathbb{C}[x, y]/I$. Then $\mathbb{C}[x, y]/I$ contains at most one copy of V_k.*

Proof. If neither of the pairs $\left(x^{k+1}y, xy^{k+1}\right)$ nor $\left(y^{2n-k}, x^{2n-k}\right)$ can be chosen as part of the basis of $\mathbb{C}[x, y]/I$ then the four polynomials

$$\begin{aligned} C &:= x^{k+1}y - cx^k, \quad D := xy^{k+1} + cy^k, \\ E &:= y^{2n-k} - dx^k, \quad F := x^{2n-k} - (-1)^k dy^k \end{aligned}$$

belong to I.

Suppose that there exists a pair $\left(x^\alpha y^\beta, x^\beta y^\alpha\right)$ in $\mathbb{C}[x, y]/I$ which belongs to V_k. We want to show that this pair cannot be linearly independent of

$\left(x^k, y^k\right)$ in $\mathbb{C}[x,y]/I$. In other words that there exist polynomials of the form

$$x^\alpha y^\beta - \gamma x^k, \quad x^\beta y^\alpha - (-1)^\beta y^k$$

in I.

Suppose $\alpha \geq 2$ and $\beta \geq 2$. The polynomial $Q \in I$ and so $x^{\alpha-2}y^{\beta-2}Q \in I$ and

$$x^{\alpha-2}y^{\beta-2}Q = x^\alpha y^\beta - qx^{\alpha-2}y^{\beta-2} \in I.$$

So $x^\alpha y^\beta$ is equal to a complex multiple of $x^{\alpha-2}y^{\beta-2}$ in $\mathbb{C}[x,y]/I$ so we may replace $x^\alpha y^\beta$ with $x^{\alpha-2}y^{\beta-2}$ in the basis. Continuing this way we may assume that either $\alpha < 2$ or $\beta < 2$. Assume that $\beta < 2$.

Again if $\alpha > 2n+1$ then the polynomial $A \in I$ can be used to replace $x^\alpha y^\beta$ with a polynomial having monomial terms of the form $x^\gamma y^\delta$ where $\gamma \leq 2n+1$ and $\delta < 2$ or $\gamma < 2$ and $\delta \leq 2n+1$.

More precisely

$$x^{\alpha-2n}y^\beta A = x^\alpha y^\beta + x^{\alpha-2n}y^{\beta+2n} - ax^{\alpha-2n}y^\beta \in I.$$

If $\alpha - 2n \geq 2$ then repeated use of the polynomial Q replaces $x^{\alpha-2n}y^{\beta+2n}$ with a monomial of the form $x^\gamma y^\delta$ where $\gamma < 2$ or $\delta < 2$. Therefore we may assume that $\alpha \leq 2n+1$ and $\beta < 2$ or $\alpha < 2$ and $\beta \leq 2n+1$.

However the only pairs $\left(x^\alpha y^\beta, y^\alpha x^\beta\right)$ in V_k where $\alpha \leq 2n+1$ and $\beta < 2$ or $\alpha < 2$ and $\beta \leq 2n+1$ are

$$\begin{gathered} \left(x^k, y^k\right), \qquad \left(x^{k+1}y, xy^{k+1}\right), \\ \left(y^{2n-k}, x^{2n-k}\right), \quad \left(xy^{2n-k+1}, x^{2n-k+1}y\right), \\ (x^{2n+1}, y^{2n+1}) \in V_1. \end{gathered}$$

Now we are assuming that $\left(x^{k+1}y, xy^{k+1}\right)$ and $\left(y^{2n-k}, x^{2n-k}\right)$ are not basic so the only other possibilities are $\left(xy^{2n-k+1}, x^{2n-k+1}y\right)$ and (x^{2n+1}, y^{2n+1}).

However

$$\left.\begin{aligned} xyE + dC &= xy^{2n-k+1} - cdx^k, \\ xyF + (-1)^i dD &= x^{2n-k+1}y + (-1)^i cdy^k \end{aligned}\right\} \in I$$

so $\left(xy^{2n-k+1}, x^{2n-k+1}y\right)$ cannot be basic. If k=1 then

$$\left.\begin{aligned} x^2F - dC &= x^{2n+1} + cdx \\ y^2E + dD &= y^{2n+1} + cdy \end{aligned}\right\} \in I$$

so (x^{2n+1}, y^{2n+1}) cannot be basic either. Therefore $\mathbb{C}[x,y]/I$ contains at most one copy of V_k. $\square$

2.4 List of possible G-graphs

In this section we construct a list of G-graphs and show that every ideal that defines a G-cluster has a G-graph belonging to this list. We do this systematically by considering each representation and choosing a polynomial or pair of polynomials belonging to this representation.

Let I be an ideal that defines a G-cluster and Γ a G-graph in $\Gamma(I)$. Begin by considering L_+. By Lemma 2.8 we may always assume that $1 \in L_+$ belongs to Γ. Now $\mathbb{C}[x,y]/I$ contains only one copy of the trivial representation so I contains the polynomials

$$A := x^{2n} + y^{2n} - a, \quad Q := x^2y^2 - q.$$

By Lemma 2.8 we may assume that $(x,y), \dots, (x^{n-1}, y^{n-1}) \in \Gamma$.

2.4.1 Case A

Next we consider L_-. If $xy \in I$ then $xy \notin \Gamma$. By Corollary 2.6 $x^{2n} - y^{2n} \notin I$ therefore take $x^{2n} - y^{2n} \in \Gamma$. Since $xy \in I$ all of the pairs $\left(x^{k+1}y, xy^{k+1}\right)$

belong to I and $xy\,(x^n \pm i^n y^n)$ belongs to I these cannot be in Γ. By Lemma 2.9 there exist no polynomials in I of the form

$$y^{2n-k} - cx^k, \quad x^{2n-k} - (-1)^k cy^k$$

for $1 \leq k \leq n-1$ and so $\left(y^{2n-k}, x^{2n-k}\right) \in \Gamma$. By Lemma 2.9 $x^n + i^n y^n \notin I$ and $x^n - i^n y^n \notin I$ and so we may take these in Γ.

Therefore $\Gamma_A \in \Gamma(I)$ where

$$\Gamma_A = \begin{array}{lllll} [\,\cdot\,] & & & & \\ y^{2n} & & & & \\ \vdots & & & & \\ y^2 & & & & \\ y & [\,\cdot\,] & & & \\ 1 & x & x^2 & \cdots & x^{2n} \quad [\,\cdot\,]. \end{array}$$

In this G-graph we take $x^{2n} - y^{2n}$ but not $x^{2n} + y^{2n}$ and take $x^n + i^n y^n$ and $x^n - i^n y^n$ rather than x^n and y^n. Clearly this satisfies the definition of a G-graph.

2.4.2 Case B

From now on we assume that $xy \notin I$ and so we can take $xy \in \Gamma$. Since $(x, y) \in \Gamma$ it remains to consider the other pair in V_1. If

$$x^2 y - cx, \quad xy^2 + cy$$

belong to I then the polynomials

$$x^{k+1} y - cx^k, \quad xy^{k+1} + cy^k$$

belong to I so $\left(x^{k+1} y, xy^{k+1}\right) \notin \Gamma$ for all $k \geq 1$. By Lemma 2.9 the pair $\left(y^{2n-k}, x^{2n-k}\right) \in \Gamma$ for $1 \leq k \leq n-1$.

Also the polynomials

$$xy(x^n \pm i^n y^n) - c(x^n \mp i^n y^n) \in I.$$

If $x^n + i^n y^n$ or $x^n - i^n y^n$ is in I then $xy(x^n - i^n y^n)$ or $xy(x^n + i^n y^n)$ belong to I. By Lemma 2.7 this cannot happen so we may assume $x^n + i^n y^n$ and $x^n - i^n y^n$ can be chosen in Γ.

