# How to calculate $A$-Hilb $\mathbb{C}^3$

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**Abstract**

Nakamura [N] introduced the $G$-Hilbert scheme $G$-Hilb $\mathbb{C}^3$ for a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$, and conjectured that it is a crepant resolution of the quotient $\mathbb{C}^3/G$. He proved this for a diagonal Abelian group $A$ by introducing an algorithm that calculates $A$-Hilb $\mathbb{C}^3$ explicitly. This note calculates $A$-Hilb $\mathbb{C}^3$ much more simply, in terms of fun with continued fractions plus regular tesselations by equilateral triangles.

# 1 Statement of the result

## 1.1 The junior simplex and three Newton polygons

Let $A \subset \mathrm{SL}(3,\mathbb{C})$ be a diagonal subgroup acting on $\mathbb{C}^3$. Write $L \supset \mathbb{Z}^3$ for the overlattice generated by all the elements of $A$ written in the form $\frac{1}{r}(a_1, a_2, a_3)$. The junior simplex $\Delta$ (compare [IR], [R]) has 3 vertexes

$$e_1 = (1,0,0), \quad e_2 = (0,1,0) \quad \text{and} \quad e_3 = (0,0,1).$$

Write $\mathbb{R}^2_\Delta$ for the affine plane spanned by $\Delta$, and $\mathbb{Z}^2_\Delta = L \cap \mathbb{R}^2_\Delta$ for the corresponding affine lattice. Taking each $e_i$ in turn as origin, construct the Newton polygons obtained as the convex hull of the lattice points in $\Delta \setminus e_i$ (see Figure 1.a):

$$f_{i,0}, f_{i,1}, f_{i,2}, \dots, f_{i,k_i+1}, \tag{1.1}$$

where $f_{i,0}$ is the primitive vector along the side $[e_i, e_{i-1}]$, and $f_{i,k_i+1}$ that along $[e_i, e_{i+1}]$. (The indices $i, i \pm 1$ are cyclic. Also, since $e_i$ is the origin, the notation $f_{i,j}$ denotes both the lattice point of $\Delta$ and the corresponding vector $e_i f_{i,j}$.) The vectors $f_{i,j}$ out of $e_i$ are subject to the Jung–Hirzebruch continued fraction rule:

$$f_{i,j-1} + f_{i,j+1} = a_{i,j} \cdot f_{i,j} \quad \text{for } j = 1, \dots, k_i, \tag{1.2}$$

where $a_{i,j} \geq 2$. Here $\frac{r_i}{\alpha_i} = [a_{i,1}, \dots, a_{i,k_i}]$ comes from expressing $\mathbb{Z}^2_\Delta$ in terms of the cone at $e_i$, writing

$$\mathbb{Z}^2_\Delta = \mathbb{Z}^2(f_{i,0}, f_{i,k_i+1}) + \mathbb{Z} \cdot f_{i,1} = \mathbb{Z}^2 + \mathbb{Z} \cdot \tfrac{1}{r_i}(\alpha_i, 1),$$

with $\alpha_i < r$ and coprime to $r$. Write $L_{ij}$ for the line out of $e_i$ extending or equal to the initial segment $[e_i, f_{ij}]$ (*line* is *line segment* throughout). The resulting fan at $e_i$ corresponds to the Jung–Hirzebruch resolution of the surface singularity $\mathbb{C}^2_{(x_i=0)}/A$. The picture so far is the simplex $\Delta$ with a number of lines $L_{ij}$ growing out of each of the 3 vertexes (Figure 1.a).

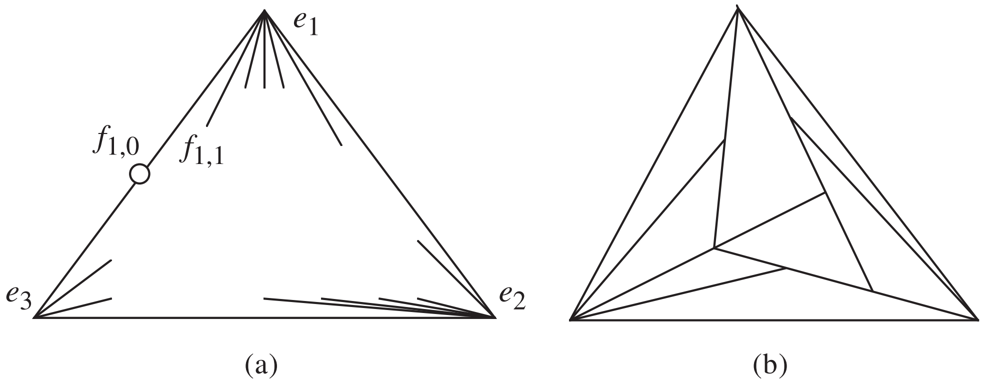


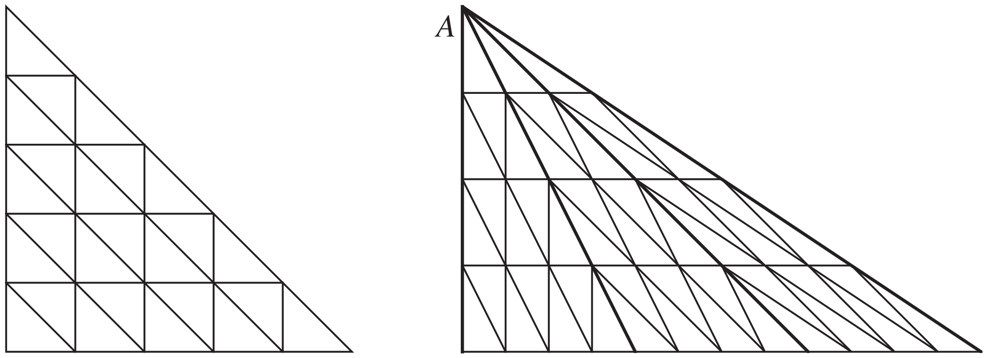

Figure 1: (a) Three Newton polygons; (b) subdivision into regular triangles

## 1.2 Regular triangles

Write $\mathbb{Z}^2$ for the group of translations of the affine lattice $\mathbb{Z}^2_\Delta$. A *regular triple* is a set of three vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$, any two of which form a basis of $\mathbb{Z}^2$, and such that $\pm v_1 \pm v_2 \pm v_3 = 0$. (The standard regular triple is $\pm(1,0), \pm(0,1), \pm(1,1)$; it appears all over elementary toric geometry, for example, as the fan of $\mathbb{P}^2$ or the blowup of $\mathbb{A}^2$.) We are only concerned with regular triples among the vectors $f_{i,j}$ introduced in 1.1.

As usual, a *lattice triangle* $T$ is a triangle $T \subset \mathbb{R}^2_\Delta$ with vertexes in $\mathbb{Z}^2_\Delta$. We say that $T$ is a *regular triangle* if each of its sides is a line $L_{ij}$ extending some $[e_i, f_{i,j}]$ and the 3 primitive vectors $v_1, v_2, v_3 \in \mathbb{Z}^2$ pointing along its sides form a regular triple.

It is easy to see that a regular triangle $T$ is affine equivalent to the triangle with vertexes $(0,0), (r,0), (0,r)$ for some $r \geq 1$, called the *side* of $T$. Its *regular tesselation* is that shown in Figure 2.a: a regular triangle of side $r$ subdivides into $r^2$ basic triangles with sides parallel to $v_1, v_2, v_3$.




(a) (b)

Figure 2: (a) A 5-regular triangle; (b) a $(4,12)$-semiregular triangle (see 2.8.3)

A regular triangle is the thing you get as the junior simplex for the group

$$A = \mathbb{Z}/r \oplus \mathbb{Z}/r = \left\langle \frac{1}{r}(1,-1,0), \frac{1}{r}(0,1,-1), \frac{1}{r}(-1,0,1) \right\rangle \subset \mathrm{SL}(3,\mathbb{C})$$

(the maximal diagonal subgroup of exponent $r$). The tesselation consists of basic triangles with vertexes in $\Delta$, so corresponds to a crepant resolution of the quotient singularity. It is known (see 3.2 below and [R], Example 2.2) that in this case $A$-Hilb $\mathbb{C}^3$ is the toric variety associated with its regular tesselation.

## 1.3 The main result

**Theorem 1.1** *The regular triangles partition the junior simplex $\Delta$.*

Section 2 gives an easy continued fraction procedure determining the partition; Figure 1.b illustrates the rough idea, and worked examples are given in 2.6 below[1] (see Figures 6–8).

[1] Homework sheets are on the lecturer's website www.maths.warwick.ac.uk/~miles.

**Theorem 1.2** *Let $\Sigma$ denote the toric fan obtained by taking the regular tesselation of all regular triangles in the junior simplex $\Delta$ as described in 1.2. The associated toric variety $Y_\Sigma$ is Nakamura's $A$-Hilbert scheme $A\text{-Hilb}\,\mathbb{C}^3$.*

**Corollary 1.3 (Nakamura)** *$A\text{-Hilb}\,\mathbb{C}^3 \to \mathbb{C}^3/A$ is a crepant resolution.*

**Corollary 1.4** *Every compact exceptional surface in $A\text{-Hilb}\,\mathbb{C}^3$ is either $\mathbb{P}^2$, a scroll $\mathbb{F}_n$ or a scroll blown up in one or two points (including $\mathrm{dP}_6$, the del Pezzo surface of degree 6).*

## 1.4 Thanks

This note is largely a reworking of original ideas of Iku Nakamura, and MR had access over several years to his work in progress and early drafts of the preprint [N]. MR learned the continued fraction tricks here from Jan Stevens (in a quite different context). We are grateful to the organisers of two summer schools at Levico in May 1999 and Lisboa in July 1999 which stimulated our discussion of this material, and to Victor Batyrev for the question that we partially answer in 2.8.4.

## 1.5 Recent developments

There has been considerable progress in our understanding of the $G$-Hilbert scheme since the preprint of this article (math.AG/9909085). The most significant development is the work of Bridgeland, King and Reid [BKR], that used derived category and Fourier–Mukai transform methods to establish that $G\text{-Hilb}\ \mathbb{C}^3 \to \mathbb{C}^3/G$ is a crepant resolution for a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$, not necessarily Abelian. In fact [BKR] settled many of the outstanding issues concerning $G\text{-Hilb}\ \mathbb{C}^3$, at least in a formal sense; for instance, it established an isomorphism between the K theory of $G\text{-Hilb}\ \mathbb{C}^3$ and the representation ring of $G$, and settled the problem of the "dynamic" versus "algebraic" definition of $G\text{-Hilb}\ \mathbb{C}^3$ discussed in Section 4.1 below.

