Investigating the sporadic finite subgroups of SL(3,C) and their McKay quivers

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Background

- Representations, G-modules, and characters for a (finite) group G
- Irreducible characters form an orthonormal basis for class functions CF(G) with sesquilinear form on C[G]:

$$(f_1, f_2)_G = \frac{1}{|G|} \sum_{x \in G} f_1(x) \overline{f_2(x)}$$

- #Irreducible characters = #Conjugacy Classes

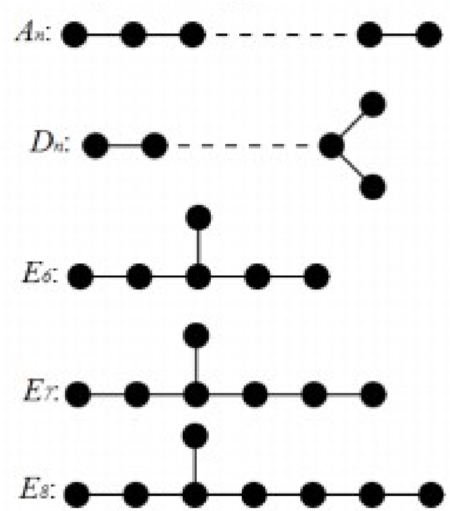
Motivation

Definition A *quiver* is a directed graph; a collection of vertices with arrows between them instead of undirected edges. The *McKay quiver* of a group G with a given representation Q is the quiver with a vertex for each irreducible representation ρ of G and having the number of arrows between ρ and ρ' as the multiplicity with which ρ' occurs in the decomposition of $\rho \otimes Q$ into irreducibles. This is $\dim \mathrm{Hom}_G(\rho', \rho \otimes Q)$.

Theorem 2.3. *(McKay correspondence in 2D) The finite subgroups of* $\mathrm{SL}(2, \mathbb{C})$ *are in bijective correspondence with the Dynkin diagrams* A_n, D_n, E_6, E_7, E_8 *via* $G \rightsquigarrow$ *the Du Val singularity* $\mathbb{C}^2/G$*. The corresponding resolution diagram (that is, the appropriate Dynkin diagram) is the truncated McKay quiver of* G*.*

McKay Correspondence in 2D

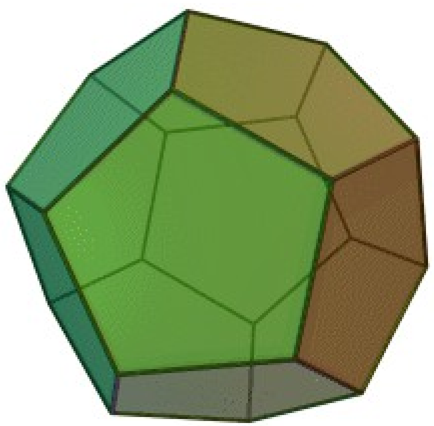
Dynkin Diagrams	Finite Subgroup of $SL(2,\mathbb{C})$	
A_n	Cyclic	
D_n	Dihedral	BD_{4n}
E_6	Tetrahedral	BT_{24}
E_7	Octahedral	BO_{48}
E_8	Icosahedral	BI_{120}

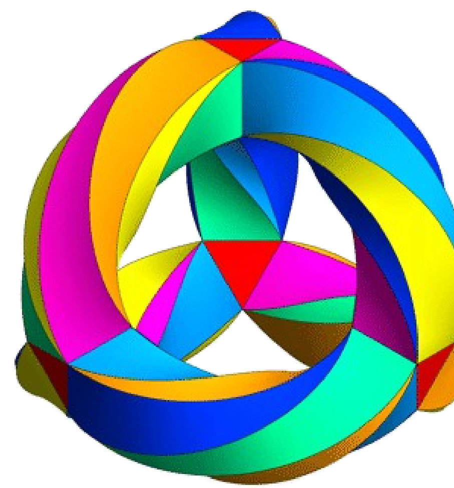


[1] [2]

Idea

- To learn about the finite subgroups of SL(3,C) and compute their McKay quivers





[3]

Literature

- A classification of the finite subgroups of SL(3,C) can be found in [Yau-Yu, 1993][4] or [Carrasco, 2014][2]
- There are four infinite families and eight exceptional cases. Explicit generators are provided.