Therefore $\Gamma_B \in \Gamma(I)$ where

$$\Gamma_B := \begin{array}{llllll} [\cdot] & & & & & \\ y^{2n-1} & & & & & \\ \vdots & & & & & \\ y^2 & [\cdot] & & & & \\ y & xy & [\cdot] & & & \\ 1 & x & x^2 & \cdots & x^{2n-1} & [\cdot]. \end{array}$$

Again $x^n + i^n y^n$ and $x^n - i^n y^n$ belong to Γ_B rather than x^n and y^n.

2.4.3 Case $\mathbf{C}_k$

Fix $k \in [2, n-1]$ and assume that k is the smallest integer so that we have the polynomials of the form

$$x^{k+1}y - cx^k, \quad xy^{k+1} + cy^k \in I.$$

Then we can take $(x^{j+1}y, xy^{j+1}) \in \Gamma$ for all $j < k$. Since $\left(x^k, y^k\right) \in \Gamma$ it remains to consider the other pair in V_k. The polynomials

$$x^{i+1}y - cx^i, \quad xy^{i+1} + cy^i \in I$$

for $i \geq k$. Therefore $(x^{i+1}y, xy^{i+1}) \notin \Gamma$ for $i \geq k$ and so we can take $(y^{2n-i}, x^{2n-i}) \in \Gamma$ for $k \leq i < n$. As in Case B $x^n + i^n y^n$ and $x^n - i^n y^n$ do not belong to I so can be taken in Γ.

Thus $\Gamma_{C_k} \in \Gamma(I)$ where

$$\Gamma_{C_k} := \begin{array}{ccccccccc}
[\,\cdot\,] & & & & & & & & \\
y^{2n-k} & & & & & & & & \\
\vdots & & & & & & & & \\
y^{k+1} & [\,\cdot\,] & & & & & & & \\
y^k & xy^k & & & & & & & \\
\vdots & \vdots & & & & & & & \\
y^2 & xy^2 & [\,\cdot\,] & & & & & & \\
y & xy & x^2y & \cdots & x^ky & [\,\cdot\,] & & & \\
1 & x & x^2 & \cdots & x^k & x^{k+1} & \cdots & x^{2n-k} & [\,\cdot\,].
\end{array}$$

Again $x^n + i^n y^n$ and $x^n - i^n y^n$ belong to Γ_{C_k} rather than x^n and y^n.

2.4.4 Case D

In each of the previous cases we were assuming that there exists some $j < n$ such that the polynomials

$$x^{j+1}y - cx^j, \quad xy^{j+1} + cy^j$$

belong to I. This implies that

$$xy\,(x^n - i^n y^n) - c(x^n + i^n y^n), \quad xy\,(x^n - i^n y^n) - c(x^n + i^n y^n)$$

belong to I. If $x^n \pm i^n y^n \in I$ then $xy\,(x^n \mp i^n y^n) \in I$ and by Lemma 2.7 $\mathbb{C}[x, y]\,/I$ contains no copy of $M_{\mp}$. For this reason in each of the previous cases we took $x^n \pm i^n y^n \in \Gamma$.

From now on we assume that xy, (x^j, y^j) and $(x^{j+1}y, xy^{j+1})$ belong to Γ for all $1 \le j \le n-1$. Consider the representation M_+. If $x^n - i^n y^n \in I$ then by Lemma 2.7 $xy(x^n + i^n y^n) \notin I$ and so $x^n + i^n y^n \notin I$. Therefore these can be taken in Γ. Since $x^n - i^n y^n \in I$ it follows that $y\left(x^n - i^n y^n\right), x\left(x^n - i^n y^n\right) \in I$. Therefore we can choose $(y(x^n + i^n y^n), x(x^n + i^n y^n))$ to be basic instead of $(x^n y, xy^n)$.

In this case $\Gamma_{D_+} \in \Gamma(I)$ where

$$\Gamma_{D_+} = \begin{array}{ccccccc} [\,\cdot\,] & & & & & & \\ y^{n+1} & xy^{n+1} & & & & & \\ y^n & xy^n & & & & & \\ \vdots & \vdots & & & & & \\ y^2 & xy^2 & [\,\cdot\,] & & & & \\ y & xy & x^2y & \cdots & x^n y & x^{n+1}y & \\ 1 & x & x^2 & \cdots & x^n & x^{n+1} & [\,\cdot\,]\,. \end{array}$$

This time $x^n + i^n y^n$, $x\left(x^n + i^n y^n\right)$, $y\left(x^n + i^n y^n\right)$ and $xy\left(x^n + i^n y^n\right)$ are in Γ_{D_+} but $x^n - i^n y^n$, $x\left(x^n - i^n y^n\right)$, $y\left(x^n - i^n y^n\right)$ and $xy\left(x^n - i^n y^n\right)$ are not.

Similarly consider the representation M_-. If $x^n + i^n y^n \in I$ then by Lemma 2.7 $xy(x^n - i^n y^n) \notin I$ and so $x^n - i^n y^n \notin I$. Therefore these can be taken in Γ. Again the pair $(x(x^n - i^n y^n), y(x^n - i^n y^n))$ can be chosen to be basic rather than $(x^n y, xy^n)$.

In this case $\Gamma_{D_-} \in \Gamma(I)$ where Γ_{D_-} has the same picture as Γ_{D_+}. This time the polynomials $x^n - i^n y^n$, $x\left(x^n - i^n y^n\right)$, $y\left(x^n - i^n y^n\right)$ and $xy\left(x^n - i^n y^n\right)$ are in Γ_{D_-} but $x^n + i^n y^n$, $x\left(x^n + i^n y^n\right)$, $y\left(x^n + i^n y^n\right)$ and $xy\left(x^n + i^n y^n\right)$ are not.

2.4.5 Case E

Finally we may assume xy, $x^n + i^n y^n$, $x^n - i^n y^n$, (x^j, y^j) and $(x^{j+1}y, xy^{j+1})$ belong to Γ for $1 \le j \le n-1$. Then $\Gamma_{E_0} \in \Gamma(I)$ where

$$\Gamma_{E_0} := \begin{array}{llllll} [\cdot] & & & & & \\ y^n & xy^n & & & & \\ \vdots & \vdots & & & & \\ y^2 & xy^2 & [\cdot] & & & \\ y & xy & x^2y & \cdots & x^n y & \\ 1 & x & x^2 & \cdots & x^n & [\cdot]. \end{array}$$

As in Case D, we may assume either the pair $(y(x^n + i^n y^n), x(x^n + i^n y^n))$ or $(y(x^n - i^n y^n), y(x^n - i^n y^n))$ is basic. If $(x(x^n + i^n y^n), y(x^n + i^n y^n))$ is basic then $\Gamma_{E_+} \in \Gamma(I)$ where

$$\Gamma_{E_+} = \begin{array}{lllllll} [\cdot] & & & & & & \\ y^{n+1} & [\cdot] & & & & & \\ y^n & xy^n & & & & & \\ \vdots & \vdots & & & & & \\ y^2 & xy^2 & [\cdot] & & & & \\ y & xy & x^2y & \cdots & x^n y & [\cdot] & \\ 1 & x & x^2 & \cdots & x^n & x^{n+1} & [\cdot]. \end{array}$$

Here $x^n + i^n y^n$, $x^n - i^n y^n$, $y\,(x^n + i^n y^n)$ and $x(x^n + i^n y^n)$ are in Γ_{E_+} but $y(x^n - i^n y^n)$ and $x(x^n - i^n y^n)$ are not.

If $(x(x^n - i^n y^n), y(x^n - i^n y^n))$ is basic then $\Gamma_{E_-} \in \Gamma(I)$ where Γ_{E_-} has the same picture as Γ_{E_+} but $x^n + i^n y^n$, $x^n - i^n y^n$, $y\,(x^n - i^n y^n)$ and $x(x^n - i^n y^n)$ are in Γ_{E_-} but $y(x^n + i^n y^n)$ and $x(x^n + i^n y^n)$ are not.