The explicit calculation of the fan $\Sigma$ of $A\text{-Hilb}\ \mathbb{C}^3$ introduced in the current article enabled AC to establish a geometric construction of the McKay correspondence [C2]: as conjectured in [R], a certain cookery with the Chern classes of the Gonzalez-Sprinberg and Verdier sheaves $\mathcal{F}_\rho$ leads to a $\mathbb{Z}$-basis of the cohomology $H^*(Y_\Sigma, \mathbb{Z})$ for which the bijection

$$\Big\{\text{irreducible representations of } A\Big\} \longleftrightarrow \text{basis of } H^*(Y_\Sigma, \mathbb{Z})$$

holds, with $Y_\Sigma = A$-Hilb $\mathbb{C}^3$ (see [C1] and [C2] for more details). Also, work in progress of Rebecca Leng [B] extends the explicit calculations in the current article to some non-Abelian subgroups of $\mathrm{SL}(3,\mathbb{C})$.

Our understanding of the construction of $G$-Hilb $\mathbb{C}^3$ as a variation of GIT quotient of $\mathbb{C}^3/G$ has also improved: as explained to us by Alastair King and Akira Ishii, $G$-Hilb $\mathbb{C}^3$ is just one of many moduli spaces of representations of the McKay quiver arising from different stability weightings in GIT. The method of proof of [BKR] proves that any "sufficiently general" stability weighting leads to a moduli space that is again a crepant resolution of $\mathbb{C}^3/G$. One expects these moduli spaces to provide every flop of $G$-Hilb $\mathbb{C}^3$. Work of Ishii and Craw (summarised in [C2], Chapter 5) opened the way to a toric calculation of moduli of representations of the McKay quiver (also called moduli of *$G$-constellations* to stress the link with $G$-clusters). As experimental evidence, [C2] proves that every flop of $G$-Hilb $\mathbb{C}^3$ is obtained as such a moduli space in the case $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ (see 1.2), and for the cyclic quotient singularities $\frac{1}{6}(1,2,3)$ and $\frac{1}{11}(1,2,8)$.

# 2 Concatenating continued fractions

## 2.1 Propellor with three blades

The key to Theorem 1.1 is the observation that easy games with continued fractions provide all the regular triples $v_1, v_2, v_3$ (see 1.2) among the vectors $f_{i,j}$. First translate the three Newton polygons at $e_1, e_2, e_3$ to a common vertex, to get the propellor shape of Figure 3, in which three hexants (the blades of the propellor) have convex basic subdivisions. The primitive vectors are read in cyclic order

$$f_{1,0}, f_{1,1}, \dots, f_{1,k}, f_{1,k+1} = -f_{2,0}, f_{2,1}, \quad \text{etc.}$$

Inverting any blade (that is, multiplying it by $-1$) makes the three hexants into a basic subdivision of a half-space. Taking plus or minus all three blades gives a basic subdivision of the plane invariant under $-1$.

## 2.2 Two complementary cones

This digression on well-known material (see for example [Rie], §3, pp. 220–3) illustrates several points. Let $L$ be a 2-dimensional lattice, and $e_1, e_2 \in L$

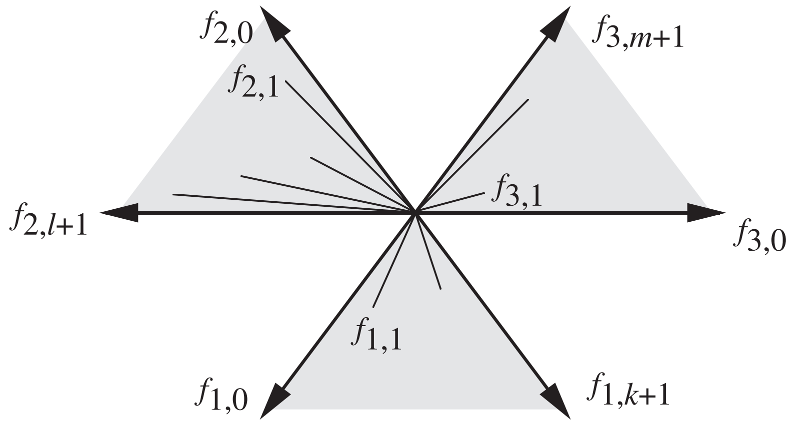



Figure 3: "Propellor" with three "blades"

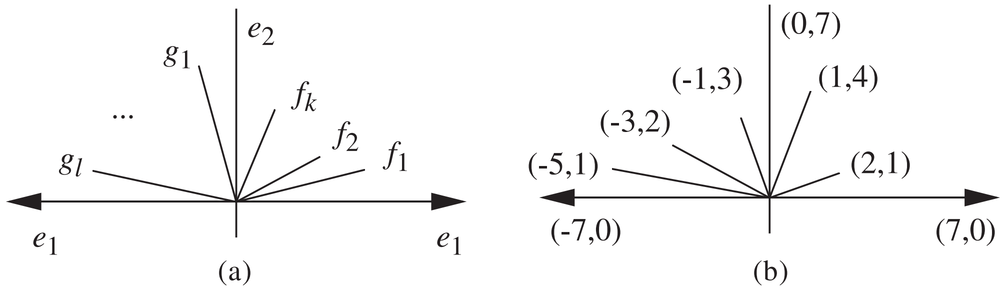



Figure 4: Complementary cones $\langle e_1, e_2\rangle$ and $\langle e_2, -e_1\rangle$

primitive vectors spanning a cone in $L_{\mathbb{R}}$. Then $\mathbb{Z}^2 = \mathbb{Z}\cdot e_1 + \mathbb{Z}\cdot e_2 \subset L$ is a sublattice with cyclic quotient $L/\mathbb{Z}^2 = \mathbb{Z}/r$; assume for the moment that $r > 1$. The *reduced* generator is $f_1 = \frac{1}{r}(\alpha, 1)$ with $1 \le \alpha < r$ and $\alpha, r$ coprime, so that $L = \mathbb{Z}^2 + \mathbb{Z}\cdot\frac{1}{r}(\alpha, 1)$. The continued fraction expansion $\frac{r}{\alpha} = [a_1, \dots, a_k]$ with $a_i \ge 2$ gives the convex basic subdivision $\langle e_1, f_1\rangle$, $\langle f_i, f_{i+1}\rangle$, $\langle f_k, e_2\rangle$ in the first quadrant of Figure 4.a.

Repeat the same construction for the cone $\langle e_2, -e_1\rangle$; for this, write the extra generator $\frac{1}{r}(\alpha, 1)$ as $\frac{1}{r}(\alpha e_2, (r-1)(-e_1))$. The reduced normal form is $\frac{1}{r}(1, \beta)$ with $\alpha\beta = (r-1) \bmod r$, or $\beta = 1/(r-\alpha) \bmod r$. The corresponding continued fraction $\frac{r}{\beta} = [b_1, \dots, b_l]$ gives the basic subdivision $e_2, g_1, \dots, g_l, -e_1$ in the top left quadrant of Figure 4.a. (In the literature, this is usually given as $\frac{r}{r-\alpha} = [b_l, \dots, b_1]$, but we want this cyclic order.)

Now the vectors $e_1, f_1, \dots, f_k, e_2, g_1, \dots, g_l, -e_1$ form a basic subdivision of the upper half-space of $L$. The whole trick is the trivial observation that this cannot be convex (downwards) everywhere, so that at $e_2$,

$$f_k + g_1 = ce_2 \quad \text{with } c \in \mathbb{Z} \text{ and } 0 \le c \le 1. \tag{2.1}$$

For vectors $f_k, g_1$ in the closed upper half-space, $c = 0$ is only possible if $f_k = e_1$ and $g_1 = -e_1$. Then $r = 1$; this is the "trivial case" with empty continued fractions, at which induction stops. Otherwise, $f_k + g_1 = e_2$. In view of this relation, put a 1 against $e_2$, and concatenate the two continued fractions as

$$[a_1, a_2, \dots, a_k, 1, b_1, \dots, b_l] \quad (= 0).$$

Because of the relation $e_2 = f_k + g_1$, the cone $\langle f_k, g_1 \rangle$ is also basic. Thus we can delete the vector $e_2$ and still have a basic subdivision of the upper half-space of $L$. A trivial calculation shows that in this subdivision, the newly adjacent vectors $f_{k-1}, f_k, g_1, g_2$ are related by

$$f_{k-1} + g_1 = (a_k - 1) f_k \quad \text{and} \quad f_k + g_2 = (b_1 - 1) g_1.$$

In other words, in the continued fraction we can replace

$$a_k, 1, b_1 \quad \text{by} \quad a_k - 1, b_1 - 1.$$

(The calculation can be seen as the matrix identity

$$\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a-1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & b-1 \end{pmatrix}.$$

The combinatorics is the same as a chain of rational curves on a surface with self-intersection the negatives of $a_1, a_2, \dots, a_k, 1, b_1, \dots, b_l$; deleting $e_2$ corresponds to "contracting" a $-1$-curve.)

Now it must be the case that at least one of $a_k - 1, b_1 - 1$ is again 1. Else the chain of vectors $e_1, f_1, \dots, f_k, g_1, \dots, g_l, -e_1$ is convex, which is absurd. If say $a_k = 2$ then consider the new cone $\langle e_1, f_k \rangle$.

Figure 4.b shows the example $\frac{1}{7}(1,2)$, where we get

$$[4, \underline{2, 1, 3}, 2, 2] \to [\underline{4, 1, 2}, 2, 2] \to [\underline{3, 1, 2}, 2] \to [\underline{2, 1, 2}] \to [1, 1]. \tag{2.2}$$

The steps express $(0,7), (1,4), (-1,3), (-3,2)$ as the sum of two neighbours. The end $[1,1]$ describes the relations

$$(2,1) = (7,0) + (-5,1) \quad \text{and} \quad (-5,1) = (2,1) + (-7,0)$$

among the final four vectors (this counts as one regular triple because we identify $\pm v$).

## 2.3 Remarks

1. In the trivial case $r = 1$ we have $c = 0$ in (2.1). There is always a 1 to contract. You always end up with $[1, 1] = 0$.

2. The regular triples $v_1, v_2, v_3$ among $e_1, f_1, \ldots, e_2, g_1, \ldots, -e_1$ correspond one-to-one with the 1's that occur during the chain of contractions, as we saw in Figure 4.b.

3. The order the vectors are contracted and the regular triples among them is determined in the course of an induction; but they might be tricky to decide a priori without running the algorithm.

4. The continued fractions keep track of successive change of basis between adjacent basic cones. Following $(e_1, f_1)$, $(f_1, f_2)$, etc. all the way around to $(g_l, -e_1)$, and on cyclically to $(-e_1, -f_1)$ gives

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & a_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \times \\ \times \begin{pmatrix} 0 & 1 \\ -1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & b_l \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

In what follows, we consider continued fractions concatened in this cyclic way. Then $[1, 1, 1]$ stands for $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which makes sense of the number $[1, 1, 1] = 1 - \frac{1}{0} = \infty$.

## 2.4 Long side

To concatenate the three continued fractions arising from the propellor of Figure 3 as a cyclic continued fraction, we study the change of basis from the last basis $f_{1,k}, f_{1,k+1}$ of the $e_1$ hexant to the first basis $f_{2,0}, f_{2,1}$ of the $e_2$ hexant. Clearly $f_{2,0} = -f_{1,k+1}$, and we claim there is a relation

$$f_{2,1} - f_{1,k} = c f_{2,0} \quad \text{with } c \geq 1. \tag{2.3}$$

Indeed, $-f_{1,k}, f_{2,0}$ and $f_{2,0}, f_{2,1}$ are two oriented bases (the usual argument).