- (A) diagonal Abelian groups;
- (B) groups coming from finite subgroups of $\mathrm{GL}(2,\mathbb{C})$;
- (C) groups generated by (A) and T;
- (D) groups generated by (C) and Q;

- (E) group of order 108 generated by S, T, V;
- (F) group of order 215 generated by (E) and $P = UVU^{-1}$;
- (G) Hessian group of order 648 generated by (E) and U;
- (H) group of order 60 isomorphic to the alternating group of degree five, A_5;
- (I) group of order 168 isomorphic to the permutation group generated by $(1234567), (142)(356), (12)(35)$;
- (L) group G of order 1080 whose quotient G/F is isomorphic to alternating group A_6;

where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, V = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix}, P = \frac{1}{i\sqrt{3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}, Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix},$$

$$F = \left\{ I_3, \omega I_3, \omega^2 I_3 \right\},$$

and $abc = -1, \omega = \exp\frac{2\pi i}{3}, \varepsilon = \left(\exp\frac{2\pi i}{9}\right)^2 = \exp\frac{4\pi i}{9}$.

But they missed two types of subgroups which were given by Yau and Yu in [YY, 1993] in 1993, completing the classification (up to conjugacy):

- (J) Group of order 180 generated by (H) and F;
- (K) Group of order 504 generated by (I) and F.

[2]

Sporadic Cases

or UVU^{-1} in the cases considered. We therefore have the following types:

(E) Group of order 108 generated by S_1, T, of (C) and V of (3).

(F) Group of order 216 generated by S_1, T, V and UVU^{-1} of (3).

(G) Group of order 648 generated by S_1, T, V and U of (3).

These groups are all primitive, and they all contain (D) as a normal subgroup (in fact (C) $\triangleleft$ (D) $\triangleleft$ (E) $\triangleleft$ (F) $\triangleleft$ (G)). The group (G) is called the Hessian group.

[4]

Order	3	27	54	108	216	648
Group	(A)	(C)	(D)	(E)	(F)	(G)

$$(A) \trianglelefteq (C) \trianglelefteq (D) \trianglelefteq (E) \trianglelefteq (F) \trianglelefteq (G)$$

60 180

$$(H) \trianglelefteq (J)$$

168 504

$$(I) \trianglelefteq (K)$$

60 1080

$$(H) \leq (L)$$

Methodology

Definition A *quiver* is a directed graph; a collection of vertices with arrows between them instead of undirected edges. The *McKay quiver* of a group G with a given representation Q is the quiver with a vertex for each irreducible representation ρ of G and having the number of arrows between ρ and ρ' as the multiplicity with which ρ' occurs in the decomposition of $\rho \otimes Q$ into irreducibles. This is $\dim \mathrm{Hom}_G(\rho', \rho \otimes Q)$.

[1]

- Tensor product of modules/representations is simply pointwise product for characters

$$\dim \mathrm{Hom}_G(\rho_j, \rho_i \otimes Q) = (\chi_j, \chi_i . \chi_Q)_G$$

- Character method seems most accessible
- Found character tables for simple groups (H), (I). [5]