2.4.6 Conclusion

Theorem 2.10 *Let I be an ideal which defines a G-cluster then there exist G-graphs Γ_1 and Γ_2 such that*

$$\Gamma_1 \in \left\{\Gamma_A,\ \Gamma_B,\ \Gamma_{C_2},\ \ldots,\ \Gamma_{C_{n-2}},\ \Gamma_{C_{n-1}},\ \Gamma_{D_+},\ \Gamma_{D_-},\ \Gamma_{E_0}\right\}$$
$$\Gamma_2 \in \left\{\Gamma_A,\ \Gamma_B,\ \Gamma_{C_2},\ \ldots,\ \Gamma_{C_{n-2}},\ \Gamma_{D_+},\ \Gamma_{D_-},\ \Gamma_{E_+},\ \Gamma_{E_-}\right\}$$

and $\Gamma_1, \Gamma_2 \in \Gamma(I)$.

Proof. We have shown that any ideal I which defines a G-cluster has $\Gamma \in \Gamma(I)$ where $\Gamma \in \left\{\Gamma_A, \Gamma_B, \Gamma_{C_2}, \ldots, \Gamma_{C_{n-1}}, \Gamma_{D_+}, \Gamma_{D_-}, \Gamma_{E_0}\right\}$. To prove the theorem it remains to show that if $\Gamma_{C_{n-1}} \in \Gamma(I)$ or $\Gamma_{E_0} \in \Gamma(I)$ then $\Gamma_{E_+} \in \Gamma(I)$ or $\Gamma_{E_-} \in \Gamma(I)$.

We prove that if $\Gamma_{C_{n-1}} \in \Gamma(I)$ then either Γ_{E_+} or Γ_{E_-} is in $\Gamma(I)$. The proof of the other case is identical. Let I be an ideal that defines a G-cluster which has $\Gamma_{C_{n-1}} \in \Gamma(I)$. If neither $(y(x^n + i^n y^n), x(x^n + i^n y^n))$ nor $(y(x^n - i^n y^n), x(x^n - i^n y^n))$ can be basic in $\mathbb{C}[x, y]/I$ then there exist polynomials of the form

$$C := y(x^n + i^n y^n) - cx^{n-1}, \quad D := x(x^n + i^n y^n) + i^n c y^{n-1}$$
$$E := y(x^n - i^n y^n) - dx^{n-1}, \quad F := x(x^n - i^n y^n) - i^n d y^{n-1}$$

in I. However

$$C - E = 2i^n y^{n+1} - (c - d)x^{n-1}$$
$$D + F = 2x^{n+1} + i^n (c - d) y^{n-1}$$

Therefore (y^{n+1}, x^{n+1}) cannot form part of a basis for $\mathbb{C}[x, y]/I$ which contradicts the assumption that $\Gamma_{C_{n-1}} \in I(\Gamma)$. $\square$

2.5 Construction of open sets contained in G-Hilb $\mathbb{C}^2$

In this section we take a G-graph given in the previous section and list generators for any ideal I that defines a G-cluster and has this G-graph. This will give an open subset in G-Hilb $\mathbb{C}^2$ of points whose defining ideal I has $\Gamma \in \Gamma(I)$.

2.5.1 Case A

Let I be any ideal that defines a G-cluster such that $\Gamma_A \in \Gamma(I)$. Then I contains the polynomials

$$\begin{aligned} A &:= x^{2n} + y^{2n} - a, & B &:= xy - b\left(x^{2n} - y^{2n}\right), \\ C &:= x^{2n+1} - cx - dy^{2n-1}, & D &:= y^{2n+1} - cy + dx^{2n-1}, \\ E &:= xy^{2n} - ex - fy^{2n-1}, & F &:= x^{2n}y - ey + fx^{2n-1}, \end{aligned}$$
$$Q := x^2y^2 - q$$

for some $a, \dots, f, q \in \mathbb{C}$.

Lemma 2.11 *Using only these polynomials every monomial in $\mathbb{C}[x, y]$ can be written as a linear combination of elements in Γ_A. In other words Γ_A spans $\mathbb{C}[x, y] / \langle A, B, C, D, E, F, Q \rangle$.*

Proof. Consider a monomial $x^\alpha y^\beta$. We want to show that this can be written as a linear combination of elements in Γ_A.

If $\alpha \geq 2$ and $\beta \geq 2$ then as in Lemma 2.9 we can use Q to write

$$x^\alpha y^\beta = qx^{\alpha-2}y^{\beta-2}.$$

Therefore it remains to prove the lemma for monomials of the form x^α, $x^\alpha y$, y^α and xy^β.

We show that x^α can be written as a linear combination of elements in Γ_A. If $\alpha \leq 2n$ then either $x^\alpha \in \Gamma_A$ or $\alpha = n, 2n$. However $x^n = ((x^n + i^n y^n) + (x^n - i^n y^n))/2$ and $(x^n + i^n y^n), (x^n - i^n y^n) \in \Gamma_A$. Now $x^{2n} = ((x^{2n} + y^{2n}) + (x^{2n} - y^{2n}))/2$ and $(x^{2n} - y^{2n}) \in \Gamma_A$ but $(x^{2n} + y^{2n}) \notin \Gamma_A$. The polynomial A allows us to replace $(x^{2n} + y^{2n})$ with $a \cdot 1$. So $x^{2n} = (a \cdot 1 + (x^{2n} - y^{2n}))/2$. Therefore the lemma holds for x^α when $\alpha \leq 2n$.

To prove the lemma holds for x^α when $\alpha > 2n$ we use induction on α. It holds for $\alpha = 2n + 1$ by the polynomial C.

Write $\alpha = \lambda 2n + \mu$ where $\lambda \geq 1$ and $0 \leq \mu \leq 2n - 1$. Assume that

$$x^\alpha = s x^\mu + t y^{2n-\mu}.$$

If $\mu \neq 0, n, 2n$ then $x^\mu, y^{2n-\mu} \in \Gamma_A$. If $\mu = 0$, n or $2n$ then as shown above x^n, y^n x^{2n} and y^{2n} can be written as a linear combinations of elements in Γ_A.

Then

$$x^{\alpha+1} = s x^{\mu+1} + t x y^{2n-\mu}. \tag{2.4}$$

If $\mu < 2n - 1$ then $x^{\mu+1} \in \Gamma_A$. If $\mu = 2n - 1$ then as remarked previously $x^{\mu+1}$ can be written as linear combinations of elements in Γ_A. Consider the monomial $x y^{2n-\mu} \notin \Gamma_A$. Now

$$\begin{aligned} y^{2n-\mu-1} B &= x y^{2n-\mu} - b x^{2n} y^{2n-\mu-1} + b y^{4n-\mu-1} \\ y^{2n-\mu-1} A &= y^{4n-\mu-1} + x^{2n} y^{2n-\mu-1} - a y^{2n-\mu-1} \end{aligned}$$

Again $y^{2n-\mu-1} \in \Gamma_A$.

If μ is odd then

$$x^{\mu+1} q_{(2n-\mu-1)/2} Q = x^{2n} y^{2n-\mu-1} - q^{(2n-\mu-1)/2} x^{\mu+1}.$$

If μ is even then

$$\begin{aligned} y^{2n-\mu-2} F &= x^{2n} y^{2n-\mu-1} - e y^{2n-\mu-1} + f x^{2n-1} y^{2n-\mu-2} \\ x^{\mu+1} q_{(2n-\mu-2)/2} Q &= x^{2n-1} y^{2n-\mu-2} - q^{(2n-\mu-2)/2} x^{\mu+1}. \end{aligned}$$

Substituting these values back into Equation 2.4 we have

$$x^{\alpha+1} = s'x^{\mu+1} + t'y^{2n-\mu-1}$$

where $x^{\mu+1}, y^{2n-\mu-1} \in \Gamma_A$ or can be written as linear combinations of elements in Γ_A. Thus the lemma is proved for x^α. The proof in the other three cases is identical. $\square$

Consider a linear combination L of the polynomials $A, \ldots, F, Q$ with coefficients in $\mathbb{C}[x, y, a, b, c, d, e, f, q]$ which is equal to an expression $\sum a_i f_i$, where $f_i \in \Gamma_A$ and $a_i \in \mathbb{C}[a, b, c, d, e, f, q]$. Because $A, \ldots, F, Q$ belong to I it follows that $\sum a_i f_i \in I$. The elements of Γ_A are linearly independent in $\mathbb{C}[x, y]/I$ so we know that each of the coefficients a_i equals zero. Therefore $L = 0$ and the relation L is a *syzygy* between the generators. The syzygies correspond to the fact that the parameters $a, \ldots, f, q$ are not independent.