We define the side $e_i e_{i+1}$ of the simplex $\Delta$ to be a *long side* if $c \geq 2$. See Figure 5. A long side $e_1 e_2$ is obviously not a primitive vector, so never occurs for "coprime" groups. The presence of a long side is a significant dichotomy in the construction (see Remark 2.8.2).

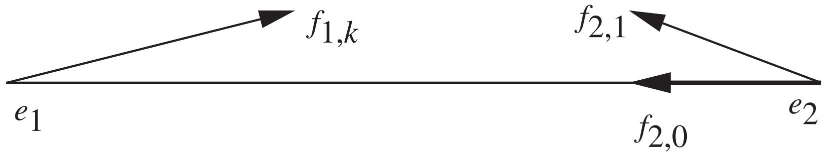



Figure 5: A long side of $\Delta$: $f_{2,1} - f_{1,k} = cf_{2,0}$ with $c \geq 2$

**Lemma** *$\Delta$ has at most one long side.*

If $e_1e_2$ and $e_1e_3$ (say) are both long sides, the basic subdivision of the upper half-space obtained by inverting the bottom blade of the propellor in Figure 3 would be convex at each ray; this is a contradiction, as usual. $\square$

## 2.5 Concatenating three continued fractions

Suppose that $e_1e_3$ and $e_2e_3$ are not long sides, and that $e_1e_2$ has $c \geq 1$ in (2.3). Consider the cyclic continued fraction:

$$[1, a_{1,1}, \dots, a_{1,k_1}, \underline{c}, a_{2,1}, \dots, a_{2,k_2}, 1, a_{3,1}, \dots, a_{3,k_3}]. \tag{2.4}$$

As above, the meaning of this is the successive change of bases anticlockwise around the figure, from $f_{1,0}, f_{1,1}$ to $f_{1,1}, f_{1,2}$ to $f_{1,k}, f_{1,k+1}$, then inverting to $-f_{1,k}, f_{1,k+1} = f_{2,0}$ etc., and on to $-f_{1,0}, -f_{1,1}$. For most purposes, we can afford to be sloppy, and not distinguish between $\pm f_{ij}$, especially in view of the definition of regular triple in 1.2. The continued fraction (or any cyclic permutation of it) evaluates to $\infty = 1 - \frac{1}{0}$, as explained in Remark 2.3.4.

## 2.6 Examples

**An example with no long side:** $\frac{1}{11}(1,2,8)$ The three continued fractions (see Figure 6.a) are

$$\begin{aligned}
&\text{at } e_1: \quad \frac{11}{4} = [3,4] \quad \left(\text{because } \frac{1}{11}(2,8) = \frac{1}{11}(1,4)\right),\\
&\text{at } e_2: \quad \frac{11}{7} = [2,3,2,2] \quad \left(\text{because } \frac{1}{11}(8,1) = \frac{1}{11}(1,7)\right),\\
&\text{at } e_3: \quad \frac{11}{2} = [6,2].
\end{aligned}$$

Since the group is coprime, there is no long side, and these concatenate to

$$[1,3,4,1,2,3,2,2,1,6,2] \quad (=\infty). \tag{2.5}$$

The contraction rule $\underline{a,1,b} \to a-1, b-1$ is as in 2.2. After any number of contractions, a 1 means a regular triple $v_1, v_2, v_3$ among the $f_{i,j}$.

Each 1 in (2.5) corresponds to one of the sides $e_3e_1$, $e_1e_2$ and $e_2e_3$. A chain of contractions with only one 1 allowed to eat its neighbours corresponds to deleting regular triangles along that side (see Figure 6.a): contractions along different sides "commute", in the sense that they can be done independently of one another. Thus starting afresh from $[1,3,4,1,2,3,2,2,1,6,2]$ each time (and numbering the steps as in Figure 6.a), we can do

$$\begin{array}{lrll}
 & \text{Step a} & f_{2,0} = f_{2,1} - f_{1,2}: & \to \ [1,3,3,1,3,2,2,1,6,2] \\
 & \text{Step b} & f_{2,1} = f_{2,2} - f_{1,2}: & \to \ [1,3,2,2,2,2,1,6,2] \\
 & & & \\
\text{or} & \text{Step c} & f_{2,5} = f_{2,4} - f_{3,1}: & \to \ [1,3,4,1,2,3,2,1,5,2] \\
 & \text{Step d} & f_{2,4} = f_{2,3} - f_{3,1}: & \to \ [1,3,4,1,2,3,1,4,2] \\
 & \text{Step e} & f_{2,3} = f_{2,2} - f_{3,1}: & \to \ [1,3,4,1,2,2,3,2] \\
 & & & \\
\text{or} & \text{Step f} & f_{1,0} = f_{1,1} - f_{3,2}: & \to \ [2,4,1,2,3,2,2,1,6,1] \\
 & \text{Step g} & f_{3,2} = f_{3,1} - f_{1,1}: & \to \ [1,4,1,2,3,2,2,1,5] \\
 & \text{Step h} & f_{1,1} = f_{1,2} - f_{3,1}: & \to \ [3,1,2,3,2,2,1,4]
\end{array}$$

Carrying out all of these in this order finally gives $[1,1,1]$, which corresponds to the regular triple $f_{1,2} + f_{2,2} + f_{3,1} = 0$. (There is no uniqueness here, but this is obviously a sensible choice; this end-point is a meeting of champions as in Remark 2.8.2.)

**Example of a long side:** $\frac{1}{15}(1,2,12)$ Note that $\operatorname{hcf}(15,12) = 3$, and the primitive vector along $e_1e_2$ is $f_{1,3} = -f_{2,0} = (-5,5,0)$ (I omit denominators $\frac{1}{15}$ throughout); see Figure 6.b. Since $f_{1,2} = (-6,3,3)$, $f_{2,1} = (4,-7,3)$ we see that $f_{2,1} - f_{1,2} = 2f_{2,0}$ and $e_1e_2$ is a long side with $c = 2$. In this case, because of the common factor, the cones at $e_1$ and $e_2$ are $\frac{1}{15}(1,6) \sim \frac{1}{5}(1,2) = [3,2]$ and $\frac{1}{5}(4,1) = [2,2,2,2]$. At $e_3$ we have $\frac{1}{15}(2,1) = [8,2]$.

Thus the concatenation (2.4) is

$$[1,3,2,\underline{2},2,2,2,2,1,8,2].$$

A chain of 5 contractions centred around the second 1 corresponds to deleting the 5 basic triangles along the bottom Figure 6.b, and reduces the continued

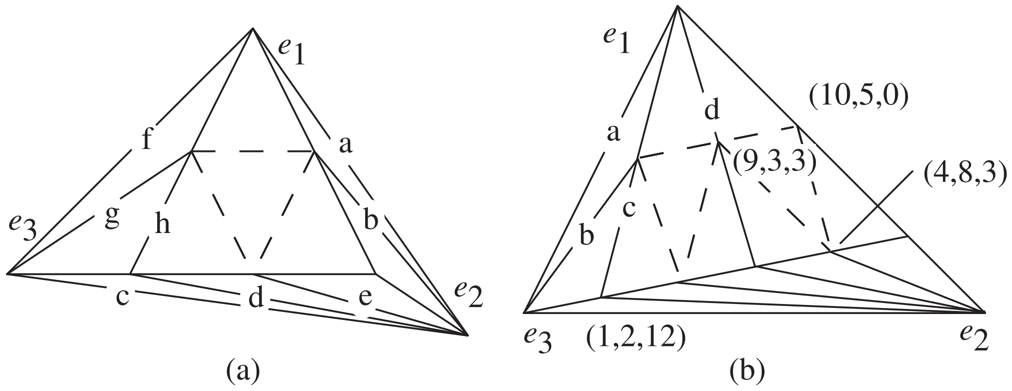



Figure 6: Deconstructing (a) $\frac{1}{11}(1,2,8)$ and (b) $\frac{1}{15}(1,2,12)$: at each step, delete a regular triangle with side the condemned vector

fraction to [1,3,2,1,3,2]. The last of these contractions cuts the long side down to ordinary size by deleting the bottom right triangle. Alternatively, starting from the first 1, the 4 steps

$$\begin{aligned}&[1,3,2,2,2,2,2,2,1,8,2] \to [2,2,2,2,2,2,2,1,8,1]\\&\qquad \to [1,2,2,2,2,2,2,1,7] \to [1,2,2,2,2,2,1,6] \to [1,2,2,2,2,1,5]\end{aligned}$$

deletes the top 4 regular triangles (two of them of side 2) in the order indicated in Figure 6.b, the last step also cutting the long side down to size. Doing all of these steps deletes all the triangles. Note that there are no regular triangles along the long side $e_1e_2$.

## 2.7 MMPs and regular triples

**Lemma** *For brevity, call a chain of contractions taking a cyclic continued fraction (2.4) down to* $[1,1,1]$ *an* MMP.

*(i) Every contraction of* 1 *in an MMP corresponds to a regular triple.*

*(ii) For every regular triple, there is MMP ending at it.*

*(iii) Every regular triple appears in every MMP.*

**Proof** In this proof, view the $\{f_{ij}\}$ as defining a fan of basic cones invariant under $-1$; we completely ignore the given "propellor", and identify $\pm v$.

A 1 corresponds to a relation $v_2 = v_1 + v_3$, which is (i). (ii) is clear: if $v_2 = v_1 + v_3$ is a regular triple, then $v_1, v_2, v_3$ and their minuses subdivide $\mathbb{R}^2$ into 6 basic cones. The chain of vectors $f_{ij}$ within any cone is a nonminimal basic subdivision, so contracts down.

We prove (iii): given a regular triple $v_1, v_2, v_3$ and a choice of MMP, suppose that the first step affecting any of the $v_i$ contracts $v_3$, and choose signs so that $v_3 = v_1 + v_2$. Then $v_1, v_2$ span a basic convex cone, and the original vectors $f_{ij}$ (including $v_3$) form a basic subdivision. After contracting some of these, the step under consideration contracts $v_3$, and thus writes it as the sum of two adjacent integral vectors, which must be in the cones $\langle v_1, v_3\rangle$ and $\langle v_2, v_3\rangle$. Since we are asking for a solution to $(1,1) = (a,b) + (c,d)$ with integers $a > b \geq 0$ and $d > c \geq 0$, it is clear that the only possible such expression is $v_3 = v_1 + v_2$. $\square$

**Alternative proof of (iii):** Count the number of regular triples and the number of contractions in an MMP. It's clear from the MMP algorithm that each vector $v_i$ appears in precisely $c_i$ regular triples, where $c_i$ is the strength of $v_i$. It follows that the disjoint union of all regular triangles has $\sum c_i$ edges, so there are $\frac{1}{3}\sum c_i$ distinct regular triples. On the other hand, in a given MMP each contraction reduces the total strength (that is, the sum of the numbers in the continued fraction) by three so there are $\frac{1}{3}\sum c_i$ contractions. The result follows from the observation that a regular triple cannot correspond to more than one contraction in a given MMP. $\square$

The lemma says that $\Delta$ has a unique subdivision into regular triangles, and any MMP computes it. This completes the proof of Theorem 1.1. $\square$

## 2.8 Remarks

Before proceeding to $G$-Hilb and the proof of Theorem 1.2, there is still a lot of fun to be derived from regular triples and the subdivision of Theorem 1.1.