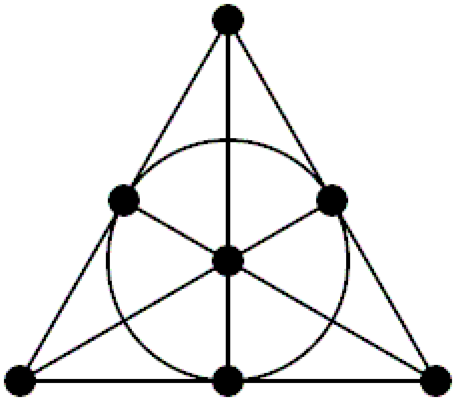
Enter Magma

- Implementing the exceptional cases in Magma
- Ambiguity in [YY] arising from [Mi-Bl-Di] typo
- Kronecker – Weber Theorem

```
// Groups I, K as per Yau, Yu.
K21<m> := CyclotomicField(21);
// Constants om = exp(2*pi*i/3), be = exp(2*pi*i/7)
om := m^7;
be := m^3;
GL3K := GeneralLinearGroup(3, K21);

// Generators
s := GL3K ! [be,0,0, 0,be^2,0, 0,0,be^4];
t := GL3K ! [0,1,0, 0,0,1, 1,0,0];
// Constant c = -1/sqrt(-7)
c := 1/7 * (be + be^2 + be^4 - be^3 - be^5 - be^6);
r := GL3K ! [c*(be^4 - be^3),c*(be^2 - be^5),c*(be -
w := GL3K ! [om,0,0, 0,om,0, 0,0,om];

// Simple group of order 168
I := MatrixGroup< 3, K21 | [s,t,r] >;
// Group of order 504
K := MatrixGroup< 3, K21 | I,w >;
```



Working with Magma

- Character Tables and Tensor Decomposition

```
---------------------------------------
Character Table of Group E of Order 108
-------------------------------------------------------------------
Class |    1  2     3     4  5  6  7  8    9  10    11    12    13    14
Size  |    1  9     1     1 12 12  9  9    9   9     9     9     9     9
Order |    1  2     3     3  3  3  4  4    6   6    12    12    12    12
-------------------------------------------------------------------
p  =  2    1  1     4     3  5  6  2  2    3   4    10     9     9    10
p  =  3    1  2     1     1  1  1  8  7    2   2     7     8     7     8
-------------------------------------------------------------------
X.1   +    1  1     1     1  1  1  1  1    1   1     1     1     1     1
X.2   +    1  1     1     1  1  1 -1 -1    1   1    -1    -1    -1    -1
X.3   0    1 -1     1     1  1  1  I -I   -1  -1    -I     I    -I     I
X.4   0    1 -1     1     1  1  1 -I  I   -1  -1     I    -I     I    -I
X.5   0    3 -1  3*J-3-3*J  0  0 -1 -1 1+J  -J   1+J    -J    -J   1+J
X.6   0    3 -1-3-3*J  3*J  0  0 -1 -1  -J 1+J    -J   1+J   1+J    -J
X.7   0    3 -1-3-3*J  3*J  0  0  1  1  -J 1+J     J  -1-J  -1-J     J
X.8   0    3 -1  3*J-3-3*J  0  0  1  1 1+J  -J  -1-J     J     J  -1-J
X.9   0    3  1-3-3*J  3*J  0  0 -I  I   J-1-J    Z1-Z1#5  Z1#5   -Z1
X.10  0    3  1  3*J-3-3*J  0  0 -I  I-1-J   J  Z1#5   -Z1   Z1-Z1#5
X.11  0    3  1-3-3*J  3*J  0  0  I -I   J-1-J   -Z1  Z1#5-Z1#5    Z1
X.12  0    3  1  3*J-3-3*J  0  0  I -I-1-J   J-Z1#5    Z1   -Z1  Z1#5
X.13  +    4  0     4     4  1 -2  0  0    0   0     0     0     0     0
X.14  +    4  0     4     4 -2  1  0  0    0   0     0     0     0     0

Explanation of Character Value Symbols
---------------------------------------
J = RootOfUnity(3)
I = RootOfUnity(4)
Z1     = (CyclotomicField(12: Sparse := true)) ! [ RationalField() | 0, 0, 0, 1]
```

```
// Simple group of order 60
H := MatrixGroup< 3, K15 | [s,u,t] >;

CT := CharacterTable(H);
Decomposition(CT, CT[2]*CT[2]);
Decomposition(CT, CT[2]*CT[3]);
Decomposition(CT, CT[2]*CT[4]);
Decomposition(CT, CT[2]*CT[5]);

Decomposition(CT, CT[3]*CT[3]);
Decomposition(CT, CT[3]*CT[4]);
Decomposition(CT, CT[3]*CT[5]);
```

```
[
    1,
    1,
    0,
    0,
    1
]
( 0, 0, 0, 0, 0 )
[
    0,
    0,
    0,
    1,
    1
]
( 0, 0, 0, 0, 0 )
```