Lemma 2.12 *If Γ_A is linearly independent in $\mathbb{C}[x, y]/\langle A, B, C, D, E, F, Q\rangle$ then the parameters $a, \ldots, f, q$ must satisfy the following relations.*

$$e = a - c,\;\; f = -d,\;\; q = abd,\;\; d = 2bc - ab,\;\; c = a + 2bfq^{n-1} \qquad (2.5)$$

Proof. Consider the relation

$$xyB - Q = -bxy\left(x^{2n} - y^{2n}\right) + q \qquad (2.6)$$

in I. Now $xy\left(x^{2n} - y^{2n}\right) \in L_+$ and 1 is our element in Γ_A belonging to the representation L_+ so the polynomial $xy\left(x^{2n} - y^{2n}\right)$ can be written as a multiple of 1. Since

$$yC - xD = xy\left(x^{2n} - y^{2n}\right) - d\left(y^{2n} + x^{2n}\right)$$

and the polynomial A writes $(x^{2n}+y^{2n})$ as a multiple of 1 we have

$$xyB - Q + byC - bxD + bdA \;\; = \;\; q - abd \in I.$$

This relation implies that $(q - abd)1 \in I$. Now $1 \notin I$ if Γ_A are linearly independent in $\mathbb{C}[x,y]/I$ so $q = abd$. In fact this syzygy implies that Q is not needed as a generator of the ideal as $Q = xyB + byC - bxD + bdA$. However we continue to include it since it will simplify the other syzygies.

Next consider the relations

$$\begin{aligned} xA - C - E &= (c - a + e)x + (d + f)y^{2n-1} \\ yA - D - F &= (c - a + e)y - (d + f)y^{2n-1} \end{aligned} \tag{2.7}$$

in I. If x and y^{2n-1} are linearly independent in $\mathbb{C}[x,y]/I$ then the coefficient of x and of y^{2n-1} in Equation 2.7 must be zero. Therefore we have the relations $c = a - e$ and $d = -f$. Again these syzygies imply that we do not need the generators E and F since $E = xA - C$ and $F = yA - D$.

Again

$$xyA - yC - xD = -xya + 2cxy - d(x^{2n} - y^{2n})$$

belongs to I. Now $x^{2n} - y^{2n} \in L_-$ belongs to Γ_A but $xy \notin \Gamma_A$. The polynomial B allows xy to be replaced with $x^{2n} - y^{2n}$ and so

$$xyA - yC - xD + (a - 2c)B = (-d - ab + 2bc)(x^{2n} - y^{2n}).$$

Again since $x^{2n} - y^{2n} \notin I$ it must be that $d = -ab + 2bc$.

Finally

$$xA - y^{2n-1}B - C = -ax + bx^{2n}y^{2n-1} - by^{4n-1} + cx + dy^{2n-1}.$$

Now

$$\begin{aligned} y^{2n-1}A &= x^{2n}y^{2n-1} + y^{4n-1} - ay^{2n-1}, \\ y^{2n-2}F &= x^{2n}y^{2n-1} - ey^{2n-1} + fx^{2n-1}y^{2n-2}, \\ xq_{n-1}Q &= x^{2n-1}y^{2n-2} - q^{n-1}x. \end{aligned}$$

Therefore

$$xA - y^{2n-1}B - C + by^{2n-1}A - 2by^{2n-2}F + 2bfq_{n-1}xQ = \left(-a + c - 2bfq^{n-1}\right)x + (d - ab + 2be)\,y^{2n-1}.$$

In the same way the final syzygy we will consider

$$yA - x^{2n-1}B - D - bx^{2n-1}A + 2bx^{2n-2}E + 2bfyq_{n-1}Q = \left(-a + c - 2bfq^{n-1}\right)y + (-d + ab - 2be)\,x^{2n-1}.$$

These syzygies can be written in a matrix M as follows.

$$\begin{pmatrix} bd & xy & by & -bx & 0 & 0 & -1 \\ x & 0 & -1 & 0 & -1 & 0 & 0 \\ y & 0 & 0 & -1 & 0 & -1 & 0 \\ xy & a-2c & -y & -x & 0 & 0 & 0 \\ x + by^{2n-1} & -y^{2n-1} & -1 & 0 & 0 & -2by^{2n-2} & 2bfq_{n-1}x \\ y - bx^{2n-1} & -x^{2n-1} & 0 & -1 & 2bx^{2n-2} & 0 & 2bfq_{n-1}y \end{pmatrix}$$

This is called the matrix of syzygies and means that

$$M\begin{pmatrix} A & B & C & D & E & F & Q \end{pmatrix}^t = 0.$$

Together these syzygies correspond to the following relations between the parameters.

$$e = a - c,\ f = -d,\ q = abd,\ d = 2bc - ab,\ c = a + 2bfq^{n-1}$$

□

Proposition 2.13 *Let I be an ideal generated by $A, \ldots, F, Q$ where the parameters $a, \ldots, f, q$ are subject to the relations given in 2.5. Then Γ_A forms a basis for $\mathbb{C}[x, y]/I$ and so I defines a G-cluster.*

Proof. By Lemma 2.11 Γ_A spans $\mathbb{C}[x,y]/I$. It remains to show that Γ_A is linearly independent in $\mathbb{C}[x,y]/I$. Now suppose that Γ_A is not linearly independent. Then there exists a G-equivariant linear combination of the elements in Γ_A which belongs to I.

Suppose $x^{2n}-y^{2n}$ is not basic then $x^{2n}-y^{2n}, xy \in I$. Now $xE \in I$ which implies $ex^2 \in I$. If $e=0$ then $a=0$ or $b=0$ by Relations 2.5. If $a=0$ then

$$x^{2n}+y^{2n}, \quad xy-b(x^{2n}-y^{2n}), \quad x^{2n+1}, \quad y^{2n+1}$$

are generators for I. If $b=0$ then

$$x^{2n}+y^{2n}-a, \quad xy$$

are generators for I. In each case only one term in the generators divides $x^{2n}-y^{2n}$ and so these generators do not imply that $x^{2n}-y^{2n} \in I$. Therefore we can take $x^{2n}-y^{2n}$ in the basis.

Otherwise $x^2 \in I$ which implies $x^{2n}+y^{2n} \in I$ and so $a \in I$. We have already shown that if $a=0$ then $x^{2n}-y^{2n} \notin I$. If $a \neq 0$ then $1 \in I$. Again only one term in the generators A, B, C, D divides 1 and so the generators do not imply that $1 \in I$.

Suppose that (y^{2n-i}, x^{2n-i}) and (x^i, y^i) are not linearly independent. Now $y^{2n-i}, x^{2n-i}, x^i, y^i \notin I$ because $x^{2n}-y^{2n} \notin I$. Therefore there exist polynomials of the form

$$y^{2n-i}-\alpha x^i, \quad x^{2n-i}-(-1)^i \alpha y^i$$

in I where $\alpha \neq 0$. If i is even then these polynomials imply that $x^{2n}-y^{2n} \in I$. We have already shown this is not possible. If i is odd then as in the proof of Lemma 2.11

$$xy^{2n-i}-\alpha x^{i+1} = -aby^{2n-i-1} - \left(\alpha - 2bq^{(2n-i-1)/2}\right) x^{i+1}.$$

However we have shown that there exists no such relation when i is even so $ab = \alpha - 2bq^{(2n-i-1)/2} = 0$. If $ab = 0$ then $q = 0$ so $ab = \alpha - 2bq^{(2n-i-1)/2} = 0$ is impossible since $\alpha \neq 0$.