### 2.8.1 It's a knock-out!

The MMP in cyclic continued fractions has an entertaining interpretation as a contest between the lines $L_{i,j}$ out of the 3 vertices $e_i$. The fan $\Sigma$ of $A$-Hilb $\mathbb{C}^3$ can be calculated using a simple 3-step procedure:

1. Draw lines $L_{ij}$ out of the corners of $\Delta$ (as illustrated in Figure 1.a). Record the strength $a_{ij}$ determined by the Jung–Hirzebruch continued fraction rule (1.1) on each line.

2. Extend the lines $L_{ij}$ until they are 'defeated' by lines $L_{kl}$ from $e_k$ with $i \neq k$ according to the following rule: when two or more lines meet at a point, the stronger line extends, but its strength decreases by 1 for every rival it defeats. Lines which meet with equal strength all die. As a consequence, strength 2 lines always die.

3. Step 2 produces the partition of $\Delta$ into regular triangles of Theorem 1.1. The regular tesselation of the regular triangles gives $\Sigma$.

**Example $\frac{1}{11}(1,2,8)$ revisited:** Consider the cyclic quotient singularity of type $\frac{1}{11}(1,2,8)$. The three continued fractions are

$$\tfrac{11}{4} = [3,4] \text{ at } e_1; \quad \tfrac{11}{7} = [2,3,2,2] \text{ at } e_2; \quad \tfrac{11}{2} = [6,2] \text{ at } e_3.$$

Figure 7(a) illustrates the result of Step 1 of the procedure. The solid lines in Figure 7(b) show the result of Step 2. For example, the line from $e_1$ with strength 3 intersects the line from $e_3$ with strength 2; the procedure

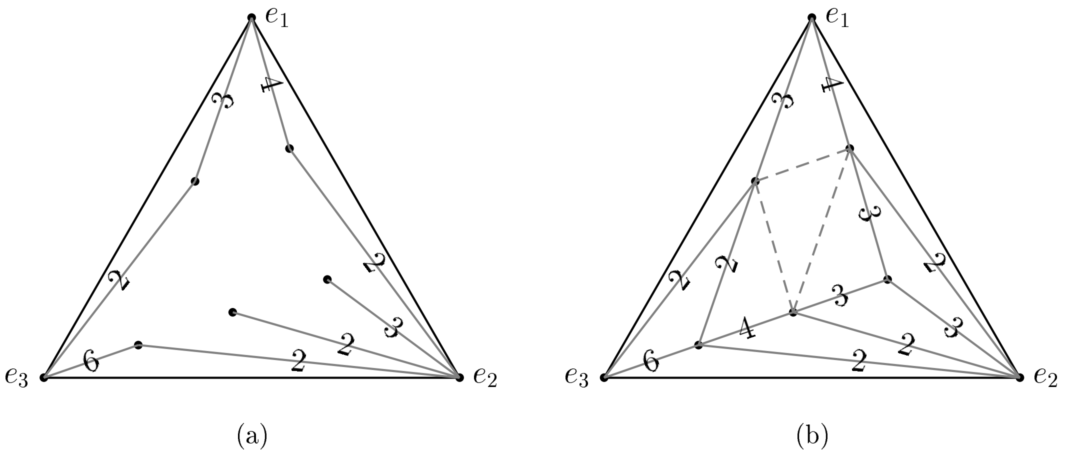



Figure 7: (a) Step 1; (b) Step 2 (solid lines) and Step 3 (dotted lines)

says that the line from $e_1$ extends with strength 2 while the line from $e_3$ dies. The resulting partition of $\Delta$ contains only one regular triangle of side $r > 1$. To perform Step 3 simply add the dotted lines to Figure 7(b).

**Another long sided example:** $\frac{1}{30}(25,2,3)$ Consider the cyclic quotient singularity of type $\frac{1}{30}(25,2,3)$. Note that $\mathrm{hcf}(30,25)=5$ and, because of the common factor, the three continued fractions are

$$\frac{1}{30}(2,3) \sim \frac{1}{5}(1,1) = [5] \quad \text{at } e_1, \quad \frac{1}{30}(25,2) \sim \frac{1}{2}(1,1) = [2] \quad \text{at } e_2$$
$$\text{and} \quad \frac{1}{30}(25,2) \sim \frac{1}{3}(2,1) = [2,2] \quad \text{at } e_3.$$

The solid lines in Figure 8, each marked with the appropriate strength, show the partition of the junior simplex of $\frac{1}{30}(25,2,3)$ into regular triangles of side two and three. The dotted lines tesselate the regular triangles.

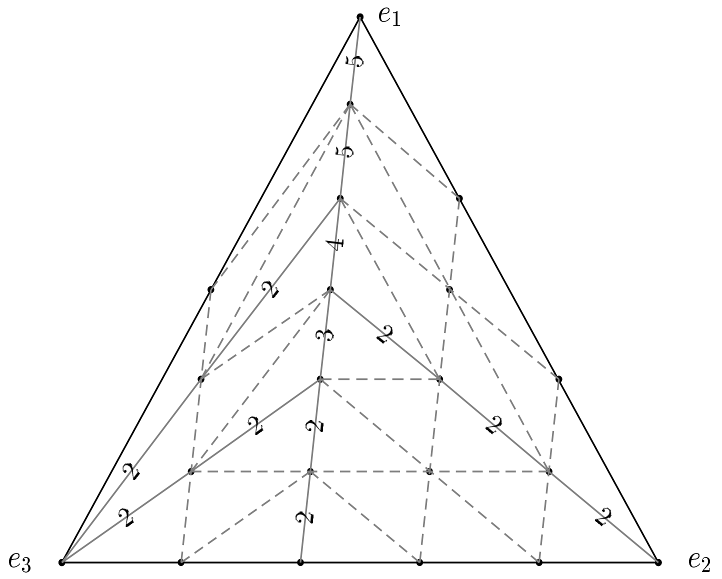



Figure 8: "It's a knock-out!" for the example $\frac{1}{30}(25,2,3)$

To have some fun, make some extra photocopies of p. 32 to distribute to the class. This is a special homework sheet doing the example $\frac{1}{101}(1,7,93)$. All the ideas of the paper can be worked out in detail on it (solutions not provided).

### 2.8.2 Meeting of champions

A regular triple is in one of two possible orientations:

**Type 1:** two consecutive vectors in the same closed blade of the propellor, for example, $f_{1,2} = f_{1,1} + f_{3,1}$ of Figure 3; or

**Type 2:** an interior vector in each blade, for example $f_{1,2} + f_{2,2} + f_{3,1} = 0$.

If there is a long side $e_1e_2$, it is subdivided by a line from $e_3$, and Type 2 cannot occur. We claim that if there is no long side, there is a unique regular triple of Type 2, giving either 3 concurrent vectors or a cocked hat as in Figure 9; both cases occur (see Figure 6.a and [R], Figure 10). These three

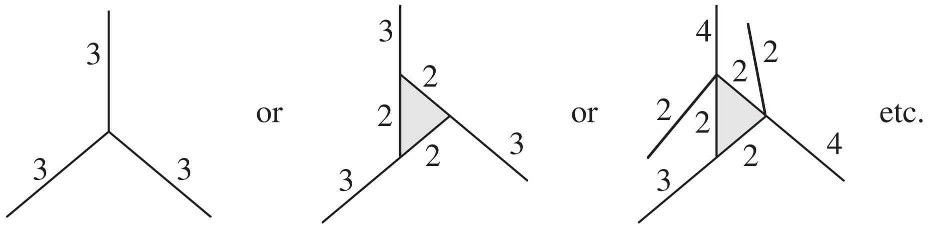



Figure 9: Meeting of champions

are the champions of the knock-out competition, that meet after eliminating all their less successful rivals.

**Proof of claim** Uniqueness is almost obvious from the topology: if it exists, a meeting of champions divides $\Delta$ into 4 regions (one possibly empty), and any other line is confined to one region (it is knocked out by any champion it meets).

For the existence, the idea is that it is natural to deconstruct $\Delta$ by eating in from one side, as we did in the examples of 2.6. The cyclic continued fraction (2.4) has three 1's, so that each side of $\Delta$ takes part in one regular triangle. Choose one side (say $e_1e_3$) and, preserving the other two, eat as many regular triangles as we can along $e_1e_3$ (that is, with sides through $e_1$ or $e_3$, as in Figure 10.a). Every regular triple of Type 1 is associated with a well defined side of $\Delta$, and is eaten in this way starting from that side. The union of regular triangles along each side forms its *catchment area*.

We now view a MMP as successively deleting dividing lines of the subdivision of Figure 3. Eating triangles in the catchment area of side $e_1e_3$ only deletes lines in the two hexants in the top right of Figure 3, between $f_{2,0}$ and $f_{3,0}$. Deleting a line joins two old cones to make a new cone, which is always basic; we conclude that the two vectors $v, v'$ bounding the catchment area of

$e_1e_3$ form a basis. After this, by assumption, no remaining line in these two hexants is marked with 1, so that the cone $\langle f_{2,0}, f_{3,0}\rangle$ now has its standard Newton polygon subdivision.

If we now complete an MMP anyhow from this position, the same two vectors $v, v'$ must occur in some regular triple. By what we have said, the remaining vector must be in the interior of the third hexant. This proves that a regular triple of Type 2 exists. $\square$

### 2.8.3 Semiregular triangles

The following definition is not logically part of Theorems 1.1–1.2, but it helps to understand complicated examples: a triangle $T = \Delta ABC$ (with preferred vertex $A$) is $(r, cr)$-*semiregular* if it is equivalent to the triangle with vertexes $(r, 0), (0, 0), (0, cr)$. Its *semiregular tesselation* is that shown in Figure 2.b. View a $(r, cr)$-semiregular triangle as made up of $c$ adjacent $r$-regular triangles with vertex at $A$; its semiregular triangulation is obtained by taking regular triangulations of each of these. (Note that we work with the affine group of $\mathbb{Z}^2$, so that each regular triangulation is a perspective view of a tesselation by equilateral triangles.) If $v_1, v_2, v_3$ are the primitive vectors along the sides of $T$ (in cyclic order, with $v_1$ the preferred side opposite $A$), the diagnostic test for semiregularity is that $v_1, v_2$ base $\mathbb{Z}_\Delta$ and $cv_1 + v_2 + v_3 = 0$. A semiregular triangle relates in the same way as in 1.2 above to the group $\mathbb{Z}/r \oplus \mathbb{Z}/cr = \left\langle \frac{1}{r}(1, -1, 0), \frac{1}{cr}(0, 1, -1)\right\rangle$. The cyclic continued fraction of a $(r, cr)$-semiregular triangle is $[1, 2, 2, \dots, 2, 1, c]$ with a chain of $c-1$ repeated 2's.

