A-Hilb $\mathbb{A}^4$, some computations, counterexamples and conjectures

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Abstract

For a finite diagonal Abelian subgroup $A \subset \mathrm{SL}(4,\mathbb{C})$, assume a crepant resolution exists; there is then an established McKay correspondence on the level of Euler characteristic, Betti numbers or even motivic integration, but the BKR result does not even begin to make sense for an equivalence of derived categories. A necessary preliminary to any progress on this question is a crepant resolution representing a good functor. With this in view, I show how to compute A-Hilb $\mathbb{A}^4$, and establish a small catalogue of things that go wrong with it – discrepancy, singularity, reducibility.

I'm mostly interested in computations, so it is convenient to stick to the subgroup $\frac{1}{r}(1,a,b,c)$ with $1+a+b+c=r$, that is, the group μ_r of rth roots of unity acting with the characters $\varepsilon\colon (x,y,z,t) \mapsto (\varepsilon x, \varepsilon^a y, \varepsilon^b z, \varepsilon^d t)$. The case of a general finite diagonal Abelian subgroup $A \subset \mathrm{SL}(4,\mathbb{C})$ is presumably no harder, but involves disproportionately tedious notation. (For example, a scheme for numbering the elements and characters of A; in my restricted case, these are $0,\dots,r-1$, with no thought involved.)

1 The condition JunSuff

As usual, the elements of the group correspond one-to-one with points of the lattice $L = \mathbb{Z}^4 + \mathbb{Z}\cdot\frac{1}{r}(1,a,b,c)$ in the unit cube of $\mathbb{R}^4$ with the standard basis (after I choose a primitive root of 1). Explicitly, the r points are $P_i = \frac{1}{r}(\bar{i}, \overline{ai}, \overline{bi}, \overline{ci})$ for $i = 0,\dots,r-1$ where $\overline{}$ is smallest positive residue mod r; they divide as usual by *age*:

$$\operatorname{age} P_i = \frac{1}{r}(\bar{i} + \overline{ai} + \overline{bi} + \overline{ci}) = 0, 1, 2, 3,$$

with the identity (or origin) at age 0, the juniors Jun at age 1, the middle-aged Mid at age 2, and the seniors at age 3.

I make the extra assumption to ensure enough juniors to do the work:

$$\text{every } P_i \in \text{Mid is a sum of two juniors} \qquad \text{(JunSuff)}$$

The following is an experimental result: *JunSuff is a necessary and sufficient condition for the existence of a crepant resolution.* More precisely, when JunSuff holds, the barycentric subdivision at a suitably chosen junior point leads to a fan of four cones still satisfying JunSuff; this holds for all my $\frac{1}{r}(1,a,b,c)$ cases with $r \leq 100$.

1.1 When does JunSuff hold?

When JunSuff holds is currently slightly mysterious. Experimental evidence (you see the pattern?): for $r \gg 0$,

$$\begin{aligned}
\text{JunSuff}(r,[1,2,3,r-6]) &\iff 6 \mid r \\
\text{JunSuff}(r,[1,2,4,r-7]) &\iff \\
& r \in [0,2,4,7,8,11,14,15,16,18,22,23] \mod 28 \\
\text{JunSuff}(r,[1,2,5,r-8]) &\iff r \in [0,4,8,10,18,20,24,28,34] \mod 40 \\
\text{JunSuff}(r,[1,2,6,r-9]) &\iff 3 \mid r \\
\text{JunSuff}(r,[1,2,7,r-10]) &\iff \\
& r \in [0,10,12,14,20,24,34,40,42,52,54,62] \mod 70 \\
\text{JunSuff}(r,[1,2,8,r-11]) &\iff r \in [0,4,8,11,12,15,19,22,23,24,30,35, \\
& 44,46,48,52,55,56,59,63,67,68,70,78,79,88] \mod 88 \\
\text{JunSuff}(r,[1,2,9,r-12]) &\iff 6 \mid r \\
\text{JunSuff}(r,[1,3,4,r-8]) &\iff \\
& r \in [0,8,12,20,24,32,36,44,48,56] \mod 80 \\
\text{JunSuff}(r,[1,3,5,r-9]) &\iff r \in [0,9,12] \mod 30 \\
\text{JunSuff}(r,[1,3,6,r-10]) &\iff r \in [0,2,3,5,6,10,11,12, \\
& 15,16,20,21,22,23,26] \mod 30 \\
\text{JunSuff}(r,[1,3,7,r-11]) &\iff r \in [0,6,11,14,18,39,56,62,69,77,83, \\
& 84,95,102,105,111,116,132,144,146,161,165, \\
& 168,179,182,188,209,210,216,221] \mod 231
\end{aligned}$$

The proper solutions with $r = 20$ are listed at the start of Section 3. Those with $r = 23$ are:

$$\frac{1}{23}(1,1,3,18), \quad \frac{1}{23}(1,1,7,14), \quad \frac{1}{23}(1,2,4,16).$$

The prime (or coprime) cases, corresponding to isolated fixed points, tend to be less rich than the composite cases. They eventually show all the same complications as the composite cases, but hidden inside after blowups, and so are somewhat deceptive to work with. Typical features appear at once in composite cases, whereas in coprime cases they may only appear for large r.

The above results were obtained from the following simple Magma routines (online version for copy-and-paste at

`http://www.warwick.ac.uk/~masda/McKay/Magma_AH4`.)

