Example 1: $\frac{1}{23}(1,1,3,18)$

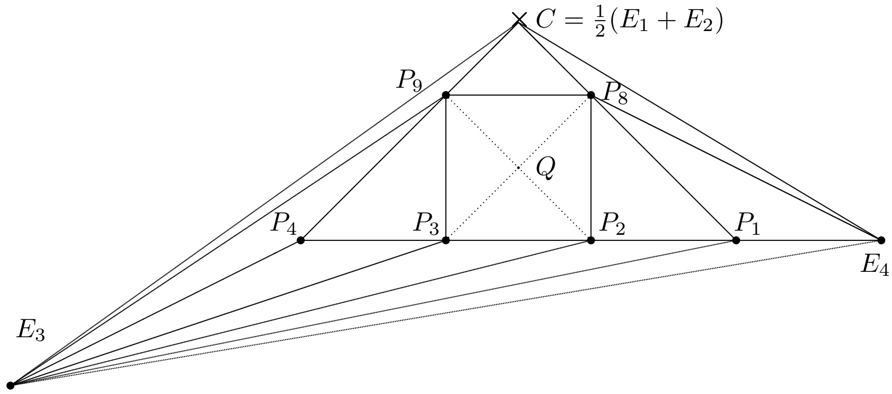

Figure 1: The median triangle for $A = \frac{1}{23}(1,1,3,18)$ and the fan computing A-Hilb $\mathbb{A}^4$. Here $P_1 = (1,1,3,18)$, $P_2 = (2,2,6,13)$, $P_8 = (8,8,1,6)$ etc. Note that E_3, P_4, P_8 are collinear. The central point $Q = P_2 + P_9 = P_3 + P_8$ has age 2, and divides the square into 4 basic cones that correspond to a discrepant divisor in A-Hilb $\mathbb{A}^4$, a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over $\mathbb{P}^1$ that has two contractions to minimal models that differ by a flop.

Figure 1 lives in the integral affine geometry of the median plane. Taking the long arithmetic progression of points $E_4, P_1, \dots, P_4$ as the horizontal axis makes this kind of figure work in plain text, and brings out some important features. Other strategically significant features of the diagram are the extreme points:

$$P_8 \text{ (with } x_3\text{-coordinate} = 1), \quad P_8 \text{ (with } x_4\text{-coordinate} = 1), \tag{0.1}$$

lying closest to the sides $x_4 = 0$ and $x_3 = 0$, and the two straight lines CP_8P_1 and CP_9P_4 from C. In bigger cases, the perspective in these figures is often misleading, and what they mean often takes some sorting out. The top triangle with the half-lattice point C as vertex appears foreshortened. Please get used to it: P_4P_9 is a primitive lattice vector, whereas $P_9C = \frac{1}{2}(P_4P_9)$ is the primitive half-lattice vector along the same line.

Example 2: $\frac{1}{23}(1,1,2,19)$

The numbers suggest this case might be similar to $\frac{1}{23}(1,1,3,18)$, but the outcome is very different. A-Hilb is computed by the fan of Figure 2. The figure is not planar: except for C, E_3 and Q_{11}, all the labelled points are in an affine plane (also containing $\frac{1}{2} \times Q_{11}$) sloping up steeply to the age 7 lattice point $Q_{72} = (72, 72, 6, 11)$. Their projection to the median triangle as drawn in the figure on the next page is much less useful. Since the figure is not planar, the position of C and E_3 is only defined after choosing a projection. The sides down from Q_{72} to E_4 on the right and to $Q_{17} = (17, 17, 11, 1)$ on the left are arithmetic progressions. Around the sides, there are 12 basic triangles with C (or the axis E_1E_2) joined to $E_3, Q_{17}, Q_{28}, \ldots, E_4$. Then E_3 joined to $E_4, P_1, \ldots, P_5, Q_{11}, Q_{17}$ and C. Each of the 10 parallelograms gives 2 ordinary nodes of A-Hilb isomorphic to $ac = bd$ on the cone joining to E_1 and E_2.