Therefore Γ_A forms a basis for $\mathbb{C}[x,y]/I$. $\square$

Therefore this open subset of G-Hilb $\mathbb{C}^2$ is parametrised by a, b, c subject to the equation $a - c + 2a^{n-1}b^{2n}(2c-a)^n$. In other words this open subset of G-Hilb $\mathbb{C}^2$ is isomorphic to the smooth surface in $\mathbb{C}^3_{a,b,c}$ given by the equation

$$a - c + 2a^{n-1}b^{2n}(2c-a)^n.$$

Proposition 2.14 $\mathrm{Soc}(I)$ *contains at most the submodule of* $\mathbb{C}[x,y]/I$ *generated by* $\langle x^{2n} - y^{2n} \rangle \cong L_-$. *Moreover* $\langle x^{2n} - y^{2n} \rangle \subset \mathrm{Soc}(I)$ *if and only if* $a = 0$.

Proof. Since $\mathrm{Soc}(I)$ is isomorphic to a direct sum of G-representations we only need to check if submodules of $\mathbb{C}[x,y]/I$ isomorphic to any of the irreducible representations belong to $\mathrm{Soc}(I)$. Clearly the submodules generated by $1, x^n + i^n y^n, x^n - i^n y^n$ do not belong to $\mathrm{Soc}(I)$. Consider the submodule generated by $x^j + \lambda y^{2n-j}, y^j + (-1)^j \lambda x^{2n-j}$. For this to be contained in $\mathrm{Soc}(I)$ it must be that $x(x^j + \lambda y^{2n-j}) = 0$. Now

$$x(x^j + \lambda y^{2n-j}) = x^{j+1} + \lambda x y^{2n-j}.$$

If j is odd then as in the proof of Lemma 2.11

$$\begin{aligned} y^{2n-j-1}B - by^{2n-j-1}A + 2bx^{j+1}q_{(2n-j-1)/2}Q \\ = xy^{2n-j} - 2bq^{(2n-j-1)/2}x^{j+1} + aby^{2n-j-1} \end{aligned}$$

and if j is even then

$$\begin{aligned} y^{2n-j-1}B - by^{2n-j-1}A + 2by^{2n-j-2}F - 2bfx^{j+1}q_{(2n-j-2)/2}Q \\ = xy^{2n-j} + 2bfq^{(2n-j-2)/2}x^{j+1} + (-2be + ab)y^{2n-j-1}. \end{aligned}$$

Therefore if j is odd

$$x(x^j + \lambda y^{2n-j}) = x^{j+1} + 2bq^{(2n-j-1)/2}\lambda x^{j+1} - ab\lambda y^{2n-j-1}.$$

This is zero if and only if $\lambda ab = 0$ and $1 + 2bq^{(2n-j-1)/2}\lambda = 0$. However if $ab = 0$ then $q = 0$ and so $1 + 2bq^{(2n-j-1)/2} \neq 0$.

Similarly if j is even

$$x(x^j + \lambda y^{2n-j}) = x^{j+1} - 2bfq^{(2n-j-2)/2}\lambda x^{j+1} - (-2be + ab)\lambda y^{2n-j-1}.$$

For this to be zero $1 - 2bfq^{(2n-j-2)/2} = 0$ and $d = (-2be + ab) = 0$. If $d = 0$ then $q = 0$ and $1 + 2bq^{(2n-j-1)/2} \neq 0$. Therefore

$$\left\langle x^j + \lambda y^{2n-j}, y^j + (-1)^j \lambda x^{2n-j} \right\rangle \not\subset \mathrm{Soc}(I).$$

It only remains to see when $x^{2n} - y^{2n} \in \mathrm{Soc}(I)$. Using the relations C and E

$$x(x^{2n} - y^{2n}) = (c+e)x + (d+f)y^{2n-1}.$$

From Equation 2.5 $d + f = 0$ and $a = c + e$ and so $x^{2n} - y^{2n} \in \mathrm{Soc}(I)$ if and only if $a = 0$. $\square$

Notice that $u = xy(x^{2n} - y^{2n}) = d(x^{2n} + y^{2n}) = ad$, $v = x^{2n} + y^{2n} = a$ and $w = x^2y^2 = abd$. The curve $a = 0$ defines the exceptional locus $u = v = w = 0$ in this affine subset. If $a = 0$ then $c = 0$ so b is the coordinate along this exceptional curve. Setting $a = 0$ and letting $b \to \infty$ we obtain the ideal which is described in the next section when the new parameters are all zero.

2.5.2 Case B

Let I be an ideal that defines a G-cluster such that $\Gamma_B \in \Gamma(I)$. Then I contains the polynomials

$$\begin{array}{ll} A := x^{2n} + y^{2n} - a, & B := x^{2n} - y^{2n} - bxy, \\ C := xy^2 - cx^{2n-1} - dy, & D := yx^2 - cy^{2n-1} + dx, \\ E := xy^3 - ex^{2n-2} - fy^2, & F := x^3y + ey^{2n-2} + fx^2 \end{array}$$
$$Q := x^2y^2 - q.$$

As in Case A we can show that using only these polynomials every monomial in $\mathbb{C}[x, y]$ can be written as a linear combination of elements in Γ_B.

To find the relations between the parameters we use syzygies. First consider

$$xC + yD + cA - 2Q \;=\; -ac + 2q \in I.$$

Again $1 \notin I$ so $q = ac/2$ and Q is not required as a generator.

Other syzygies are

$$-xC + yD + cB = -(cb + 2d)xy,$$

$$yC - E + cx^{2n-4}F - cey^2q_{n-2}Q$$
$$= (f - d - ceq^{n-2})y^2 + (e + cf)x^{2n-2},$$

$$xD - F + cy^{2n-4}E + ey^2q_{n-2}Q$$
$$= -(f - d - ceq^{n-2})x^2 - (e + cf)y^{2n-2},$$

$$yA + yB - 2x^{2n-2}D - 2cyq_{n-1}Q + bC$$
$$= \left(-a + 2cq^{n-1} - bd\right)y - (bc - 2d)x^{2n-1},$$

$$xA - xB - 2y^{2n-2}C - bD - 2cxq_{n-1}Q$$
$$= (-a - bd + 2cq^{n-1})x + (2d + bc)y^{2n-1}.$$

The matrix of syzygies is

$$\begin{pmatrix} c & 0 & x & y & 0 & 0 & -2 \\ 0 & c & x & -y & 0 & 0 & 0 \\ 0 & 0 & y & 0 & -1 & cx^{2n-4} & -cey^2q_{n-2} \\ 0 & 0 & 0 & x & cy^{2n-4} & -1 & cex^2q_{n-2} \\ y & y & b & -2x^{2n-2} & 0 & 0 & -2cq_{n-1}y \\ x & -x & -2y^{2n-2} & -b & 0 & 0 & -2cq_{n-1}x \end{pmatrix}$$

and these correspond to the following relations between the parameters

$$\begin{gathered} 2q = ac, \quad 2d = -bc, \quad e = -cf, \\ f - d - ceq^{n-2} = 0, \quad 0 = a + bd - 2cq^{n-1}. \end{gathered} \tag{2.8}$$

Proposition 2.15 *Let I be an ideal generated by $A, \dots, F, Q$ where the parameters $a, \dots, f, q$ are subject to the relations given in 2.8. Then Γ_B forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

Proof. The proof that Γ_B spans $\mathbb{C}[x,y]/I$ is the same as the proof of Lemma 2.11. That Γ_B is linearly independent in $\mathbb{C}[x,y]/I$ is proved using the techniques in the proofs of Lemma 2.11 and Proposition 2.13.