1.2 Points of $\frac{1}{r}(a_1, a_2, a_3, a_4)$ in unit cube

```
function Pts(r,A)
  return [[a*k mod r : a in A] : k in [1..r-1]];
end function;
```

1.3 The condition JunSuff

The condition is that every middle-aged point is a sum of 2 juniors. The function returns true, or false together with the first case it fails.

```
function JunSuff(r,A)
  Points := Pts(r,A);
  Jun := [P : P in Points | &+P eq r];
  Mid := [P : P in Points | &+P eq 2*r];
  Sums := [[P1[i]+P2[i] : i in [1..4]] : P1 in Jun, P2 in Jun];
    if &and[P in Sums : P in Mid] then return true;
    else return false, [P : P in Mid | P notin Sums][1];
    end if;
end function;
```

1.4 Search for solutions of JunSuff

For each r, list $\frac{1}{r}(a_1, a_2, a_3, a_4)$ for which JunSuff holds

```
function Solutions(r);
  TempSolutions := [[0,0,0,0]]; // initiate list with nonsense
  for a1 in [i : i in [0..r div 4]],
    a2 in [i : i in [a1..(r-a1) div 3]],
      a3 in [i : i in [a2..(r-a1-a2) div 2]] do
     a4 := r-a1-a2-a3;
     A := [a1,a2,a3,a4];
      if
    (a4 ne r) and (GreatestCommonDivisor([r] cat A) eq 1) and
    (&and[A ne Sort(B) : B in Pts(r,C), C in TempSolutions])
    and (JunSuff(r,A)) then
      Append(~TempSolutions,A);
     end if;
  end for;
  return Remove(TempSolutions,1); // take out initial nonsense
end function;
```

1.5 Verifying the table

To see which $\frac{1}{r}(1,2,7,r-10)$ satisfy JunSuff, run the following routine over the intervals $[70,139]$, $[140,209]$ and $[210,279]$:

```
[r-70 : r in [70..139] | JunSuff(r,[1,2,7,r-10])];
```

1.6 JunSuff implies crepant resolution

The following routine verifies experimentally that each $\frac{1}{r}(a_1,\dots,a_4)$ satisfying JunSuff has at least one barycentric subdivision at a junior point P_i that defines a crepant partial resolution with 4 affine pieces each of which is a cyclic quotient point (such as $\frac{1}{a_1}(-r,a_2,a_3,a_4)$) still satisfying JunSuff.

```
Jun := [P : P in Pts(r,A) | &+P eq r];
[B : B in Jun | &and [JunSuff(B[i], [-r]
                        cat Remove(B,i)) : i in [1..4]]];
```

1.7 Example: $\frac{1}{23}(1,2,4,16)$

This satisfies JunSuff, and of his 6 junior points

$$(1,2,4,16),\ (2,4,8,9),\ (3,6,12,2),\ (6,12,1,4),\ (12,1,2,8),\ (13,3,6,1),$$

barycentric subdivision at P_1, P_3, P_6, P_{12} works. P_{13} does not work, because $\frac{1}{13}(3,3,6,1)$ has middle aged point $\frac{1}{13}(9,9,5,3)$ that is not a sum of two juniors.

The following routine verifies all solutions for given r.

```
time for r in [20..25] do
  &and
    [&or
      [&and [JunSuff(B[i], [-r] cat Remove(B,i)) : i in [1..4]]
                   : B in Jun ]
                   : A in Solutions(r)];
end for;
```

This takes approx 2 seconds to do [20..25], but approx 40 seconds to do the 80 cases with $r = 56$.

2 A-sets and monomial A-clusters

For a finite diagonal subgroup $A \subset \mathrm{GL}(n)$, the ring of invariants (whose Spec is the quotient $X = \mathbb{A}^n/A$) is based by invariant monomials, that make up the sublattice of monomials $M \subset \mathbb{Z}^n$ dual to the overlattice $L = \mathbb{Z}^n + \sum \mathbb{Z} \cdot \frac{1}{r}(\underline{a})$ (summed over $\frac{1}{r}(\underline{a}) \in A$). Each character $i \in \widehat{A} = \mathrm{Hom}(A, \mathbb{G}_m)$ gives rise to the eigensheaf $\mathcal{L}_i$, based by monomials forming a coset of M in $\mathbb{Z}^n$. This section turns these conceptually simple ideas into practical calculations.

2.1 Invariant monomials

It is easy to say what to do: put in all the invariant monomials in the unit cube except those that are divisible in a nontrivial way by other invariant monomials. However, I streamline a bit to avoid carrying out $O(r^3)$ tests on $O(r^4)$ monomials. I have no doubt that this could all be done more cleverly, for example, by using continued fractions to find monomials in two variables in a given eigenspace.

When testing for divisibility, it is efficient to list the invariant monomials by degree. The routines below assume that $A \subset \mathrm{SL}(4)$ and $A[1] = 1$.

```
KK := FiniteField(1009);
RR<x,y,z,t> := PolynomialRing(KK,4);
```

```
function InvariantMonomials(r,A)
  Inv := [x^r,y^r,z^r,t^r,x*y*z*t];
    for i in [0..r-1], j in [0..r-i-1], k in [0..r-i-j-1] do
      if i*j*k eq 0 then
    Append(~Inv,x^((-A[2]*i-A[3]*j-A[4]*k) mod r)*y^i*z^j*t^k);
      elif (A[2]*i+A[3]*j+A[4]*k) mod r eq 0 then
        Append(~Inv,y^i*z^j*t^k);
      end if;
    end for;
  Exclude(~Inv,1); // Omit 1 before testing divisibility.
S := [[RR!1]]; // S is a list of lists Si of mons of deg i.
  for i in [1..r] do
    Si := [m : m in Inv | TotalDegree(m) eq i];
    Append(~S,Si); // Omit those divisible by mons of deg i.
    Inv := [m : m in Inv |
             not &or[IsDivisibleBy(m,n) : n in Si]];
  end for;
    if #Inv ne 0 then
        error("Invariant monomial has degree > r");
    end if;
  return S;
end function;
```

2.2 Little open question

I assume tacitly that *the semigroup of monomials is generated by elements* $x^{\mathbf{m}} = x^{m_1}y^{m_2}z^{m_3}t^{m_4}$ *of total degree* $\sum m_i \le r$. This holds in two variables by Jung–Hirzebruch continued fractions games, and for three variables and $A \subset \mathrm{SL}(3)$, but in my context I take this as an experimental fact (and test for it at the end of `InvariantMonomials` in case it might fail).