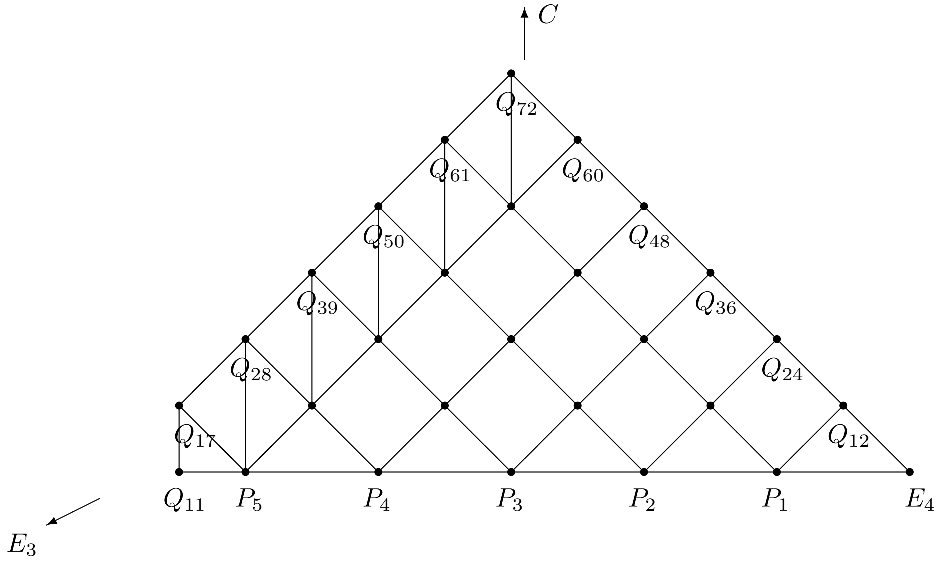

Figure 2: The median triangle for $A = \frac{1}{23}(1,1,2,19)$ and the fan computing $A\text{-Hilb}\,\mathbb{A}^4$, with 78 affine pieces, 20 of them having a curve of nodes. The figure leaves the viewer to imagine the lines and slivers of triangles around the boundary joining E_4, P_1–P_4 to E_3, and E_3, Q_{17}–Q_{61}, Q_{72}, Q_{60}–Q_{12}, E_4 to C. The top half of the figure with it array of square tiles, and the bottom half are in different planes, and it is often preferable to plot them as two separate figures.

Figure 3: The points of Figure 2 in perspective, scaled down to the median plane. In this perspective, the vertex E_3 is off the page on the bottom left.

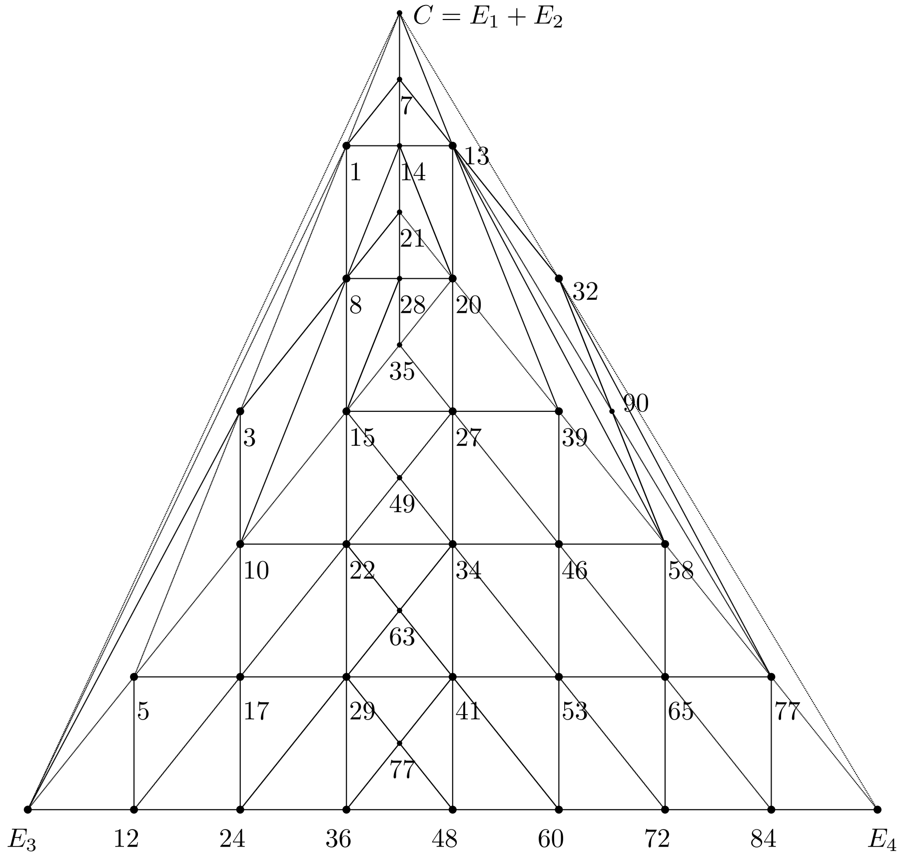

Figure 4: The median triangle for $A = \frac{1}{96}(40, 40, 15, 1)$ and the fan computing its nonsingular A-Hilb $\mathbb{A}^4$. The row of points $C, Q_7, Q_{14}, Q_{21}, Q_{28}, Q_{35}$ then Q_{49}, Q_{63}, Q_{77} on the line vertically down from C have age 2 (drawn as half-lattice points), so represent discrepant divisors. Likewise the point $Q_{90} = P_{13} + P_{77} = P_{32} + P_{58}$ at the midpoint of the tiny tile $E_4 P_{77} P_{13} P_{32}$, drawn bigger below.