The point of the definition is that it allows you to ignore a string of 2's in continued fractions. If you calculate a series of examples such as $\frac{1}{101}(1, k, 100 - k)$ for $k = 2, 3, 4, 5, 6$ you'll see that almost all the area of $\Delta$ is taken up by semiregular triangles, so this definition is a convenient way of summarising the information.

In this kind of toric geometry, the following objects correspond: (1) a string of 2's in a continued fraction; (2) the continued fraction of $\frac{r}{r-1}$ and the matrix $\left(\begin{smallmatrix} r-1 & r-2 \\ r & r-1 \end{smallmatrix}\right)$; (3) a row of collinear points in $L$; (4) a chain of $-2$-curves; (5) an $A_k$ singularity on the relative canonical model of a surface.

### 2.8.4 Description of $\Sigma$

It is not hard to read from the construction of the basic fan $\Sigma$ that every (internal) vertex has valency $3,4,5$ or $6$, and every (compact) surface of the resolution is $\mathbb{P}^2$, a scroll $\mathbb{F}_n$, or a once or twice blown-up scroll including $\mathrm{dP}_6$ (the del Pezzo surface of degree 6, the regular hexagons of [R]). This provides the foundation for an explicit construction of the McKay correspondence for $A$-Hilb $\mathbb{C}^3$ (see [C1]). The $\mathrm{dP}_6$ correspond to internal lattice points in the tesselations of the regular triangles; there are $\binom{r_i-1}{2}$ of them in each regular triangle of side $r_i$. Looking at what happens in examples, including quite complicated ones (see the Activity Pack on p. 32), seems to indicate other restrictions on $\Sigma$: for example, a twice blown up scroll usually has a twice blown up fibre with 3 components of selfintersection $-2,-1,-2$; scrolls $\mathbb{F}_a$ or blown up scrolls only glue into other $\mathbb{F}_{a'}$ with $|a-a'|\le 2$. This question deserves a more systematic study.

### 2.8.5 Inflation and further regular subdivision

Note that inflating $\Delta$ to $n\Delta$ (or equivalently, replacing $\mathbb{Z}^2_\Delta$ by $\frac{1}{n}\mathbb{Z}^2_\Delta$), which corresponds to extending $A$ to $n^2A = \{g \in \mathrm{diag}\cap\mathrm{SL}(3,\mathbb{C}) \bigm| ng \in A\}$, leaves the continued fractions at the corners unchanged, so the same picture still gives a subdivision into regular triangles, with a finer meshed regular tesselation.

# 3 Regular triangles versus invariant ratios of monomials

## 3.1 Regular triples and invariant ratios

The regular triples $v_1, v_2, v_3$ of Section 2 live in $L$. Passing to the dual lattice $M$ of invariant monomials is a clever exercise in elementary coordinate geometry in an affine lattice that plays a key role in the proof of Theorem 1.2.

**Proposition 3.1** *Every regular triangle of side $r$ gives rise to the invariant ratios of Figure 10 (we permute $x,y,z$ if necessary). Moreover,*

$$d-a=e-b-c=f=r \quad \textit{in Case a}, \tag{3.1}$$

$$d-a=e-b=f-c=r \quad \textit{in Case b}. \tag{3.2}$$

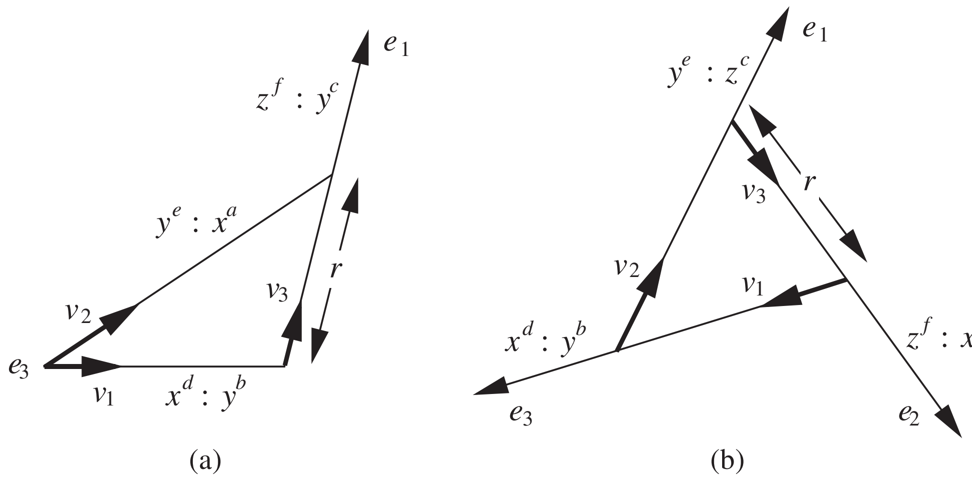



Figure 10: Regular triples versus monomials: (a) corner triangle; (b) meeting of champions

Note: $b, d$ (etc.) are not necessarily coprime; but $x^d/y^b$ is primitive in $M$, that is, not a power of an *invariant* monomial.

**Proposition 3.2** *Let $l$ be any lattice line of $\mathbb{Z}^2_\Delta$, and $\mathbf{m} \in M$ an invariant monomial that bases its orthogonal $l^\perp \cap M$ (as explained at the start of the proof of Proposition 3.1). Then the lattice lines of $\mathbb{Z}^2_\Delta$ parallel to $l$ are orthogonal to $\mathbf{m}(xyz)^i$ for $i \in \mathbb{Z}$.*

*It follows that the regular tesselations of the regular triangles of Figure 10 are cut out by the ratios*

$$x^{d-i} : y^{b+i}z^i, \quad y^{e-j} : z^j x^{a+j}, \quad z^{f-k} : x^k y^{c+k} \quad \text{in Case } a, \tag{3.3}$$

$$x^{d-i} : y^{b+i}z^i, \quad y^{e-j} : z^{c+j} x^j, \quad z^{f-k} : x^{a+k} y^k \quad \text{in Case } b, \tag{3.4}$$

*for $i, j, k = 0, \ldots, r-1$.*

**Proof** The overlattice $L$ is based by $e_i, v_1, v_2$ for any $i = 1, 2$ or $3$ and any regular triple $v_1, v_2, v_3$ (or more generally by any point of $\mathbb{Z}^2_\Delta$, together with any basis $v_1, v_2$ of the translation lattice $\mathbb{Z}^2$ of $\mathbb{Z}^2_\Delta$). In contrast, $e_1, e_2, e_3$ base $\mathbb{Z}^3 \subset L$, and $x, y, z$ base the dual lattice $\mathbb{Z}^3$ of monomials on $\mathbb{C}^3$. The invariant monomials form the sublattice $M \subset \mathbb{Z}^3$ on which $L$ is integral, so that $M = \mathrm{Hom}(L, \mathbb{Z})$. Write $R$ for one of the regular triangles of Figure 10.

Each side of $R$ defines a sublattice (say) $\{e_3, v_1\}^\perp \cap M \cong \mathbb{Z}$. The ratio $x^d : y^b$ in Figure 10, or the monomial $\xi = x^d/y^b$, is the basis of $\{e_3, v_1\}^\perp \cap M$ on which the triangle is positive, say $v_2(\xi) > 0$. So much for Figure 10.

For the equalities (3.1) in Case a, note that Figure 10.a gives $v_1, v_2, v_3$ up to proportionality:

$$\begin{aligned} v_1 &\sim (b, d, -(b+d)), \\ v_2 &\sim (e, a, -(a+e)), \\ v_3 &\sim (c+f, -f, -c). \end{aligned} \tag{3.5}$$

We claim that *the constants of proportionality are all equal, and equal to*

$$\frac{1}{de-ab} = \frac{1}{ac+af+ef} = \frac{1}{bf+cd+df}.$$

(The denominators are the $2\times 2$ minors in the array of (3.5).) For this, write

$$\xi = \frac{x^d}{y^b}, \quad \eta = \frac{y^e}{x^a}, \quad \zeta = \frac{z^f}{y^c}.$$

These 3 monomials are *not* a basis of $M$ (unless $r = 1$, when our regular triangle is basic). But any two of them are *part of a basis*. Indeed, let $e$ be any vertex of $R$ and $\pm v_i, \pm v_j$ primitive vectors along its two sides; then $\{e, \pm v_i, \pm v_j\}$ is a basis of $L$, and the two monomials along the sides are part of the dual basis of $M$. Now there are lots of dual bases around, and the claim follows at once from

$$v_1(\eta) = v_2(\xi) = v_3(\xi) = 1, \quad v_1(\zeta) = v_2(\zeta) = v_3(\eta) = -1.$$

(The signs can be read from Figure 10.)

Equating components of $v_1 + v_3 = v_2$ gives $e = b + c + f$ and $a = d - f$, the first two equalities of (3.1). For the final equality, if we start from $e_3$ and take $f$ steps along the vector $v_1$, we arrive at

$$e_3 + f v_1 = \frac{1}{de-ab}\Big(bf, df, de - ab - bf - df\Big).$$

The final entry $de - ab - bf - df$ evaluates to $cd$. Thus this point has last two entries $df, cd$ proportional to $f, c$, so lies on the third side of $R$. Therefore $r = f$.

The proof of (3.2) in Case b is similar, and left for your amusement. For Proposition 3.2, write $m, u \in M_{\mathbb{R}}$ for the linear forms on $L$ corresponding to the monomials $\mathbf{m}, xyz \in M$. The junior plane $\mathbb{R}^2_\Delta$ is defined by $u = 1$; therefore $\{(m + iu)^\perp\}_{i\in\mathbb{R}}$ is a pencil of parallel lines in $\mathbb{R}^2_\Delta$. For any lattice point $P \in \mathbb{Z}^2_\Delta$ we have $m(P) \in \mathbb{Z}$ and $u(P) = 1$, so $(m + iu)^\perp$ can only contain a lattice point for $i \in \mathbb{Z}$. $\square$

**Remark** The coordinates of points of the tesselation can be calculated in many ways: for example, in Case a, we get

$$e_3 + iv_1 + jv_2 = \frac{1}{de - ab}\big(bj + ei, dj + ai, de - ab - (a + e)i - (b + d)j\big),$$

which could be used to prove Proposition 3.2; or from the $2 \times 2$ minors of

$$\begin{pmatrix} d - i & -(b + i) & -i \\ -(a + j) & e - j & j \end{pmatrix}.$$

It is curious that these explicit calculations in the general case shed almost no light on Propositions 3.1–3.2, even when you know the answers. In contrast, practice with a few numerical examples shows at once what is going on.

**Example** Consider once again $\frac{1}{11}(1, 2, 8)$. The line from $e_3$ to the lattice point $\frac{1}{11}(1, 2, 8)$ represents a 2-dimensional cone $\tau$ in $\mathbb{R}^3$ with normal vector $\pm(2, -1, 0)$. The corresponding toric stratum is $\mathbb{P}^1$ obtained by glueing Spec $\mathbb{C}[x^2y^{-1}]$ to Spec $\mathbb{C}[x^{-2}y]$, so is parametrised by the $A$-invariant ratio $x^2 : y$. Repeat for all lines to produce Figure 11.