2.3 Eigenmonomials

Next, I put in the monomials that generate the eigensheaves $\mathcal{L}_i$; I test for divisibility by banging monomials into the quotient ring (or *coinvariant ring*) $\overline{R} = R/\{\text{invariant monomials}\}$.

```
function EigenSp(r,A)
```

```
  Inv := InvariantMonomials(r,A);
  Eig := [&cat[S : S in Inv]]; // Collate the list of lists.
  Rbar := quo<RR | Exclude(Eig[1],1)>; // invariants <> 1
    for s in [1..r-1] do
  temp := [x^((s-A[2]*i-A[3]*j-A[4]*k) mod r)*y^i*z^j*t^k :
       k in [0..r-i-j-1], j in [0..r-i-1], i in [0..r-1] |
            i*j*k eq 0] cat [y^i*z^j*t^k :
       k in [0..r-i-j-1], j in [0..r-i-1], i in [0..r-1] |
       (i*j*k ne 0) and ((A[2]*i+A[3]*j+A[4]*k) mod r eq s)];
  temp := [m : m in temp | Rbar!m ne 0];
  Append(~Eig,temp);
  end for;
return Rotate(Eig,-1); // Put invariants in final rth place.
end function;
```

2.4 Monomial A-clusters

An A-cluster Z has $H^0(\mathcal{O}_Z) = k[A]$, the regular representation of A. Therefore $H^0(\mathcal{O}_Z)$ has a monomial basis with exactly one monomial in each eigenspace. In other words, to make Z, pick one monomial in each eigenspace and set each of its compatriots to be a multiple of it, by an A-invariant rational function that is a candidate for a function on an affine piece of A-Hilb. If I can do that at all, I can certainly set all these multiples to zero, obtaining the monomial cluster that is a zero dimensional strata of the toric scheme A-Hilb. The choice of monomial basis is called an *A-set* (Nakamura's A-graph).

Listing all possible A-sets is again conceptually simple: mark one monomial in each eigenspace, let all the unmarked guys (in all eigenspaces) generate an ideal, and ask whether this generates an A-cluster. This fails if my choice kills off a whole eigenspace; for example, if $A = \frac{1}{5}(1,4)$, choosing $y^4 \in \mathrm{Eig}(1)$ kills off x, so excludes the choice of $x^2 \in \mathrm{Eig}(2)$. Or, as an even more extreme case, choosing $y^4 \in \mathrm{Eig}(1)$ and $x^4 \in \mathrm{Eig}(4)$ would kill off both x and y.

2.5 Tree search

The computational difficulty: a search over the Cartesian product of r eigenspaces, each containing $O(r^2)$ elements, is not practical. Instead, my routine below is a tree search: starting from a single choice of monomial $m_1 \in \mathrm{Eig}(1)$, make a number of appropriate choices of monomials $m_{i_j} \in \mathrm{Eig}(i_j)$ for $j = 1, \dots, \nu$. This is a typical node of my tree. From it, the algorithm construct the quotient ring $\overline{R}$ by killing off all the compatriots of my choices:

$$\overline{R} = R/(\{\mathrm{Eig}(i_j) \setminus m_{i_j} \bigm| j = 1, \dots, \nu\}).$$

For a subsequent value of i (starting from $i_\nu + 1$), consider the number:

$$n_i = \#\{m \in \mathrm{Eig}(i) \bigm| 0 \neq \overline{m} \in \overline{R}\}.$$

The case $n_i = 0$ is a dead end. There is no monomial in $\mathrm{Eig}(i)$ with nonzero residue in $\overline{R}$; my choices so far are therefore contradictory (as described above for $\frac{1}{5}(1,4)$), and I backtrack to the last branching point whose possibilities are not exhausted. The case $n_i = 1$ is a straight track: $\mathrm{Eig}(i)$ contains a single appropriate monomial: there is no contradiction and no choice to make, so simply pass on to the next i. The case $n_i \geq 2$ is a branching point: I list the choices, and restart the search from this node (making the first choice from an ordered list). If I get to the end of the list $\widehat{A}$ of eigenvalues (in my $A = \mathbb{Z}/r$ case, $i = r - 1$), then each eigenspace of $\overline{R}$ is one dimensional; this means that my choices so far is an A-set, so pick the fruit (thus giving the node the logical status of dead end) and backtrack.

2.6 Preliminaries to ASets

Given a quotient ring $\overline{R}$ and a finite set of monomials S, list $m \in S$ that map to nonzero elements of $\overline{R}$.

```
function NonZeroMonomials(Rbar,S)
 return [m : m in S | Rbar!m ne 0];
end function;
```

Given a quotient ring $\overline{R}$ and a list Eig of finite sets of monomials, lists the i for which $\mathrm{Eig}[i]$ has 0 or ≥ 2 monomials.

```
function NotOne(R,E)
 return [[i] cat L : i in [1..#E] | #L ne 1 where
   L is NonZeroMonomials(R,E[i])];
end function;
```