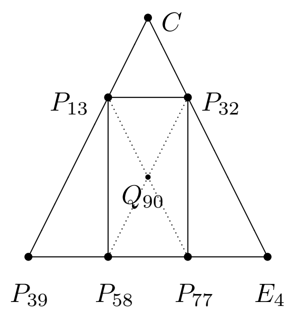

Figure 5: A better perspective on the tile $E_4 P_{77} P_{13} P_{32}$ of Figure 4. Since we are in the affine geometry of a $\mathbb{Z}^2$ lattice, we are free to redraw to any scale.

The IPMU abstract

Speaker: Miles Reid

Title: $(1,2)$-symmetric subgroups of $\mathrm{SL}(4,\mathbb{C})$

Abstract: The topic is finite diagonal subgroups $A \subset \mathrm{SL}(4,\mathbb{C})$ and their A-Hilbert schemes. As a dimension reducing assumption, I impose the additional $(1,2)$-symmetric condition. The case to bear in mind is $\frac{1}{r}(1,1,a,b)$ with $r = a+b+2$. The "junior end and all-even" conditions for the quotient $X = \mathbb{A}^4/A$ to have a crepant resolution are known from Sarah Davis's thesis [D].

Studying the A-Hilbert scheme A-Hilb $\mathbb{A}^4$ in the general $(1,2)$-symmetric case is interesting in its own right, and provides more detailed insight into crepant resolutions. The variety $Y = A$-Hilb $\mathbb{A}^4$ is toric, a union of affine pieces corresponding to monomomial ideals $I \subset k[\mathbb{A}^4] = k[x,y,z,t]$, and can be constructed by my 2009 computer algebra routine [M]. In very many cases Y is nonsingular, and is a resolution $Y \to X$ with exceptional divisors of discrepancy 0 or 1.

The calculation of A-Hilb $\mathbb{A}^4$ mirrors the classical construction of Nakamura [A] and Craw–Reid [CR], with some remarkable modifications. In $(1,2)$-symmetric coordinates x,y,z,t, an A-regular monomal ideal I has quotient the regular representation of A, so $k[x,y,z,t]/I = kA$. Thus its monomial basis MB has one monomial in each character space. Now x and y have the same character, so at most one of them can be basic. If $x \in$ MB then the SL(4) condition says that x^2zt is A-invariant, but MB may contain multiples of xzt. This means that as a subset of the monomial lattice $\mathbb{Z}^4$, MB has a "double decker" structure as the union of 4 planar sets:

- the x,z-sector $\{x^iz^j\}$
- the x,t-sector $\{x^it^j\}$
- the lower deck of the z,t-sector $\{z^it^j\}$
- the top deck $\{xz^it^j\}$, that is, multiples of xzt.

I want to classify *all of these quotients for all $(1,2)$-symmetric groups*, by analogy with Nakamura's tripods, that tesselate the trihedral plane of the quotient lattice $\mathbb{Z}^3/\mathbb{Z}\cdot(1,1,1)$. I replace this with the odd-looking plane of the lattice quotient

$$\mathbb{Z}^4/(\mathbb{Z}\cdot(1,1,1,1) \oplus \mathbb{Z}\cdot(1,-1,0,0))$$

where the x,z- and x,t-sectors occupy angles of 135° degrees each, and the z,t-sector the remaining 90°. Mapping the monomials of MB to this plane keeps track of their characters, but overlays the two decks of the z,t-sector on the 90° angle. However, since the character of xzt is -1 times the character of x,

the monomials of the top deck interleaves with those of the lower deck of the z,t-sector like black and white squares on a chessboard.

I will provide pictures of typical cases on my website, plus examples and guided exercises on how to use the computer algebra routines of [M].