Working with Magma

- Tabulation of Data in Excel, some shortcuts

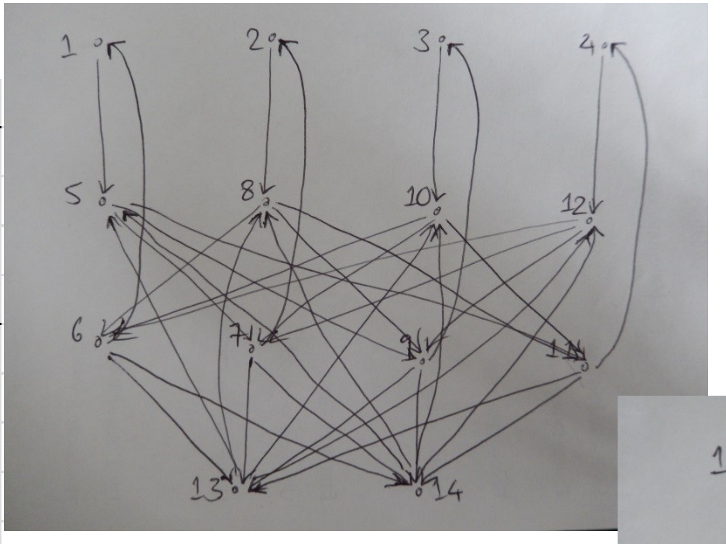
	A	B	C	D	E	F	G	H
1	Group E	X1	X2	X3	X4	X5	X6	X7
2	X1	X1	X2	X3	X4	X5	X6	X7
3	X2	X2	X1	X4	X3	X8	X7	X6
4	X3	X3	X4	X2	X1	X10	X9	X11
5	X4	X4	X3	X1	X2	X12	X11	X9
6	X5	X5	X8	X10	X12	X7 + X9 + X11	X1 + X13 + X14	X2 + X13 + X14
7	X6	X6	X7	X9	X11	X1 + X13 + X14	X8 + X10 + X12	X5 + X10 + X12
8	X7	X7	X6	X11	X9	X2 + X13 + X14	X5 + X10 + X12	X8 + X10 + X12
9	X8	X8	X5	X12	X10	X6 + X9 + X11	X2 + X13 + X14	X1 + X13 + X14
10	X9	X9	X11	X7	X6	X3 + X13 + X14	X5 + X8 + X12	X5 + X8 + X10
11	X10	X10	X12	X8	X5	X6 + X7 + X11	X3 + X13 + X14	X4 + X13 + X14
12	X11	X11	X9	X6	X7	X4 + X13 + X14	X5 + X8 + X10	X5 + X8 + X12
13	X12	X12	X10	X5	X8	X6 + X7 + X9	X4 + X13 + X14	X3 + X13 + X14
14	X13	X13	X13	X13	X13	X5 + X8 + X10 + X12	X6 + X7 + X9 + X11	X6 + X7 + X9 + X11
15	X14	X14	X14	X14	X14	X5 + X8 + X10 + X12	X6 + X7 + X9 + X11	X6 + X7 + X9 + X11
16								
17	Group F	X1	X2	X3	X4	X5	X6	X7
18	X1	X1	X2	X3	X4	X5	X6	X7
19	X2	X2	X1	X4	X3	X5	X8	X9
20	X3	X3	X4	X1	X2	X5	X7	X6
21	X4	X4	X3	X2	X1	X5	X9	X8
22	X5	X5	X5	X5	X5	X1 + X2 + X3 + X4	X14	X14
23	X6	X6	X8	X7	X9	X14	X10 + X15	X11 + X15
24	X7	X7	X9	X6	X8	X14	X11 + X15	X10 + X15
25	X8	X8	X6	X9	X7	X14	X13 + X15	X12 + X15