2.7 The main ASets function

```
function ASets(r,A)
  Eig := EigenSp(r,A);
// initialise dynamic constructions
Bag := []; // To store the fruit.
finished := false;
I := []; // List of indexes i1, i2, ... where I make choices.
C := []; // List of lists of candidates mi in Eig[i], i in I.
      // Cycle through monomials in last list then delete it.
M := []; // List of number of current choices from C.
  // In other words, current choices are C[i,M[i]] for i in I.

while not finished do
   // Make the current quotient ring by killing all monomials
   // in all Eig[i] except the current choices mi for i in I.
   Rbar := quo< RR | Exclude(Eig[r],1) cat
           &cat[Exclude(Eig[I[i]],C[i,M[i]]) : i in [1..#I]] >;
   // Analyse whether it is an A-cluster.
  tlist := NotOne(Rbar,Eig);
    if tlist eq [] then
       Include(~Bag,Basis(DivisorIdeal(Rbar))); end if;
   // If so, bag the fruit (intending to backtrack).
   // If each tlist[i] is [i, plus >= 1 monomial], so no
   // contradiction or result, go forward: add new index to I,
   // new list of candidates to C, and first choice to M.
     if (tlist ne []) and (&and[#l ne 1 : l in tlist]) then
       Append(~I,Integers()!tlist[1,1]);
       Append(~C,Remove(tlist[1],1)); Append(~M,1);
     end if;
  // I,M; // Uncomment this line for a diagnostic display.
  if (tlist eq [])
     or ((tlist ne []) and (&or[#l eq 1 : l in tlist])) then
  // Backtrack, include setting finished to true if can't.
      ss := #I;
      if &and[M[i] eq #(C[i]) : i in [1..ss]] then
        finished := true;
      else tt := Max([i : i in [1..ss] | M[i] ne #(C[i])]);
```

```
      I:=I[1..tt]; M:=M[1..tt]; M[tt]:=M[tt]+1; C:=C[1..tt];
    end if;
  end if;
end while;
return Bag;
end function;
```

For example, `ASets(20,[1,4,5,10])` gives 27 sets of monomials ideals of A-clusters, of which one specimen is

$$[x^2, xy^3, xyz, xt, y^4, y^3z^2, y^2t, z^3, z^2t, t^2].$$

3 Deforming A-clusters, analysing the results

3.1 Main aim

My function `ASets` lists A-sets, each named by the monomials S generating the ideal of a cluster Z_S, so that $H^0(\mathcal{O}_Z) = R_S = k[x, y, z, t]/S$ is the regular representation of A. An A-set S defines an affine toric piece of A-Hilb parametrising A-cluster having the same eigenbasis as the monomial cluster $R_S = k[x, y, z, t]/S$; these are flat deformations of R_S, but not necessarily *small* deformations in cases when the open piece is reducible. I now show how to treat questions such as how bad A-Hilb is (discrepant, singular, reducible, and so on) in terms of algorithms starting from S. At some points I am guided by numerological hints as to what A-Hilb should look like coming from the Mackay correspondence in motivic integration.

3.2 Eigenbasis of R_S

The quotient is a direct sum

$$H^0(\mathcal{O}_Z) = R_S = \bigoplus_{i \in \widehat{A}} k \cdot n_i$$

of 1-dimensional eigenspaces, each based by a monomial n_i. Another way of saying this (following Nakamura) is that the monomials under the Newton polygon of S form a fundamental domain in $\mathbb{Z}^n$ for translation by the lattice M of invariants.

The function EigenBasis returns a monomial basis of R_S sorted into eigenspaces. It assumes S is a finite set of mononials defining a finite dimensional quotient R/S (for example, an A-set), and the polynomial ring is divided up into a list $E = \mathrm{Eig}$ of eigenspaces (as produced by my function EigenSp(r,A)).

```
function EigenBasis(S,E)
 Rbar := quo<RR | S>;
 MonB := MonomialBasis(Rbar);
 return [[n : n in MonB | n in E[i]][1] : i in [1..#E]];
end function;
```

3.3 Matrix of relations

An A-set S generates the monomial ideal of R_S; the affine piece Y_S of A-Hilb replaces each monomial relation $m = 0$ for $m \in S$ by a nearby binomial relation $m = \lambda_m n$, where n is the element of the eigenbasis in the same character as m, and λ_m is some scalar.

The birational view of this is as a set of invariant rational monomials $\lambda_m = m/n \in M$. (Birational methods are not always in order, since A-Hilb may be reducible.) These monomials generate a cone in M, whose Spec is the corresponding affine piece Y_S of A-Hilb. In cases where the cone $\langle\lambda_m\rangle \subset M_{\mathbb{R}}$ contains a whole line, there is a relation $M \cdot (\prod \lambda_m^{\nu_m} - 1)$ with M a monomial, so the monomial ideal (with all $\lambda_m = 0$) is not in the closure of the birational component, and A-Hilb is reducible.

The birational case is standard toric geometry: given S, for each $m \in S$, calculate its character (eigenvalue) and look up its mate n in the eigenbasis of R/S, giving rise to the set of invariant monomials $\lambda_m = m/n$. Assume $\frac{1}{r}(A)$ is given and that Eig = EigenSp(r,A) is already in place.

```
function RelMatrix(rr,AA,SS,Eig)
 Rbar := quo<RR|SS>;
 EB := EigenBasis(SS,Eig);
 M := Matrix(4,[Integers()|1,1,1,1]); // add invariant x*y*z*t
for m in SS do
 if m eq x*y*z*t then; // don’t add invariant x*y*z*t twice
 else
   B := [Degree(m,RR.i) : i in [1..4]];
```

```
    c := &+[AA[i]*B[i] : i in [1..4]] mod rr;
    if c eq 0 then c := rr; end if;
    n := EB[c];
    C := [Degree(n,RR.i) : i in [1..4]];
    M := VerticalJoin(M,Matrix(4,[B[i]-C[i] : i in [1..4]]));
  end if;
end for;
 return M;
end function;
```

3.4 Example: The coprime case $A = \frac{1}{23}(1, 2, 4, 16)$

This case illustrates how to calculate A-Hilb and one of the mechanisms that can make it nonsingular but discrepant, a simple blowup of a crepant resolution.