References **[D]** Sarah Davis, Crepant resolutions and A-Hilbert schemes in dimension four, Warwick 2012 PhD thesis
`http://webcat.warwick.ac.uk/record=b2584703~S1`

[M] Magma AH4 ("legacy software" – I know how to use it, and what to do when it breaks down).
`https://homepages.warwick.ac.uk/~masda/McKay/Magma_AH4`

[N] Iku Nakamura, Hilbert schemes of abelian group orbits, J. of Algebraic Geometry **10** (2001) 757-779

[CR] Alastair Craw and Miles Reid, How to calculate A-Hilb $\mathbb{C}^3$, in Ecole d'été sur les variétes toriques (Grenoble, 2000), collection Séminaires et Congrès, SMF 2001, arXiv preprint math.AG/9909085 32 pp.

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For IPMU conference talk. Nice typeset pictures for
AHilb for 1/23(1,1,3,18)  and  1/23(1,1,2,19)

The second has lots of nodes and lots of divisors of
large discrepancy.



==== Group 1/9(1,1,2,5), Affine piece No 4 out of 15 ====
false
[ 2  0 -1  0]
[ 1  0  2 -1]
[ 1  0 -3  1]
[-1  1  0  0]
[-1  0  5  0]
[-1  0  0  2]

[ 1  0 -2  0 -1  1]
[ 0  1 -1  0 -1  1]

->
[1  -1 -1  0  0  0]


[ 1  1  1  1]
[ 2  0 -1  0]
[ 1  0  2 -1]
```

```
[ 1  0 -3  1]
[-1  1  0  0]
[-1  0  5  0]
[-1  0  0  2]

[ 1  0  0 -3 -1 -2  1]
[ 0  1  0 -2  0 -1  1]
[ 0  0  1 -1  0 -1  1]

x*y*z*t = a*1,
x^2 = b*z,
x*z^2 = c*t,
x*t = d*z^3,
y = e*x,
z^5 = f*x,
t^2 = g*x,

b = c*d   (given by row 2 - row 3) of the final kernel
a = c^2*d*e*g (given by row 1 - 2*row 3)

Conclusion:
The parameter space = affine piece of AHilb
is singular c*g - d*f. The construction over
it has z^2*t - c*g as additional equation.
Monomial basis:
[.]
   z^4
      z^3
         z^2 [.]
     [.]    z x*z
        z*t    1   x  [.]
            t  [.]
         [.]
in character spaces
[.]
   8
      6
         4
            2  3
         7  !  0  1 [.]
            5
        [.]

The ! marks the lhs of the relation
x*z*t = z^4. This is the first hint
of "interleaving" of "top deck".
```

```
==== Group 1/11(1,1,2,7), Affine piece No 11 out of 30 ====
false
[ 1  1  1  1]
[ 2  0 -1  0]
[ 1  0  3 -1]
[ 1  0 -4  1]
[-1  1  0  0]
[-1  0  6  0]
[-1  0 -1  2]

[ 1  0  0 -3 -1 -2  1]
[ 0  1  0 -2  0 -1  1]
[ 0  0  1 -1  0 -1  1]

x*y*z*t - a*1,
x^2 - b*z,
x*z^3 - c*t,
x*t - d*z^4,
y - e*x,
z^6 - f*x,
t^2 - g*x*z

Conclusion is the same: c*g - d*f in AHilb direction
(last line of the ker matrix), extra z^2*t - c*g

[.]
z^5
   z^4
      z^3 [.]
         z^2 x*z^2
     [.]    z x*z
        z*t    1   x  [.]
               t
          [.]
in character spaces
[.]
10
   8
      6
         4  5
            2  3
```

```
         9  !  0  1 [.]
            7
        [.]

==== Group 1/13(1,1,2,9), Affine piece No 5 out of 25 ====
false
[ 1  1  1  1]
[ 2  0 -1  0]
[ 1  0  2 -2]
[ 1  0 -3  2]
[-1  1  0  0]
[-1  0  5 -1]
[-1  0  0  3]    same but one more power of t throughout

[ 1  0  0 -3 -1 -2  1]
[ 0  1  0 -2  0 -1  1]
[ 0  0  1 -1  0 -1  1]

[.]
   z^4
      z^3
         z^2 [.]
     [.]    z x*z
       z*t xzt   1   x  [.]
   z*t^2    t  x*t
         t^2 [.]
      [.]
in character spaces
[.]
   8
      6
         4  [.]
            2  3
        11 12  0  1 [.]
     7   !  9  10
         5  [.]
      [.]

This is the first occurrence of top deck for a _singular_ affine piece.
the monomial x*z*t is basic
The ! marks the lhs of the relation
x*z*t = z^4.
of "interleaving" of "top deck".
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