The edges of $\Sigma$ are not cut out by ratios; rather, the edges determine a single copy of $\mathbb{C}$ with coordinate an invariant monomial. That is, the image of the $x$, $y$ or $z$-axis of $\mathbb{C}^3$ under the quotient map $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3/A$.

## 3.2 Basic triangles and their dual monomial bases

The regular tesselation of a regular triangle $R$ of side $r$ is a simple and familiar object. A moment's thought shows that every basic triangle $T$ is one of the following two types (see Figure 12 for the subgroup $\mathbb{Z}/r^2 \subset \mathrm{SL}(3, \mathbb{Z})$):

**"up"** For $i, j, k \geq 0$ with $i + j + k = r - 1$, push the three sides of $R$ inwards by $i$, $j$ and $k$ lattice steps respectively. (There are $\binom{r+1}{2}$ choices.) We

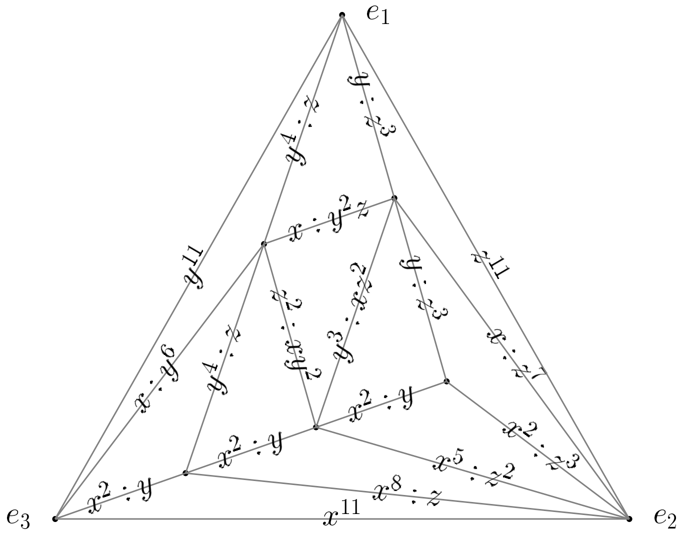



Figure 11: Ratios on the exceptional curves in $A$-Hilb $\mathbb{C}^3$ for $\frac{1}{11}(1,2,8)$

visualise three shutters closing in until they leave a single basic triangle $T$. Note that $T$ is a scaled down copy of $R$, parallel to $R$ and in the same orientation; in other words, up to a translation, it is $\frac{1}{r}R$.

**"down"** For $i,j,k>0$ with $i+j+k=r+1$, push the three sides of $R$ inwards by $i$, $j$ and $k$ lattice steps (giving $\binom{r}{2}$ choices). Now the shutters close over completely, until they have a triple overlap consisting of a single basic triangle $T$, in the opposite orientation to $R$; up to translation, it is $-\frac{1}{r}R$.

A basic triangle $T$ has a basic dual cone in the lattice $M$, based by 3 monomials perpendicular to the 3 sides of $T$. These monomials are given by Proposition 3.2, or more explicitly as follows.

**Corollary 3.3** *Let $R$ be one of the regular triangle of Figure 10. Its up basic triangles have dual bases*

$$\xi = x^{d-i}/y^{b+i}z^i, \quad \eta = y^{e-j}/z^j x^{a+j}, \quad \zeta = z^{f-k}/x^k y^{c+k} \quad \textit{in Case a}$$
$$\xi = x^{d-i}/y^{b+i}z^i, \quad \eta = y^{e-j}/z^{c+j} x^j, \quad \zeta = z^{f-k}/x^{a+k} y^k \quad \textit{in Case b}$$

*for $i,j,k \geq 0$ with $i+j+k=r-1$. Its down basic triangles have dual bases*

$$\begin{aligned} &\lambda = y^{b+i}z^i/x^{d-i}, \quad \mu = z^j x^{a+j}/y^{e-j}, \quad \nu = x^k y^{c+k}/z^{f-k} \quad \textit{in Case a} \\ &\lambda = y^{b+i}z^i/x^{d-i}, \quad \mu = z^{c+j} x^{j}/y^{e-j}, \quad \nu = x^{a+k} y^{k}/z^{f-k} \quad \textit{in Case b} \end{aligned}$$

*for $i,j,k > 0$ with $i+j+k=r+1$.*

**Example** $A = \mathbb{Z}/r \oplus \mathbb{Z}/r$ The lattice is

$$\mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1,-1,0) + \mathbb{Z} \cdot \frac{1}{r}(0,1,-1),$$

and $\Delta$ is spanned as usual by $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. We omit denominators as usual, writing lattice points of $\Delta$ as $(a,b,c)$ with $a+b+c=r$.

An up triangle $T$ has vertexes $(i+1,j,k)$, $(i,j+1,k)$ and $(i,j,k+1)$ for some $i,j,k \geq 0$ with $i+j+k=r-1$ as in Figure 12.a. Since $T$ is basic, so is its dual cone in the lattice of monomials, so the dual cone has the basis

$$\xi = x^{r-i}/y^i z^i, \quad \eta = y^{r-j}/x^j z^j, \quad \zeta = z^{r-k}/x^k y^k.$$

Thus the affine piece $Y_T = \mathbb{C}^3_{\xi,\eta,\zeta} \subset Y_\Sigma$ parametrises equations of the form

$$\begin{aligned} x^{r-i} &= \xi y^i z^i, & y^{i+1}z^{i+1} &= \eta\zeta x^{r-i-1}, & & \\ y^{r-j} &= \eta x^j z^j, & x^{j+1}z^{j+1} &= \xi\zeta y^{r-j-1}, & xyz &= \xi\eta\zeta. \\ z^{r-k} &= \zeta x^k y^k, & x^{k+1}y^{k+1} &= \xi\eta z^{r-k-1}, & & \end{aligned} \tag{3.6}$$

A down triangle $T$ has vertexes $(i-1,j,k)$, $(i,j-1,k)$ and $(i,j,k-1)$ for some $i,j,k \geq 0$ with $i+j+k=r+1$ as in Figure 12.b. The sides of $T$ again correspond to the invariant ratios $x^{r-i} : y^i z^i$ etc., and its dual has basis

$$\lambda = y^i z^i/x^{r-i}, \quad \mu = x^j z^j/y^{r-j}, \quad \nu = x^k y^k/z^{r-k}.$$

The affine piece $Y_T = \mathbb{C}^3_{\lambda,\mu,\nu} \subset Y_\Sigma$ parametrises the equations

$$\begin{aligned} y^i z^i &= \lambda x^{r-i}, & x^{r-i+1} &= \mu\nu y^{i-1}z^{i-1}, & & \\ x^j z^j &= \mu y^{r-j}, & y^{r-j+1} &= \lambda\nu x^{j-1}y^{j-1}, & xyz &= \lambda\mu\nu. \\ x^k y^k &= \nu z^{r-k}, & z^{r-k+1} &= \lambda\mu x^{k-1}y^{k-1}, & & \end{aligned} \tag{3.7}$$

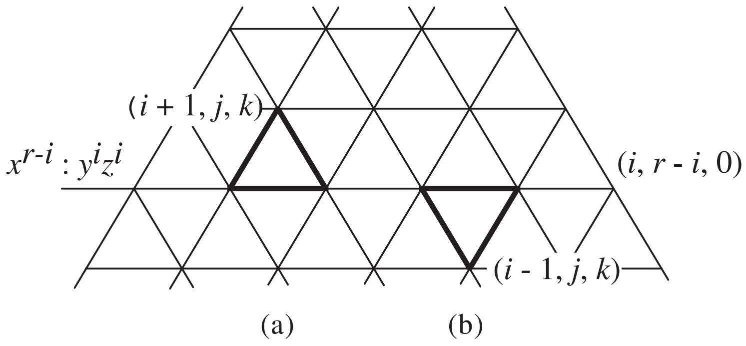



Figure 12: (a) Up triangle; (b) down triangle (same $i$, nonspecific $j, k$)

**Example: regular corner triangle of side $r = 1$** The invariant ratios corresponding to the sides of a corner triangle $T$ are shown in Figure 10.a, where the integers $r, a, b, c, d, e, f$ are related as in Proposition 3.2. If $T$ has side $r = 1$, it is basic, as is the dual cone in the lattice of monomials. The basis consists of the invariant ratios

$$\xi = x^{a+1}/y^b, \quad \eta = y^{b+c+1}/x^a, \quad \zeta = z/y^c.$$

It follows that $\mathbb{C}^3_T = \mathbb{C}^3_{\xi,\eta,\zeta} \subset Y_\Sigma$ parametrise the system of equations (of which several are redundant):

$$\begin{aligned} x^{a+1} &= \xi y^b & y^{b+1}z &= \eta\zeta x^a \\ y^{b+c+1} &= \eta x^a, & x^{a+1}z &= \xi\zeta y^{b+c}, \quad xyz = \xi\eta\zeta. \\ z &= \zeta y^c & xy^{c+1} &= \xi\eta \end{aligned} \tag{3.8}$$

## 3.3 Remarks

### 3.3.1 Rough proof of Theorem 1.2

The standard construction of toric geometry is that $Y_\Sigma$ is the union of the affine pieces $Y_T = \operatorname{Spec} k[T^\vee \cap M]$ taken over all the triangles $T$ making up the fan $\Sigma$. Corollary 3.3 says that $k[T^\vee \cap M] = k[\xi, \eta, \zeta]$ (respectively $k[\lambda, \mu, \nu]$), that is, $Y_T \cong \mathbb{C}^3 \subset Y_\Sigma$, with affine coordinates $\xi, \eta, \zeta$ (respectively $\lambda, \mu, \nu$). On the other hand Corollary 3.3 also causes $Y_T$ to parametrise systems of equations such as

$$x^{d-i} = \xi y^{b+i} z^i, \quad y^{e-j} = \eta z^j x^{a+j}, \quad z^{f-k} = \zeta x^k y^{c+k}, \quad \text{etc.}$$

To prove Theorem 1.2, we show that these equations determine a certain $A$-cluster of $\mathbb{C}^3$, and conversely, every $A$-cluster occurs in this way; thus $Y_T$ is naturally a parameter space for $A$-clusters. The details are given in Section 5.

### 3.3.2 The knock-out rule 2.8.1 in exponents

Suppose that two lines $L_{ij}$ from the regular subdivision intersect at an interior point of $\Delta$; they necessarily come out of different vertexes, say for clarity, $e_1$ and $e_3$. Thus they correspond to primitive ratios $z^f : y^c$ and $y^e : x^a$. Then

$$\text{a line continues beyond the crossing point if and only if it has the strictly smaller exponent of } y. \tag{3.9}$$

The proof follows from Figure 10 and the equalities of Proposition 3.1; we leave the details as an exercise.