The conclusion in this case is that A-Hilb is nonsingular, a toric union of 26 copies of $\mathbb{A}^4$, and has discrepancy an irreducible divisor E_9 corresponding to $P_9 = \frac{1}{23}(9, 18, 13, 6)$; this divisor appears on 6 affine pieces. Omitting P_9 from the fan of A-Hilb gives a toric blowdown to a crepant resolution Y, contracting the discrepancy divisor to a toric surface $\mathbb{P}^2 \subset Y$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$. In this case, the mechanism causing A-Hilb to blow up P_9 also has a reasonable interpretation (see the end of 3.6).

To start, use `AH23 := ASets(23,[1,2,4,16])` to list the A-sets. There are 26 of them, with 4 to 9 generators in each set. For example, the biggest is No. 11

$$\texttt{AH23[11]} = [x^2, xyzt, xyt^2, xz^2, y^2, yz^2, z^4, z^2t, t^3].$$

These define a monomial ideal, and I want its neighbours in A-Hilb, generated by the corresponding binomials

$$x^2 - ay, \quad xyzt - \pi, \quad xyt^2 - \gamma z, \quad \text{etc.}$$

Apart from my naming scheme for scalars (see below), this is given by

$$\texttt{RelMatrix(r,A,AH23[11],Eig)} = \begin{matrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & -3 & 2 \\ 1 & 0 & 2 & -2 \end{matrix} \quad \begin{matrix} 0 & 2 & -1 & 0 \\ -1 & 1 & 2 & -2 \\ 0 & 0 & 4 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 0 & 3 \end{matrix}$$

The nonsingularity result I want corresponds to the fact that the 9 rows of this matrix are all positive linear combinations of rows $2, 3, 6, 9$, which is a basis of M dual to the basis

$$P_{13} = \frac{1}{23}(13, 3, 6, 1), \quad P_6 = \frac{1}{23}(6, 12, 1, 4),$$
$$P_9 = \frac{1}{23}(9, 18, 13, 6), \quad P_2 = \frac{1}{23}(2, 4, 8, 9).$$

A simple-minded but effective way of discovering this is by linear algebra: do

```
> M := RelMatrix(r,A,AH23[11],Eig);
> KernelMatrix(M);
[ 1  0  0  0 -1  0 -1  1 -1]
[ 0  1  0  0  0  0 -1  2 -1]
[ 0  0  1  0 -1  0  0  1 -1]
[ 0  0  0  1  0  0 -1  1  0]
[ 0  0  0  0  0  1  0 -1  1]
```

This consists of the linear relations between the rows of M. For example, the -1 in n_{47} and n_{58} express rows $7, 8$ as sums of the other rows. The function `IsNonSingularPiece` below does this automatically (when it works).

3.5 Checking nonsingularity

For any monomial ideal, the rows of its RelMatrix are a set of monomials in the toric lattice M. The preferred case is when they span a basic monomial cone; this means that 4 of the monomials are a basis of M, and the others are positive integral combinations of them. The following Magma function (while somewhat kludgy) usually works for this.

Given a matrix N, ask for the index j of a positive entry n_{ij} in a row with all the other entries $n_{ij'} < 0$, or vice versa. Return 0 if none.

```
function convex_row(N)
  n := NumberOfRows(N);
  J := 0;
  for i in [1..n] do
      if J eq 0 then
         pos := [j : j in [1..n+4] | N[i,j] gt 0];
         neg := [j : j in [1..n+4] | N[i,j] lt 0];
```

```
          if (#neg eq 1) and (#pos gt 1) then
             found := true; J := neg[1];
          elif (#pos eq 1) and (#neg gt 1) then
             found := true; J := pos[1];
          end if;
      end if;
  end for;
  return J;
end function;
```

The nonsingular case is when 4 rows of $M =$ `RelMatrix` base the cone generated by all rows. In other words, all but 4 rows of M are eliminated as convex integral combinations of the basis. In this case M_0 is formed by these rows, standing for monomials, and plus-or-minus Adjoint(M_0) is the dual basis of L, part of the fan of A-Hilb.

```
function IsNonSingularPiece(r,A,Si,Eig)
M := RelMatrix(r,A,Si,Eig);
  M0 := M;
  good := true;
  while good and (NumberOfRows(M0) gt 4) do
    N := KernelMatrix(M0);
    j := convex_row(N);
       if j ne 0 then RemoveRow(~M0,j);
       else good := false;
       end if;
  end while;
if good then
  N0 := Transpose(Adjoint(M0)); // dual cone
    if &and[N0[i,j] le 0 : j in [1..4], i in [1..4]]
      then N0 := -1*N0;
    end if;
  Discr := [1/r*&+[N0[i,j] : j in [1..4]]-1 : i in [1..4]];
  // return true, M0, N0, Discr;
  return true, N0, Discr;
else
  return false, M0, KernelMatrix(M0);
end if;
end function;
```

The above routine seems to give the right answer in all the nonsingular cases, i.e., for basic cones of monomials. In the singular cases, it seems to detect that the cone is not basic, but does not find all the convex dependence relations, so the output is sometimes longer than necessary. The general toric functions in Magma will be available in the next release, and there is no point in improving the current functions.

3.6 Application to $\frac{1}{23}(1,2,4,16)$

The six discrepant affine pieces of A-Hilb come from

```
for i in [4,5,9,10,11,12] do
  IsNonSingularPiece(r,A,AH23[i],Eig);
end for;
```

The output leads to the following neat conclusion: consider the lattice points

$$e_2 = (0,1,0,0), \quad P_2 = \frac{1}{23}(2,4,8,9), \quad P_3 = \frac{1}{23}(3,6,12,2),$$
$$P_{13} = \frac{1}{23}(13,3,6,1), \quad P_6 = \frac{1}{23}(6,12,1,4), \quad P_9 = \frac{1}{23}(9,18,3,6).$$

All but the last are junior, whereas P_9 has age 2. Observe that

$$P_9 = P_3 + P_6 \quad \text{and} \quad e_2 + P_2 + P_{13} = P_3 + 2P_6.$$

That is, in the junior affine 3-space, the line P_3P_6 passes through the centroid of triangle $\Delta e_2P_2P_{13}$ (draw P_3 as lying over Δ and P_6 under).