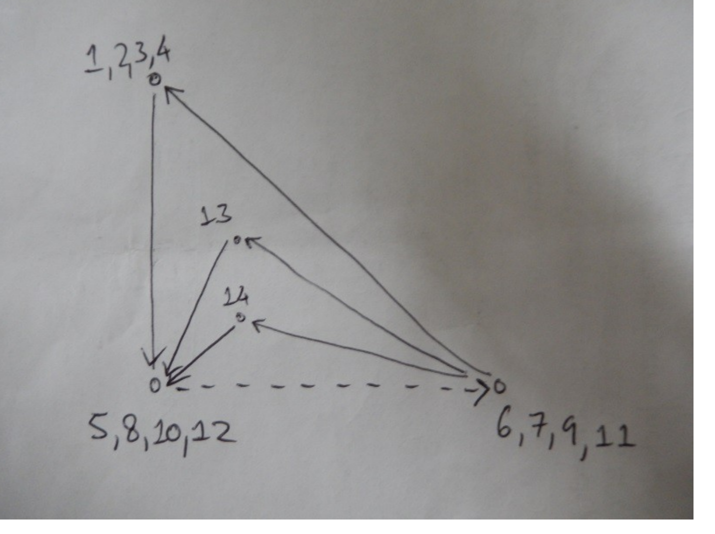
E,F,G H,J H,L I,K List Sheet1

Results

- McKay Quiver where Q is the inclusion of G into SL(3,C)

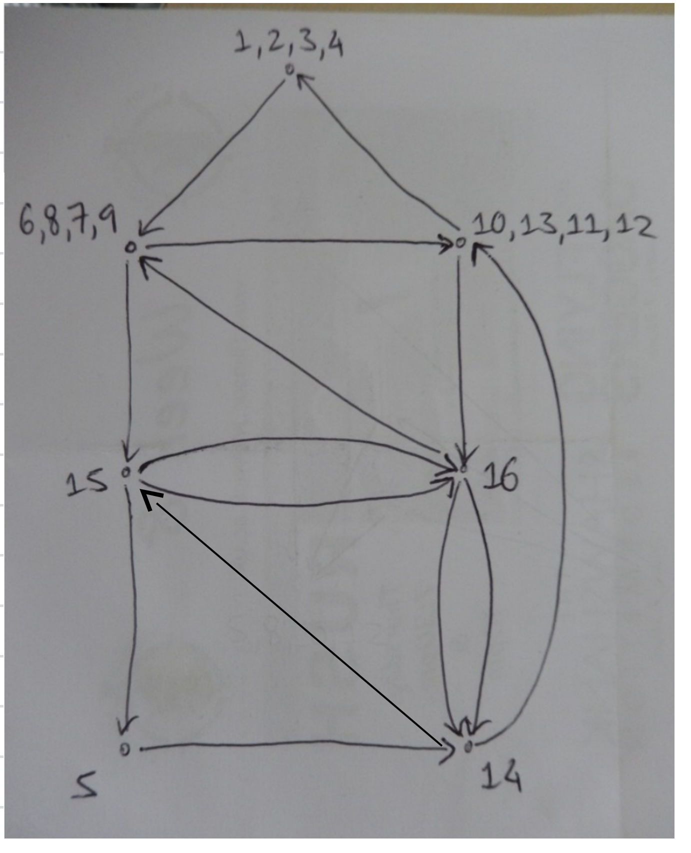
	Group E	X5
1	X1	X5
1	X2	X8
1	X3	X10
1	X4	X12
3	X5	X7 + X9 + X11
3	X6	X1 + X13 + X14
3	X7	X2 + X13 + X14
3	X8	X6 + X9 + X11
3	X9	X3 + X13 + X14
3	X10	X6 + X7 + X11
3	X11	X4 + X13 + X14
3	X12	X6 + X7 + X9
4	X13	X5 + X8 + X10 + X12
4	X14	X5 + X8 + X10 + X12