Suppose that (x,y) and (y^{2n-1}, x^{2n-1}) are not linearly independent in $\mathbb{C}[x,y]/I$. Then there exist polynomials of the form

$$\lambda x - \mu y^{2n-1}, \quad \lambda y + \mu x^{2n-1}$$

in I. Therefore $\mu(x^{2n} - y^{2n}) \in I$. If $\mu = 0$ then $x, y \in I$ and so $a = 0$ or $1 \in I$. As in Case A the generators $A, \dots, F, Q$ do not imply $x, y \in I$ when $a = 0$ and do not imply $1 \in I$ in any case.

Otherwise $x^{2n} - y^{2n} \in I$ and so $xy \in I$ or $b = 0$. As in case A the generators $A, \ldots, F, Q$ do not imply a relation between x and y^{2n-1} when $b = 0$. If $xy \in I$ then relations C and D imply that $dx^2, dy^2 \in I$ and so either $d = 0$ or $x^2, y^2 \in I$. As above we can show that neither case is possible.

Finally we show by induction that (x^i, y^i) and (y^{2n-i}, x^{2n-i}) are linearly independent in $\mathbb{C}[x, y]/I$ for $1 \leq i \leq n$. We have shown that it is true when $i = 1$. Assume that it is not true for some $1 < i < n$. Then there exist polynomials of the form

$$y^{2n-i} - \alpha x^i, \quad x^{2n-i} - (-1)^i \alpha y^i$$

in I and $\alpha \neq 0$ by induction. These polynomials imply that

$$y^{2n-i+1} - \alpha x^i y \in I. \tag{2.9}$$

If i is even then

$$x^i y = cx^{i-2} y^{2n-1} - dx^{i-1} = cq^{(i-2)/2} y^{2n-i+1} - dx^{i-1}.$$

Substituting this back into Equation 2.9 implies

$$\alpha dx^{i-1} + (-\alpha cq^{(i-2)/2} + 1) y^{2n-i+1} \in I.$$

By induction there exists no relation between x^{i-1} and y^{2n-i+1} in I so $\alpha d = \alpha cq^{(i-2)/2} - 1 = 0$. This implies that $d = 0$. However if $d = 0$ then we can see that the ideal contains no polynomials of the form $y^{2n-i} - \alpha x^i$ and $x^{2n-i} - (-1)^i \alpha y^i$.

If i is odd then

$$x^i y = ex^{i-3} y^{2n-2} - fx^{i-1} = eq^{(i-3)/2} y^{2n-i+1} - fx^{i-1}.$$

As above we can show that this leads to a contradiction. □

Therefore these parameters define a smooth surface in $\mathbb{C}^4_{a,b,e,f}$ with equations

$$a - b^2c/2 - a^{n-1}c^n/2^{n-2} = f + bc/2 + a^{n-2}c^n f/2^{n-1} = 0$$

Proposition 2.16 $\mathrm{Soc}(I)$ *contains at most the submodules of* $\mathbb{C}[x,y]/I$ *generated by* $x^{2n} - y^{2n}$ *and* $\langle y^{2n-1}, x^{2n-1}\rangle$. *Moreover* $\langle x^{2n} - y^{2n}\rangle \cong L_- \subset \mathrm{Soc}(I)$ *if and only if* $c = 0$ *and* $\langle y^{2n-1}, x^{2n-1}\rangle \subset V_1 \in \mathrm{Soc}(I)$ *if and only if* $a = b = 0$.

Proof. The proof is similar to the proof of Proposition 2.14 □

Notice that $a = b = 0$ is a Cartier divisor since on the open set given by $c \neq 0$ it has the equation $a = 0$ and on the open set given by $1 - a^{n-2}c^n/2^{n-2} \neq 0$ it is given by the equation $b = 0$.

Again $u = abc/2$, $v = a$ and $w = ac/2$. Therefore $a = 0$ defines the exceptional locus. This decomposes into two prime divisors $a = b = 0$ and $c = 0$. If $c = 0$ then b is the coordinate along this exceptional curve. Setting $c = 0$ and letting $b \to \infty$ we obtain the ideal that was described in the previous section when the parameters were zero. If $a = b = 0$ then c is the coordinate along this exceptional curve. Setting $a = b = 0$ and letting $c \to \infty$ then we obtain the ideal which is described in the next section when $k = 2$ and the new parameters are all zero.

2.5.3 Case $\mathbf{C}_k$

Fix $k \in [2, n-2]$. Let I be an ideal that defines a G-cluster such that $\Gamma_{C_k} \in \Gamma(I)$. Then I contains the polynomials

$$\begin{aligned}
B &:= x^{2n-k+1} - bxy^k - cy^{k-1},\\
C &:= y^{2n-k+1} - (-1)^k bx^k y - (-1)^{j-1} cx^{k-1},\\
D &:= xy^{k+1} - dx^{2n-k} - ey^k,\\
E &:= x^{k+1}y - (-1)^{k+1} dy^{2n-k} + ex^k,\\
F &:= xy^{k+2} - fx^{2n-k-1} - gy^{k+1},\\
G &:= x^{k+2}y - (-1)^k fy^{2n-k-1} + gx^{k+1},\\
Q &:= x^2y^2 - q.
\end{aligned}$$

Again we can show that using only these polynomial every monomial in $\mathbb{C}[x,y]$ can be written as a linear combination of elements in Γ_{C_k}.

As in Case A we can construct a matrix of syzygies and so generate the relations between these parameters. The matrix of syzygies is

$$\begin{pmatrix}
d & 0 & x & 0 & 0 & 0 & -y^{k-1}\\
0 & (-1)^k d & 0 & y & 0 & 0 & -x^k\\
0 & 0 & 0 & -x & (-1)^k dy^{2n-2k-2} & 1 & (-1)^k df q_{n-i-1} x^{k+1}\\
0 & 0 & y & 0 & -1 & dx^{2n-2i-2} & (-1)^k df q_{n-i-1} y^{k+1}\\
y & 0 & b & -x^{2n-2k} & 0 & 0 & (-1)^k dq_{n-k} y^k\\
0 & x & -y^{2n-2k} & (-1)^k b & 0 & 0 & -dq_{n-k} x^k
\end{pmatrix}$$

and so the relations are

$$\begin{gathered}
e = -bd, \quad q = cd, \quad f = -dg,\\
g = e + (-1)^k df q^{n-k}, \quad 0 = (-1)^k dq^{n-k} + c + be.
\end{gathered} \tag{2.10}$$

Proposition 2.17 *Let I be an ideal generated by $B, \dots, G, Q$ where the parameters $b, \dots, g, q$ are subject to the relations given in 2.10. Then Γ_{C_k} forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

PROOF. This is proved using the same techniques as the proof of Proposition 2.15. □

Again we have that this affine open set is isomorphic to a smooth surface in $\mathbb{C}^4_{b,c,d,g}$ given by the equations

$$c - b^2d + (-1)^k d^{n-k+1}c^{n-k} = g + bd + (-1)^k gc^{n-k}d^{n-k+2} = 0.$$

Proposition 2.18 $\mathrm{Soc}(I)$ *contains at most the submodules* $\langle x^k y, xy^k\rangle$ *and* $\langle y^{2n-k}, x^{2n-k}\rangle$. *Moreover* $\langle x^k y, xy^k\rangle \cong V_{k-1} \subset \mathrm{Soc}(I)$ *if and only if* $d = 0$ *and* $\langle y^{2n-k}, x^{2n-k}\rangle \cong V_k \subset \mathrm{Soc}(I)$ *if and only if* $b = c = 0$.

PROOF. The proof is similar to the proof of Proposition 2.14. □

If k is odd then $u = 2b(cd)^{(k+1)/2}$, $v = 2c(cd)^{(k-1)/2}$ and $w = cd$. If k is even then $u = 2c(cd)^{k/2}$, $v = 2b(cd)^{k/2}$ and $w = cd$. Therefore $cd = 0$ defines the exceptional locus on this affine subset. This decomposes as the two prime divisor $d = 0$ and $b = c = 0$.