The three simplexes

$$P_3P_6e_2P_2, \quad P_3P_6P_2P_{13}, \quad P_3P_6P_{13}e_2$$

are all basic, and provide the fan of a crepant resolution. Instead, the fan of A-Hilb (given by the above output) has six discrepant simplexes, obtained by slicing these simplexes along the planes

$$P_9e_2P_2, \quad P_9P_2P_{13}, \quad P_9P_{13}e_2,$$

from P_9 on the central axis P_3P_6. Interestingly, A-Hilb is unable to accept the crepant solution of joining P_3P_6 in a fan because it is obliged to mark the boundaries between (z^3 vs x^3t^2), (z^3 vs xyt^2) and (z^3 vs xt^5) in Eig[12]. In each case, P_3 prefers z^3, and P_6 prefers the other, whereas P_9 is impartial between them.

3.7 Applications to 12 cases with $r = 20$

There are 12 cases with $r = 20$ within the confines of my current $\frac{1}{r}(1,a,b,c)$ computational impediment:

```
> Sol20 := [A : A in Solutions(20) | A[1] eq 1];
> ASets20 := [ASets(20,B) : B in Sol20];
> [#Si : Si in ASets20];
[ 20, 24, 20, 20, 20, 20, 20, 42, 27, 33, 20, 30 ]
for i in [1..#Sol20] do
   A := Sol20[i]; Eig := EigenSp(20,A);
   i, #ASets20[i],
        &and[IsNonSingularPiece(20,A,ASets20[i][j],Eig)
        : j in [1..#ASets20[i]]];
end for;
```

The first line gives the twelve cases `Sol20` as

$$\begin{array}{cccc} (1,1,2,16) & (1,1,3,15) & (1,1,4,14) & (1,1,6,12) \\ (1,1,8,10) & (1,1,9,9) & (1,2,2,15) & (1,2,5,12) \\ (1,2,7,10) & (1,3,4,12) & (1,4,4,11) & (1,4,5,10) \end{array} \tag{Sol20}$$

(the only other essential case with $r = 20$ is $\frac{1}{20}(2,5,5,8)$).

The second line calculates the A-sets in these twelve cases, and reports back the number of A-sets in each case as a coarse diagnostic of how complicated the A-Hilb is. I explain how a seasoned experimentalist guess-reads this output. For 7 of these groups A there are 20 A-sets; in these cases A-Hilb should be a crepant resolution: each A-set defines a monomial cluster that is the origin of an affine toric piece, $\mathbb{A}^4$ in all the cases I have solved. The overlap between affine pieces is a toric variety with a $\mathbb{G}_m = \mathbb{C}^\times$ factor, so A-Hilb has Euler number 20, as predicted by the McKay correspondence for a crepant resolution.

In the cases with 24, 27 and 30, I expect A-Hilb to be a fairly simple blowup of a crepant resolution, as in 3.6. For example, blowing up a point in a 4-fold crepant resolution increases the number of affine pieces by 3 (and with it the Euler number); blowing up a copy of $\mathbb{P}^1$ increases it by 4; blowing up a nonsingular toric surface increases it by the Euler number of the surface. The A-Hilb itself has a good chance of being nonsingular, or having at worst $xy = ztu$ or $(xz = yt) \times \mathbb{A}^1$ as singularities.

The case $\frac{1}{20}(1, 2, 5, 12)$ with 42 affine pieces stands out, and at first sight I expected it to be reducible (by analogy with the exuberant case $\frac{1}{30}(1, 6, 10, 13)$ treated below); for example, a component isomorphic (as toric variety) to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ would contribute 16 toric affine pieces. In this case A-Hilb turns out to be irreducible, in fact with only two affine pieces having the mildest possible singularity 3-fold node $\times\, \mathbb{A}^1$.

The third line reports back that of the twelve cases, Nos. 8, 10 and 12 are singular. Taking apart the logical conjunction `&and` one can say which affine pieces are singular and how badly.

I now test these guesses by computations in particular cases. The following case, while too simple to be of genuine interest, illustrates most of the terms; not much else happens in the 7 good cases with $r = 20$ and 20 affine pieces. The example also illustrates the way in which the A-set itself determines an affine toric piece of A-Hilb.

3.8 Example

For $\frac{1}{20}(1, 1, 2, 16)$, one affine piece of A-Hilb is $\mathbb{A}^4$ with coordinates a, b, c, d; it parametrises the clusters

$$x = ay, \quad y^2 = bz, \quad z^8 = ct, \quad t^2 = dz^6, \quad z^2t = e, \quad \text{with} \quad e = cd. \tag{1}$$

The right-hand side of the equations (x, y^2, z^8, t^2, z^2t) names the A-set. The quotient by these has monomial basis

$$\begin{array}{cccccc@{\qquad}ccccc}
1 & z & z^2 & \dots & z^7 & E & & & & & \\
t & zt & E & & & & y & yz & z^2 & \dots & yz^7 \\
E & & & & & & yt & yzt & & &
\end{array} \tag{2}$$

(where E denotes an equation) as has $R_S = H^0(\mathcal{O}_Z)$ for every Z in this affine piece. The right-hand side of (1) are obtained by looking up the unique monomial in (2) in the same eigenspace as the left-hand side; these relations are necessarily present, because the two sides map down to a 1-dimensional vector space. For the same reason, there must be a fifth relation $z^2t = r_5$ for some scalar r_5; I prove $r_5 = cd$. For this, multiply $z^2t = r_5$ by t:

$$r_5t = z^2t^2 = dz^8 = cdt;$$

moreover, t is a basis of an eigenspace, so is certainly not zero.