Results

	Group F	X6
1	X1	X6
1	X2	X8
1	X3	X7
1	X4	X9
2	X5	X14
3	X6	X10 + X15
3	X7	X11 + X15
3	X8	X13 + X15
3	X9	X12 + X15
3	X10	X1 + X16
3	X11	X3 + X16
3	X12	X4 + X16
3	X13	X2 + X16
6	X14	X10 + X11 + X12 + X13 + X15
6	X15	X5 + 2X16
8	X16	X6 + X7 + X8 + X9 + 2X14



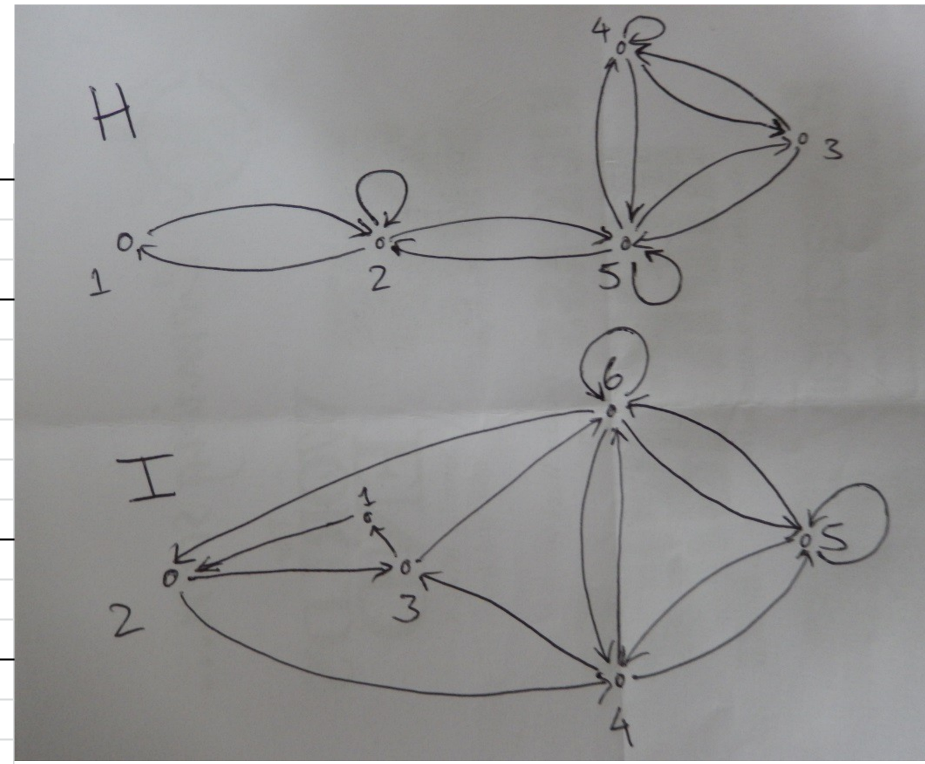
Results

	Group H	X2
1	X1	X2
3	X2	X1 + X2 + X5
3	X3	X4 + X5
4	X4	X3 + X4 + X5
5	X5	X2 + X3 + X4 + X5

	Group I	X2
1	X1	X2
3	X2	X3 + X4
3	X3	X1 + X6
6	X4	X3 + X5 + X6
7	X5	X4 + X5 + X6
8	X6	X2 + X4 + X5 + X6

	Group J	X4
1	X1	X4
1	X2	X6
1	X3	X8
3	X4	X1 + X4 + X13
3	X5	X10 + X13
3	X6	X2 + X6 + X14
3	X7	X12 + X15
3	X8	X3 + X8 + X15
3	X9	X11 + X14
4	X10	X5 + X10 + X13
4	X11	X9 + X11 + X14
4	X12	X7 + X12 + X15
5	X13	X4 + X5 + X10 + X13
5	X14	X6 + X9 + X11 + X14
5	X15	X7 + X8 + X12 + X15