If $d = 0$ then b is the coordinate along this exceptional curve. Setting $d = 0$ and $b \to \infty$ gives the gives the ideal that is described in the previous case, Case C_{k-1} or Case B, when all the parameters are zero. If $b = c = 0$ then d is the coordinate along this exceptional curve. Setting $b = c = 0$ and $d \to \infty$ gives the gives the ideal which is described in the next case, Case C_{k+1}, when all the parameters are zero.

2.5.4 Case D

An ideal I which defines a G-cluster and $\Gamma_{D_+} \in \Gamma(I)$, is generated by the polynomials

$$\begin{aligned}
B &:= x^{n+2} - bxy^{n-1} - cy^{n-2}, \\
C &:= y^{n+2} - (-1)^{n+1} byx^{n-1} - (-1)^n cx^{n-2}, \\
D &:= (x^n - i^n y^n) - dxy\,(x^n + i^n y^n)\,, \\
E &:= x\,(x^n - i^n y^n) - ey^{n-1} - fx\,(x^n + i^n y^n)\,, \\
F &:= y\,(x^n - i^n y^n) - e(-i)^n x^{n-1} + fy\,(x^n + i^n y^n)\,, \\
Q &:= x^2y^2 - q
\end{aligned}$$

and the syzygy matrix is

$$\begin{pmatrix}
-dy & 0 & x & \alpha & 0 & i^n dy^{n-1} \\
0 & i^n dx & y & 0 & \alpha & dx^{n-1} \\
-1 + dxy & 0 & x^2 & 0 & 0 & (bd + i^n)y^{n-2} + i^n dxy^{n-1} \\
0 & i^n\,(1 + xyd) & y^2 & 0 & 0 & -((-i)^n bd + 1)x^{n-2} + dyx^{n-1} \\
0 & 0 & -(-i)^n e & y & x & 0
\end{pmatrix}$$

where $\alpha = -1 - (-i)^n bd/2$. This corresponds to the relations

$$\begin{aligned}
c &= i^n q + bdq, & f &= (-i)^n ed/2, \\
b &= dc + i^n dq, & 0 &= e - cd + (-i)^n bde/2 - i^n dq
\end{aligned} \tag{2.11}$$

between the parameters.

Proposition 2.19 *Let I be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.11. Then Γ_{D_+} forms a basis for $\mathbb{C}[x, y]/I$ and so I defines a G-cluster.*

Proof. This is proved using the same techniques as the proof of Proposition 2.15. □

Thus this affine subset is the smooth surface in $\mathbb{C}^4_{b,d,e,q}$ given by the equations

$$b - 2i^n dq - bd^2 q = e - bd^2 q + (-i)^n bde/2 - 2i^n dq = 0.$$

Proposition 2.20 $\mathrm{Soc}(I)$ *contains at most the submodule* $\langle xy(x^n + i^n y^n)\rangle$. *Moreover* $\langle xy(x^n + i^n y^n)\rangle \cong M_+ \subset \mathrm{Soc}(I)$ *if and only if* $q = 0$.

PROOF. The proof is similar to the proof of Proposition 2.14. $\square$

If k is odd then $u = 2(i^n + bd)q^{(n+1)/2}$, $v = 2bq^{(n-1)/2}$ and $w = q$. If k is even then $u = 2bq^{n/2}$, $v = 2(i^n + bd)q^{n/2}$ and $w = q$. Therefore $q = 0$ defines the exceptional locus on this affine subset.

If $q = 0$ then d is the coordinate along this exceptional curve. Setting $q = 0$ and $d \to \infty$ gives the ideal that is described in Case E_+ when all the parameters are zero.

An ideal I that defines a G-cluster and $\Gamma_{D_-} \in \Gamma(I)$, is generated by the polynomials

$$\begin{aligned}
B &:= x^{n+2} - bxy^{n-1} - cy^{n-2},\\
C &:= y^{n+2} - (-1)^{n-1}bx^{n-1}y(-1)^n - c(-1)^n x^{n-2},\\
D &:= x^n + i^n y^n - dxy\left(x^n - i^n y^n\right),\\
E &:= x\left(x^n + i^n y^n\right) - ey^{n-1} - fx\left(x^n - i^n y^n\right),\\
F &:= y\left(x^n + i^n y^n\right) + e(-i)^n x^{n-1} + fy\left(x^n - i^n y^n\right),\\
Q &:= x^2y^2 - q,
\end{aligned}$$

the syzygy matrix is

$$\begin{pmatrix}
dy & 0 & x & \alpha & 0 & i^n dy^{n-1}\\
0 & -i^n dx & y & 0 & \alpha & dx^{n-1}\\
-1 + dxy & 0 & x^2 & 0 & 0 & (bd - i^n)y^{n-2} - i^n dxy^{n-1}\\
0 & -i^n\left(1 + dxy\right) & y^2 & 0 & 0 & ((-i)^n bd - 1)x^{n-2} + dyx^{n-1}\\
0 & 0 & (-i)^n e & y & -x & 0
\end{pmatrix}$$

where $\alpha = -1 + (-i)^n bd/2$. The relations between the parameters are

$$\begin{array}{ll} c = -i^n q + bdq, & b = dc - i^n dq. \\ f = -(-i)^n de/2 & 0 = e - dc - (-i)^n bde/2 + i^n dq. \end{array} \tag{2.12}$$

Proposition 2.21 *Let I be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.12. Then Γ_{D_-} forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

Proof. This is proved using the same techniques as the proof of Proposition 2.15. □

Thus this affine subset is the smooth surface in $\mathbb{C}^4_{b,d,e,q}$ given by the equations

$$b + 2i^n dq - bd^2 q = e - bd^2 q - (-i)^n bde/2 + 2i^n dq = 0.$$

Proposition 2.22 $\mathrm{Soc}(I)$ *contains at most the submodule $\langle xy(x^n - i^n y^n)\rangle$. Moreover $\langle xy(x^n - i^n y^n)\rangle \cong M_- \subset \mathrm{Soc}(I)$ if and only if $q = 0$.*

Proof. The proof is similar to the proof of Proposition 2.14. □

If k is odd then $u = 2(-i^n + bd)q^{(n+1)/2}$, $v = 2b(q)^{(n-1)/2}$ and $w = q$. If k is even then $u = 2b(q)^{n/2}$, $v = 2(-i^n + bd)q^{n/2}$ and $w = q$. Therefore $q = 0$ defines the exceptional locus on this affine subset.

If $q = 0$ then d is the coordinate along this exceptional curve. Setting $q = 0$ and $d \to \infty$ gives the ideal that is described in Case E$_-$ when all the parameters are zero.

2.5.5 Case E

If I is an ideal that defines a G-cluster and $\Gamma_{E_+} \in \Gamma(I)$ then I is generated by the polynomials of the form

$$\begin{aligned}
B &:= x\left(x^n - i^n y^n\right) - by^{n-1} - cx\left(x^n + i^n y^n\right),\\
C &:= y\left(x^n - i^n y^n\right) - (-i)^n b x^{n-1} + cy\left(x^n + i^n y^n\right),\\
D &:= xy\left(x^n + i^n y^n\right) - d\left(x^n - i^n y^n\right),\\
E &:= x^{n+2} - ey^{n-2} - fxy^{n-1},\\
F &:= y^{n+2} - (-1)^n e x^{n-2} - (-1)^{n-1} x^{n-1} y,\\
Q &:= x^2y^2 - q.
\end{aligned}$$

The syzygy matrix is

$$\begin{pmatrix}
y & -x & 2c & 0 & 0 & 0\\
y^2 & xy+2d & 2y & 0 & 0 & -4x^{n-1}\\
0 & 0 & (-i)^n f & y^2 & -i^n x^2 & -(x^n - i^n y^n)\\
0 & y & 0 & 0 & i^n(1-c) & -(1+c)x^{n-2}\\
x & 0 & 0 & -(1-c) & 0 & i^n(1+c)y^{n-2}
\end{pmatrix}$$

and corresponds to the relations

$$\begin{aligned}
b &= 2i^n cd, & q &= cd^2\\
0 &= -e - i^n q + df & 0 &= -f + b + cf
\end{aligned} \tag{2.13}$$

between the parameters.