Finally, one checks that $xyzt = abcd$ modulo (1); in fact

$$xyzt = ay^2zt = abz^2t = abcd.$$

This is the characteristic property for an affine piece of A-Hilb to be crepant over $X = \mathbb{A}^4/A$: x, y, z, t are coordinates on $\mathbb{A}^4$, and a, b, c, d are invariant rational monomials in them that form local coordinates on A-Hilb and satisfy $dx \wedge dy \wedge dz \wedge dt = da \wedge db \wedge dc \wedge dd$.

3.9 Remark: socle and reducible A-Hilb

Notice that the above deduction $r_5 = cd$ is based on the fact that the right-hand side of $z^2t = r_5$ is $r_5 \cdot 1$, where r_5 is a scalar to be determined, and $1 \in \overline{R}$ can be multiplied by some monomial t to give a basic element of $\overline{R}$ – this argument would definitely fails if the monomial on the right-hand side were in the socle of $\overline{R}$, when no deduction is possible about the value of the scalar. This should lead to exceptional components of A-Hilb, and certainly does so in sufficiently complicated cases; these components do not belong to the birational component of A-Hilb, and lie over the origin of $\mathbb{C}^4/A$ (or a small toric stratum). In other words, they parametrise A-clusters not obtained as limits of free orbits. For an example where this really happens, see 5.4.

4 Appendix: the case $\frac{1}{20}(1, 2, 5, 12)$

This has 42 A-sets, each giving an affine piece Y_i; just two of these are singular, Y_{15} and Y_{17} (see below). I calculate Y_8 as a typical example – it is nonsingular but discrepant, with a close relation between the discrepancy and the shape of the clusters parametrised by Y_8. Its origin is the monomial A-cluster defined by

$$(x^4, x^3z, x^3t, x^2z^2, xz^3, xt^2, y, z^4, z^2t, zt^3, t^5) \tag{3}$$

The quotient ring has monomial basis

$$\begin{matrix} 1 & t & t^2 & t^3 & t^4 \\ z & zt & zt^2 & & \\ z^2 & & & & \\ z^3 & & & & \end{matrix} \qquad \begin{matrix} x & xt \\ xz & xzt \\ xz^2 & \end{matrix} \qquad \begin{matrix} x^2 & x^2t \\ x^2z & x^2zt \end{matrix} \qquad x^3$$

`RelMatrix` gives the neighbouring clusters; its output (massaged a bit) is the matrix with 12 rows

$$M = \begin{matrix} 1 & 0 & -1 & 2 \\ -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 1 & 0 & 3 & -3 \\ 4 & 0 & 0 & -2 \\ 2 & 0 & 2 & -1 \end{matrix} \qquad \begin{matrix} 0 & 0 & 4 & 0 \\ 1 & 1 & 1 & 1 \\ -2 & 0 & 2 & 1 \\ 3 & 0 & -3 & 1 \\ 3 & 0 & 1 & -4 \\ -2 & 1 & 0 & 0 \end{matrix}$$

The last four rows give the invariant monomials

$$\kappa = z^2t/x^2, \quad m = x^3t/z^3, \quad l = x^3z/t^4, \quad b = y/x^2.$$

The kernel matrix (linear dependencies between rows of M)

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & -2 & -1 & -1 \end{matrix}$$

expresses the first eight rows of M as their linear combinations, or in other words, expresses

$$xt^2/z, \quad zt^3/x, \quad t^5, \quad xz^3/t^3, \quad x^4/t^2, \quad x^2z^2/t, \quad z^4, \quad xyzt,$$

as monomials in κ, m, l, b (for the proof, see below). For example,

$$xt^2/z = \kappa m \quad \text{and} \quad xyzt = \kappa^3 m^2 lb.$$

One shows eventually that Y_8 is $\mathbb{A}^4$ with coordinates κ, m, l, b. In other words, κ, m, l, b base the cone of invariant monomials that are positive on the monomial ideal (3). One calculates the dual basis of L:

$$P_{11} + e_2 = \tfrac{1}{20}(11, 22, 15, 12), \quad P_9 = \tfrac{1}{20}(9, 18, 5, 8),$$
$$P_5 = \tfrac{1}{20}(5, 10, 5, 0), \qquad e_2 = (0, 1, 0, 0),$$

4.1 Observation: discrepancy and "all the equations"

The above results for Y_8 are possibly typical of what holds whenever an affine piece is nonsingular. The basis of the dual cone in L has ages $3, 2, 1, 1$, which coincides with the exponents in

$$xyzt = \kappa^3 m^2 lb,$$

giving the Jacobian determinant and the discrepancy as $\kappa^2 m = xyzt/\kappa mlb$.