# 4 The equations of $A$-clusters

## 4.1 Two different definitions of $G$-Hilb $M$

We start with a mild warning. The literature uses two a priori different notions of $G$-Hilb: in one we set $n = |G|$, take the Hilbert scheme $\operatorname{Hilb}^n M$ of all clusters of length $n$, then the fixed locus $(\operatorname{Hilb}^n M)^G$, and finally, define $G$-Hilb $M$ as the irreducible component containing the general $G$-orbit, so birational to $M/G$. This is a "dynamic" definition: a cluster $Z$ is allowed in if it is a flat deformation of a genuine $G$-orbit of $n$ distinct points. Thus the dynamic $G$-Hilb is irreducible by definition, but we don't really know what functor it represents. Also, the definition involves the Hilbert scheme $\operatorname{Hilb}^n M$, which is almost always very badly singular. (This point deserves stressing: $\operatorname{Hilb}^n M$ is much more singular than anything needed for $G$-Hilb. As Mukai remarks, the right way of viewing $G$-Hilb should be as a *variation of GIT quotient* of $X = \mathbb{C}^3/G$.)

Here we use the algebraic definition: a *$G$-cluster* $Z$ is a $G$-invariant subscheme $Z \subset M$ with $\mathcal{O}_Z$ the regular representation of $G$. The *$G$-Hilbert scheme* $G$-Hilb $M$ is the moduli space of $G$-clusters. Ito and Nakamura prove by continuity that a dynamic $G$-cluster satisfies this condition, so that the dynamic $G$-Hilbert scheme is contained in the algebraic, but the converse is not obvious: a priori, $G$-Hilb $M$ may have exuberant components (and quite possibly does in general in higher dimensions).

Ito and Nakajima [IN, 2.1] proved that the algebraic and dynamic definitions of $A$-Hilb $\mathbb{C}^3$ coincide for a finite Abelian subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$. More recently, Bridgeland, King and Reid [BKR] proved that the definitions coincide for any finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$, not necessarily Abelian.

## 4.2 Nakamura's theorem

**Theorem 4.1 ([N])** *(I) For every finite diagonal subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$ and every $A$-cluster $Z$, generators of the ideal $\mathcal{I}_Z$ can chosen as the system of 7 equations*

$$\begin{aligned} x^{l+1} &= \xi y^b z^f, & y^{b+1}z^{f+1} &= \lambda x^l, & & \\ y^{m+1} &= \eta z^c x^d, & z^{c+1}x^{d+1} &= \mu y^m, & xyz &= \pi. \\ z^{n+1} &= \zeta x^a y^e, & x^{a+1}y^{e+1} &= \nu z^n, & & \end{aligned} \tag{4.1}$$

*Here $a,b,c,d,e,f,l,m,n \geq 0$ are integers, and $\xi,\eta,\zeta,\lambda,\mu,\nu,\pi \in \mathbb{C}$ are constants satisfying*

$$\lambda\xi = \mu\eta = \nu\zeta = \pi. \tag{4.2}$$

*(II) Moreover, exactly one of the following cases holds:*

$$\text{“up”} \quad \begin{cases} \lambda = \eta\zeta, \quad \mu = \zeta\xi, \quad \nu = \xi\eta, \quad \pi = \xi\eta\zeta \\ l = a+d, \quad m = b+e, \quad n = c+f; \quad or \end{cases} \tag{4.3}$$

$$\text{“down”} \quad \begin{cases} \xi = \mu\nu, \quad \eta = \nu\lambda, \quad \zeta = \lambda\mu, \quad \pi = \lambda\mu\nu \\ l = a+d+1, \quad m = b+e+1, \quad n = c+f+1. \end{cases} \tag{4.4}$$

**Remarks** The group $A$ doesn't really come into our arguments, which deal with *all* diagonal groups at one and the same time. For example, $A = 0$ makes perfectly good sense. The particular group for which $Z$ is an $A$-cluster is determined from the exponents in (4.1) as follows: its character group $A^*$ is generated by its eigenvalues $\chi_x, \chi_y, \chi_z$ on $x, y, z$, and related by

$$\chi_x + \chi_y + \chi_z = 0 \quad \text{and} \quad \begin{aligned} (l+1)\chi_x &= b\chi_y + f\chi_z \\ (m+1)\chi_y &= c\chi_z + d\chi_x \\ (n+1)\chi_z &= a\chi_x + e\chi_y. \end{aligned} \tag{4.5}$$

This is a presentation of $A$ as a $\mathbb{Z}$-module, as a little $4 \times 3$ matrix; all our stuff about regular triples, regular tesselations and so on, can be viewed as a classification of different presentations of $A^*$ of type (4.5).

The equations of $Z$ in Theorem 4.1 may be redundant (for example, (3.8)), and the choice of exponents $a, b, \dots, n$ is usually not unique: a cluster with $\pi \neq 0$ corresponds to a point in the big torus of $Y_\Sigma$, belonging to every affine set $Y_T$, and thus can be written in *every* form consistent with the group $A$.

Although at this point we are sober characters doing straight-laced algebra, the argument is substantially the same as that already sketched in [R], which you may consult for additional examples, pictures, philosophy and jokes. See also [N].

**Proof of (I)** By definition (see 4.1), the Artinian ring $\mathcal{O}_Z = k[x,y,z]/I_Z = \mathcal{O}_{\mathbb{C}^3}/\mathcal{I}_Z$ of $Z$ is the regular representation, so each character of $A$ has exactly a one dimensional eigenspace in $\mathcal{O}_Z$. Arguing on the identity character and using the assumption $A \subset \mathrm{SL}(3,\mathbb{C})$ provides an equation $xyz = \pi$ for some $\pi \in \mathbb{C}$.

Since $k[x,y,z]$ is based by monomials, their images span $\mathcal{O}_Z$; monomials are eigenfunctions of the $A$ action. Obviously, each eigenspace in $\mathcal{O}_Z$ contains a nonzero image of a monomial $\mathbf{m}$, and is based by any such. Moreover, if $\mathbf{m}$ is a multiple of an invariant monomial, say $\mathbf{m} = \mathbf{m}_0\mathbf{m}_1$ with $\mathbf{m}_0$ invariant under $A$, and is nonzero in $\mathcal{O}_Z$, then the other factor $\mathbf{m}_1$ is also a basis of the same eigenspace. From now on, we say *basic monomial* in $\mathcal{O}_Z$ to mean the nonzero image in $\mathcal{O}_Z$ of a monomial that is not a multiple of an invariant monomial; in particular, it is not a multiple of $xyz$, so involves at most two of $x, y, z$.

The next result shows how to choose the equations in (4.1).

**Lemma** *Let $x^r$ be the first power of $x$ that is $A$-invariant. Then there is (at least) one $l \in [0, r-1]$ such that $1, x, x^2, \dots, x^l \in \mathcal{O}_Z$ are basic monomials, and $x^{l+1}$ is a multiple of some basic monomial $y^b z^f$ in the same eigenspace, say $x^{l+1} = \xi y^b z^f$ for some $\xi \in \mathbb{C}$.*

Let's first see that the lemma gives the equations in (I). Indeed $x^{l+1}, y^b z^f$ belong to a common eigenspace, and therefore, because $xyz$ is invariant, also $x^l$ and $y^{b+1}z^{f+1}$ belong to a common eigenspace. This is based by $x^l$ by choice of $l$, hence we get the relation $y^{b+1}z^{f+1} = \lambda x^l$.

Finally, since $y^b z^f$ is a basic monomial, $\lambda\xi = \pi$ corresponds to the syzygy $\lambda(\mathrm{i}) + x(\mathrm{ii}) - y^b z^f(\mathrm{iii})$ between the three relations

$$(\mathrm{i}) \quad x^{l+1} = \xi y^b z^f, \quad (\mathrm{ii}) \quad y^{b+1} z^{f+1} = \lambda x^l, \quad (\mathrm{iii}) \quad xyz = \pi.$$

The relations involving $y^{m+1}$ and $z^{n+1}$ arise similarly.

**Proof of the lemma** If $x^{r-1} \neq 0 \in \mathcal{O}_Z$ it is a basic monomial, and one choice is to take $l = r - 1$ and $b = f = 0$, and to take the relation $x^{l+1} = x^r = \xi \cdot 1$. (Other choices arise if the eigenspace of some $x^{l'+1}$ with $l' < l$ also contain a basic monomial $y^{b'} z^{f'}$.)

If not, there is some $l$ with $0 \leq l \leq r-1$ such that $1, x, x^2, \ldots, x^l$ are basic monomials and $x^{l+1} = 0 \in \mathcal{O}_Z$. Now the eigenspace of $x^{l+1}$ must contain a basic monomial $\mathbf{m}$; under the current assumptions, we assert that $\mathbf{m}$ is of the form $y^b z^f$, which proves the lemma. We need only prove that $\mathbf{m}$ is not a multiple of $x$. If $\mathbf{m} = x\mathbf{m}'$ then $\mathbf{m}'$ must in turn be a basic monomial in the same eigenspace as $x^l$. But then $x^l = (\text{unit}) \cdot \mathbf{m}'$ contradicts $x^{l+1} = 0$ and $x\mathbf{m}' \neq 0$. $\square$

Now (I) says that, for any $A$ and any $A$-cluster $Z$, once the relations (4.1) are derived as above, $\mathcal{O}_Z$ is based by the monomials in the tripod of Figure 13, and the relations reduce any monomial $\mathbf{m}$ to one of these. We derived the relations in pairs $x^{l+1} \mapsto y^b z^f$ and $y^{b+1} z^{f+1} \mapsto x^l$. The first type reduces pure powers of $x$ higher than $x^l$. Suppose we have a further relation

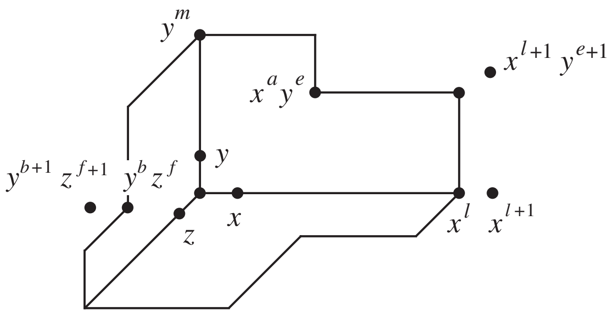



Figure 13: Tripod of monomials basing $\mathcal{O}_Z$

in the first quadrant, (say) $x^\alpha y^\varepsilon \mapsto \mathbf{m}$: if $\mathbf{m}$ involves $x$ or $y$ the new relation would be a multiple of a simpler relation. On the other hand, if $\mathbf{m} = z^\gamma$ is a pure power of $z$, the above argument shows the new relation is paired with a relation $z^{\gamma+1} \mapsto x^{\alpha-1}y^{\varepsilon-1}$, which contradicts our choice of $n$ (in the exponent of $z^{n+1}$). This concludes the proof of (I). $\square$

**Proof of (II)** The point is that a monomial just off one of the shoulders of the tripod of Figure 13 such as $x^{l+1}y^{e+1}$ or $y^{m+1}z^{f+1}$, etc., reduces to a basic monomial in two steps involving two of the $\xi, \eta, \zeta$ relations, or two of the $\lambda, \mu, \nu$ relations. (Compare [R], Remark 7.3 for a discussion.)