	Group K	X4
1	X1	X4
1	X2	X7
1	X3	X6
3	X4	X5 + X10
3	X5	X1 + X16
3	X6	X8 + X12
3	X7	X9 + X11
3	X8	X3 + X17
3	X9	X2 + X18
6	X10	X5 + X13 + X16
6	X11	X9 + X15 + X18
6	X12	X8 + X14 + X17
7	X13	X10 + X13 + X16
7	X14	X12 + X14 + X17
7	X15	X11 + X15 + X18
8	X16	X4 + X10 + X13 + X16
8	X17	X6 + X12 + X14 + X17
8	X18	X7 + X11 + X15 + X18



Observations

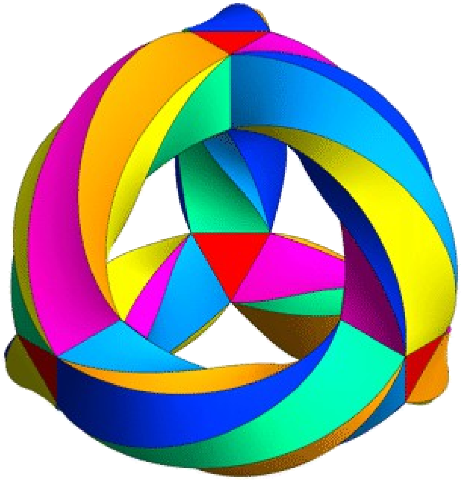
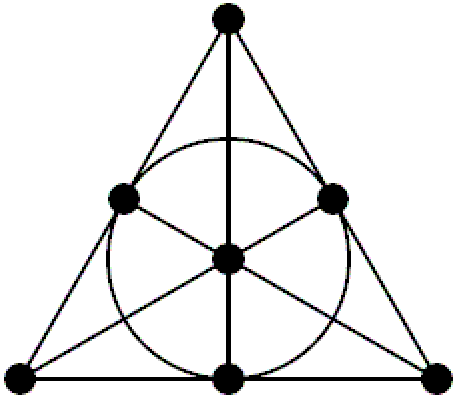
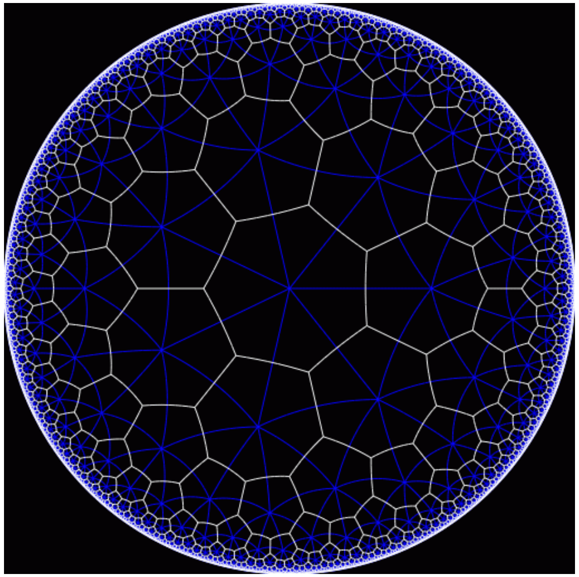
- Linear (degree 1) characters form a group which acts on the set of irreducible characters
- Inclusion representation was used for quivers, but the quivers by the dual or twists of the chosen representation are isomorphic
- High levels of symmetry in most cases
- Tripling in the cases (H,J) and (I,K)

Potential Future Directions

- Presenting difficult quivers in a meaningful way
- Understanding quivers at the level of representations/modules
- Further study into the four infinite families
- Significance of orbits by linear character action
- Significance of (normal) (abelian) subgroups on quivers (w.r.t. induction)
- McKay Quiver generator program for groups with known character table

Conclusion

- Independent research
- Working with Magma
- Relations between different areas of research

References

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