Proposition 2.23 *Let I be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.13. Then Γ_{E_+} forms a basis for $\mathbb{C}[x,y]/I$ and so I defines a G-cluster.*

Proof. This is proved using the same techniques as the proof of Proposition 2.15. □

This affine open set is isomorphic to a smooth surface in $\mathbb{C}^3_{c,d,f}$ given by the equation

$$-f(1-c)+2i^n cd=0.$$

Notice that on the open set given by $c\neq 1$ then f can be expressed as a regular function of c and d. Therefore G-Hilb $\mathbb{C}^2$ contains the open set given by $c\neq 1$ in $\mathbb{C}^2_{c,d}$. Thus G-Hilb $\mathbb{C}^2$ is rational.

Proposition 2.24 $\mathrm{Soc}(I)$ *contains at most the submodules isomorphic to* V_{n-2}*,* V_{n-1} *and* M_+*. Moreover* $V_{n-2}\subset \mathrm{Soc}(I)$ *if and only if* $c=1$*,* $V_{n-1}\subset \mathrm{Soc}(I)$ *if and only if* $d=0$ *and* $M_+\subset \mathrm{Soc}(I)$ *if and only if* $c=0$*.*

PROOF. The proof that $\mathrm{Soc}(I)$ contains at most the submodules isomorphic to V_{n-2}, V_{n-1} and M_+ is similar to the proof of Proposition 2.14.

If $d=0$ then it follows that $x^{n+2}(1+c)=y^{n+2}(1+c)=0$ and so if $c\neq -1$ the submodule $\langle y\left(x^n+i^ny^n\right), x\left(x^n+i^ny^n\right)\rangle\cong V_{n-1}$ is in $\mathrm{Soc}(I)$. If $c=-1$ then we have the polynomials $2x^{n+1}-by^{n-1}$ and $2y^{n+1}-b(-1)^nx^{n-1}$ in I and when $d=0$ the submodule $\langle yx^n, xy^n\rangle\cong V_{n-1}$ is in $\mathrm{Soc}(I)$. The point $(c,d)=(-1,0)$ is the ideal in Case E_0 given by setting every polynomial that is not in Γ_{E_0} equal to zero. Therefore if $d=0$ then V_{n-1} is in $\mathrm{Soc}(I)$.

If $c=0$ then the submodule $\langle x^n-i^ny^n\rangle\cong M_+$ is in $\mathrm{Soc}(I)$.

If $c=1$ then $d=b=q=0$. Therefore $x^2y^{n-1}=x^{n-1}y^2=xy^n=x^ny=0$ and so $\langle x^{n-1}y, xy^{n-1}\rangle\cong V_{n-2}\in \mathrm{Soc}(I)$. $\square$

If n is even then $u=2c^{n/2}d^ne$, $v=2c^{(n-2)/2}d^{n-2}e$ and $w=cd^2$ and if n is odd $u=2c^{(n-1)/2}d^{n-1}e$, $v=2c^{(n-1)/2}d^{n-1}e$ and $w=cd^2$. Either way the exceptional locus on this affine subset is defined by cd^2. This is the union of the three prime divisors $c=0$, $d=0$ and $c=1$.

If $c=0$ then d is the coordinate along this exceptional curve. Setting $c=0$ and letting $d\to\infty$ gives the ideal in Case D_+ where all the parameters are zero. If $d = 0$ then c is the coordinate along this exceptional curve. Setting $d = 0$ and letting $c \to \infty$ gives the ideal in Case E_- when all of the parameters are zero. If $c = 1$ then f is the coordinate along this exceptional curve and letting $f \to \infty$ gives the ideal in Case C_{n-2} when all of the parameters are zero.

An ideal I which defines a G-cluster and $\Gamma_{E_-} \in \Gamma(I)$, is generated by the polynomials

$$\begin{aligned}
&B := x\left(x^n + i^n y^n\right) - by^{n-1} - cx\left(x^n - i^n y^n\right),\\
&C := y\left(x^n + i^n y^n\right) + (-1)^n bx^{n-1} + cy\left(x^n - i^n y^n\right),\\
&D := xy\left(x^n - i^n y^n\right) - d\left(x^n + i^n y^n\right),\\
&E := x^{n+2} - ey^{n-2} - fxy^{n-1},\\
&F := y^{n+2} - (-1)^n ex^{n-2} - (-1)^{n-1} x^{n-1} y,\\
&Q := x^2y^2 - q.
\end{aligned}$$

The syzygy matrix is

$$\begin{pmatrix}
y & -x & 2c & 0 & 0 & 0\\
y^2 & xy+2d & 0 & 0 & 2y & -4x^{n-1}\\
0 & 0 & -(-i)^n f & y^2 & i^n x^2 & -(x^n + i^n y^n)\\
0 & y & 0 & 0 & -i^n(1-c) & (1+c)x^{n-2}\\
x & 0 & 0 & -(1-c) & 0 & -(1+c)i^n y^{n-2}
\end{pmatrix}$$

and corresponds to the relations

$$\begin{array}{ll}
b = -2i^n cd, & q = cd^2\\
e = i^n q + fd & 0 = -f + b + cf.
\end{array} \tag{2.14}$$

Proposition 2.25 *Let I be an ideal generated by $B, \ldots, F, Q$ where the parameters $b, \ldots, f, q$ are subject to the relations given in 2.14. Then Γ_{E_-} forms a basis for $\mathbb{C}[x, y]/I$ and so I defines a G-cluster.*

Proof. This is proved using the same techniques as the proof of Proposition 2.15. □

This affine open set is a smooth surface in $\mathbb{C}^3_{c,d,f}$ given by the equation

$$-f - 2i^n cd + cf = 0.$$

Notice that on the open set given by $c \neq 1$ f can be expressed as a regular function of c and d.

Proposition 2.26 $\mathrm{Soc}(I)$ *contains at most the submodules isomorphic to* V_{n-2}*,* V_{n-1} *and* M_-*. Moreover* $V_{n-2} \subset \mathrm{Soc}(I)$ *if and only if* $c = 1$*,* $V_{n-1} \subset \mathrm{Soc}(I)$ *if and only if* $d = 0$ *and* $M_- \subset \mathrm{Soc}(I)$ *if and only if* $c = 0$*.*

Proof. The proof is similar to the proof of Proposition 2.24. □

If n is even then $u = 2c^{n/2}d^n e$, $v = 2c^{(n-2)/2}d^{n-2}e$ and $w = cd^2$ and if n is odd $u = 2c^{(n-1)/2}d^{n-1}e$, $v = 2ec^{(n-1)/2}d^{n-1}$ and $w = cd^2$. Either way the exceptional locus is given by $cd^2 = 0$ this is the union of the three prime divisors $c = 0$, $d = 0$ or $c = 1$.

If $c = 0$ then d is the coordinate along this exceptional curve. Setting $c = 0$ and letting $d \to \infty$ gives the ideal in Case D_- where all the parameters are zero. If $d = 0$ then c is the coordinate along this exceptional curve. Setting $d = 0$ and letting $c \to \infty$ gives the ideal in Case E_+ when all of the parameters are zero. If $c = 1$ then f is the coordinate along this exceptional curve and letting $f \to \infty$ gives the ideal in Case C_{n-2} when all of the parameters are zero.

2.5.6 Conclusion

We have constructed an affine cover of G-Hilb $\mathbb{C}^2$ where every open set is a smooth surface in $\mathbb{C}^4$. Two of the open sets contain open subsets of $\mathbb{C}^2$

and so G-Hilb $\mathbb{C}^2$ is rational. For any ideal I that defines a G-cluster $\mathrm{Soc}(I)$ contains at most two irreducible representations of G. Therefore we have proved Parts 2 and 3 of Theorem 2.4. Theorem 2.10 proves Part 1 and so Theorem 2.4 is proved.