The first reduction applies if $b + e \geq m$:

$$x^{l+1}y^{e+1} \mapsto \xi y^{b+e+1}z^f \mapsto \xi\eta y^{b+e-m}x^d z^{c+f}$$

This implies that the monomials $x^{l-d+1}y^{m-b+1}$ and $z^{c+f}$ are in the same eigenspace, and the existence of the relation

$$x^{l-d+1}y^{m-b+1} = \xi\eta z^{c+f}$$

between them. But from the argument in (I), there is only one relation in this quadrant, namely $x^{a+1}y^{e+1} = \nu z^n$. Therefore $l - d = a$, $m - b = e$, $c + f = n$ and $\nu = \xi\eta$. Now $a + d \geq l$ and $c + f \geq n$, so that we can run the same two-step reduction to other monomials to get $\lambda = \eta\zeta$ and $\mu = \xi\zeta$.

The second type of reduction applies if $m \geq b + e + 1$

$$y^{m+1}z^{f+1} \mapsto \lambda y^{m-b}x^l \mapsto \lambda\nu x^{l-a-1}y^{m-b-e-1}z^n$$

Therefore the two monomials $y^{b+e+2}$ and $x^{l-a-1}z^{n-f-1}$ are in the same eigenspace, and $y^{b+e+2} = \lambda\nu x^{l-a-1}z^{n-f-1}$. As before, this must be identical to the $\eta$ relation, so that $m+1 = b+e+2$, $l-a-1 = d$, $n-f-1 = c$ and $\eta = \lambda\nu$. This proves the theorem. $\square$

# 5 Proof of Theorem 1.2

The point is to identify the objects in the conclusion of Corollary 3.3 and of Theorem 4.1; this is really just a mechanical translation. To distinguish between the two sets of symbols, in the monomial bases of Corollary 3.3, we first substitute for $d, e, f$ from (3.1–3.2) of Proposition 3.1, and then replace

$$a \mapsto A, \quad b \mapsto B, \quad c \mapsto C.$$

Each of the monomial bases of Corollary 3.3 gives rise to a triple of equations, either up:

$$\begin{array}{llll} x^{A+r-i}=\xi y^{B+i}z^i, & y^{B+C+r-j}=\eta z^j x^{A+j}, & z^{r-k}=\zeta x^k y^{C+k} & \text{in Case a} \\ x^{A+r-i}=\xi y^{B+i}z^i, & y^{B+r-j}=\eta z^{C+j} x^{j}, & z^{C+r-k}=\zeta x^{A+k} y^{k} & \text{in Case b} \end{array}$$

with $i,j,k\geq 0$ and $i+j+k=r-1$; or down:

$$\begin{array}{llll} y^{B+i}z^i=\lambda x^{A+r-i}, & z^j x^{A+j}=\mu y^{B+C+r-j}, & x^k y^{C+k}=\nu z^{r-k} & \text{in Case a} \\ y^{B+i}z^i=\lambda x^{A+r-i}, & z^{C+j} x^{j}=\mu y^{B+r-j}, & x^{A+k} y^{k}=\nu z^{C+r-k} & \text{in Case b} \end{array}$$

with $i,j,k>0$ and $i+j+k=r+1$.

Each triple can be completed to the equations of an $A$-cluster; for example, the first triple gives:

$$\begin{array}{rlrl} x^{A+r-i} &= \xi y^{B+i}z^i & y^{B+r-j-k}z^{r-j-k} &= \eta\zeta x^{A+j+k} \\ y^{B+C+r-j} &= \eta z^j x^{A+j} & z^{r-i-k}x^{A+r-i-k} &= \zeta\xi y^{B+C+k+i} \qquad xyz=\xi\eta\zeta. \\ z^{r-k} &= \zeta x^k y^{C+k} & x^{r-i-j}y^{C+r-i-j} &= \xi\eta z^{i+j} \end{array}$$

(The method is to multiply together any two of the equations and cancel common factors.) Since $i+j+k=r-1$, these are of the form of Theorem 4.1, with $l=A+j+k$, $b=B+i$,$f=i$, etc.. The other cases are similar. Therefore as explained in 3.3.1, each affine piece $Y_T\cong\mathbb{C}^3\subset Y_\Sigma$ parametrises $A$-clusters.

Conversely, we prove that for $A\subset \mathrm{SL}(3,\mathbb{C})$ a finite diagonal subgroup and $Z$ an $A$-cluster with equations as in Theorem 4.1, $Z$ belongs to one of the families parametrised by $Y_T$. If $Z$ is "up" its equations are determined by the first three:

$$x^{a+d+1}=\xi y^b z^f, \quad y^{b+e+1}=\eta z^c x^d, \quad z^{c+f+1}=\zeta x^a y^e. \tag{5.1}$$

Consider first just two of the possibilities for the signs of $f-b$, $d-c$, $e-a$.

1. Suppose $b\geq f$, $d\geq c$ and $e\geq a$. We define $A,B,C,i,j,k$ by

   $$A=d-c,\quad B=b-f,\quad C=e-a,\quad i=f,\quad j=c,\quad k=a$$

   and set $r=i+j+k+1$. Then, obviously,

   $$a=k,\quad b=B+i,\quad c=j,\quad d=A+j,\quad e=C+k,\quad f=i.$$

   Substituting these values in the exponents of (5.1), puts the equations of $Z$ in the form up, Case a.

2. Similarly, if $b \geq f$, $c \geq d$ and $a \geq e$, we fix up $A, B, C, i, j, k$ so that

$$a = A + k, \quad b = B + i, \quad c = C + j, \quad d = j, \quad e = k, \quad f = i.$$

Substituting in (5.1), shows that $Z$ is up, Case b.

One sees that the permutation $y \leftrightarrow z$ leads to $b \leftrightarrow f$, $a \leftrightarrow d$ and $c \leftrightarrow e$, and the other possibilities for the signs of $e-a$, $f-b$, $d-c$ all reduce to these two cases on permuting $x, y, z$. In fact, Figure 10.a has 6 different images on permuting $x, y, z$ (corresponding to the choices of $e_1$ and $e_3$), and Figure 10.b has 2 different images (corresponding to the cyclic order).

If $Z$ is "down" its equations can be deduced from the second three:

$$y^{b+1}z^{f+1} = \lambda x^{a+d+1}, \quad z^{c+1}x^{d+1} = \mu x^{b+e+1}, \quad x^{a+1}y^{e+1} = \nu z^{c+f+1} \tag{5.2}$$

Exactly as before, if $b \geq f$, $d \geq c$ and $e \geq a$ then we can fix up $A, B, C \geq 0$ and $i, j, k > 0$ so that

$$\begin{gathered} a + 1 = k, \quad b + 1 = B + i, \quad c + 1 = j, \\ d + 1 = A + j, \quad e + 1 = C + k, \quad f + 1 = i, \end{gathered}$$

which puts (5.2) in the form down, Case a. The rest of the proof is a routine repetition. This proves Theorem 1.2. $\square$

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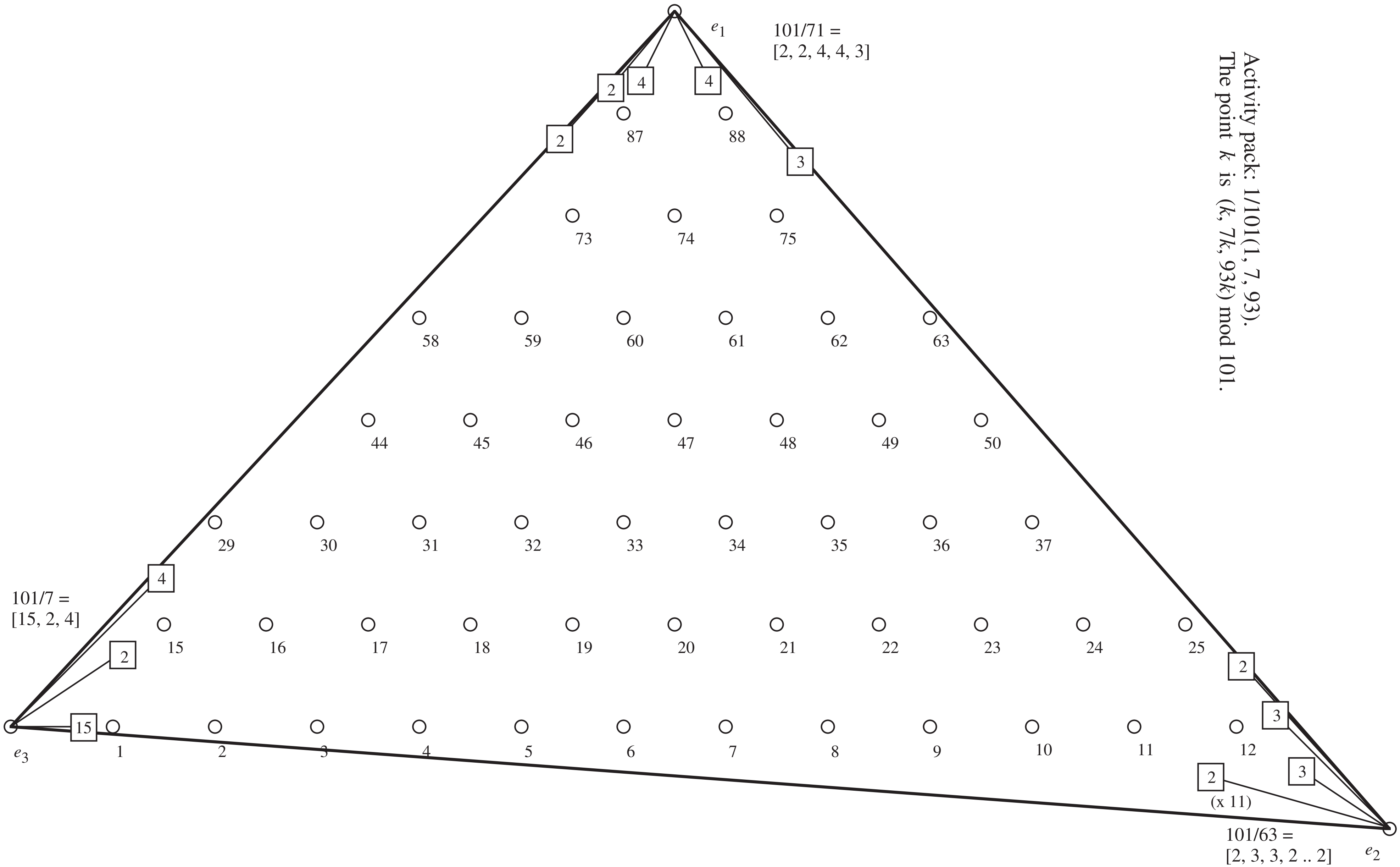


Activity pack: 1/101(1, 7, 93).
The point $k$ is $(k, 7k, 93k)$ mod 101.