The defining ideal of the A-cluster of Y_8 has minimal generators

$$\begin{gathered} z^2t = \kappa x^2, \quad x^3t = mz^3, \quad x^3z = lt^4, \quad y = bx^2, \\ xt^2 = \kappa mz, \quad zt^3 = \kappa^2 mx, \quad t^5 = \kappa^3 m^2 \cdot 1, \quad xz^3 = \kappa lt^3, \\ x^4 = \kappa mlt^2, \quad x^2z^2 = \kappa^2 mlt, \quad z^4 = \kappa^3 ml \cdot 1, \quad xyzt = \kappa^3 m^2 lb, \end{gathered}$$

(The first four are basic, and the remainder are deduced from them by cancelling a basic monomial. See below for the proof.) However, this minimal basis is part of a *whole* set of equations indexed by the $47 = 4 \cdot 3 \cdot 2 \cdot 2 - 1$ non-unit factors of $\kappa^3 m^2 lb$, with redundant relations such as

$$z^4t^2 = \kappa^2 x^4, \quad z^6t^3 = \kappa^3 x^6, \quad z^3t^4 = \kappa^3 mx^3, \quad \text{etc.}$$

Notice that in this context, each relation has a *complementary* relation, obtained by multiplying the left-hand side by $\kappa^3 m^2 lb$ and the right-hand side by $xyzt$ and cancelling, so $x^{m+1} = ay^{n_1}z^{n_2}t^{n_3}$ gives $y^{n_1+1}z^{n_2+1}t^{n_2+1} = \alpha x^m$ with $a\alpha = \pi = \kappa^3 m^2 lb$, etc. The minimal generators have complements

$$\begin{gathered} x^3y = \kappa^2 l^2 mbz, \quad yz^4 = \kappa^3 l^2 bx^2, \quad yt^5 = \kappa^3 lmbx^2, \quad x^3zt = \kappa^3 l^2 m, \\ yz^2 = \kappa^2 l^2 bt, \quad x^2y = \kappa l^2 bt^2, \quad xyz = l^2 bt^4, \quad yt^3 = \kappa^2 lmbz^2, \\ yzt^3 = \kappa^2 lbx^3, \quad yt^2 = kbxz, \quad xyt = bz^3, \quad xyzt = \kappa^3 m^2 lb \end{gathered}$$

which are already in the ideal.

Recall Nakamura's result that in the $A \subset \mathrm{SL}(3)$ case, every A cluster can be defined by exactly 7 relations, including $xyz = \pi$ and two complementary sets of three relations of the form

$$x^{m+1} = ay_1^n z_2^n, \quad y^{n_1+1}z^{n_2+1} = \alpha x^m \quad \text{with } a\alpha = \pi, \text{ etc.}$$

Proof of nonsingularity The idea is as in 3.8: I am allowed to cancel basic monomials. For the relation represented by the first row of M_0, and the first row of `KernelMatrix`(M_0): I know that a relation $xt^2 = r_1 z$ holds. Since x^2z is basic in the monomial cluster,

$$r_1 x^2 z = x^3 t^2 = m * z^3 t = \kappa m x^2 z,$$

and cancelling x^2z gives $r_1 = \kappa m$. The others are similar.

```
x*z basic, so r2 *x*z = z^2*t^3 = ka*x^2*t^2
x^2 basic, x^2*t^5 = ka^2*m^2*z^2*t
t^4 basic etc.
```

This ideal situation may hold whenever the toric cone is basic.

4.2 Singular cases

Of the 42 affine pieces of A-Hilb for $\frac{1}{20}(1,2,5,12)$, only Y_{15} and Y_{17} are singular. For Y_{15}, permuting the 13 rows of `RelMatrix` gives

$$M_0 = \begin{array}{rrrr} -1 & 0 & 1 & 3 \\ 0 & 2 & 0 & -2 \\ 1 & 0 & 3 & -3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & -1 & 2 \end{array} \qquad \begin{array}{rrrr} 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ 1 & 1 & 1 & -4 \\ 1 & 1 & -3 & 1 \end{array}$$

The last five rows of M give the invariant monomials

$$a = x^2/y, \quad \lambda = yt^2/xz, \quad \kappa = z^2t/y, \quad \delta = xyz/t^4, \quad \gamma = xyt/z^3.$$

The kernel matrix of M (linear dependencies between its rows):

$$\begin{array}{rrrrrrrrrrrrr} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{array}$$

The singularity shows up in the $1, 1, -1, -1$ in the last row, which says that $a\lambda = \kappa\gamma$ (initially under birational assumptions, proved later). Note that while this equation gives Y_{15} the singularity (3-fold node) $\times \mathbb{A}^1$, the other equations are shielded from the effect of a: the ratios appearing in them are monomials in $\lambda, \kappa, \delta, \gamma$.

$$\begin{gathered} x^2 = ay, \quad yt^2 = \lambda xz, \quad z^2t = \kappa y, \quad xyz = \delta t^4, \quad xyt = \gamma z^3 \\ zt^3 = \lambda\kappa x, \quad y^2 = \lambda\delta t^2, \quad xz^3 = \kappa\delta t^3, \quad yz^2 = \lambda\kappa\delta t \\ z^4 = \lambda\kappa^2\delta, \quad xt^2 = \kappa\gamma z, \quad t^5 = \lambda\kappa^2\gamma, \quad xyzt = \lambda\kappa^2\delta\gamma. \end{gathered}$$

with $a\lambda = \kappa\gamma$. The relation $xyzt = \lambda\kappa^2\delta\gamma = a\lambda^2\kappa\delta$ is redundant as ideal generator.

To prove $a\lambda = \kappa\gamma$, note that yz is basic, and

$$\kappa\gamma yz = \gamma z^3 t = xyt^2 = \lambda x^2 z = a\lambda yz.$$

```
Cases 17
========

> M0;
[-1  1 -1  2 ]
[ 0 -1  2  1 ]
[ 0  1  2 -1 ]
[-1  0  1  3 ]
[ 0  0  4  0 ]
[ 1  0 -1  2 ]
[ 1  1  1  1 ]
[ 2 -1  0  0 ]
[ 0  2  0 -2 ]
[-1 -1 -1  4 ]
[ 1  0  3 -3 ]
[ 1  1 -3  1 ]

> KernelMatrix(M0);
[ 1  0  0  0  0  0  0  0 -1 -1  0  0 ]
[ 0  1  0  0  0  0  0  0  0 -1 -1  0 ]
[ 0  0  1  0  0  0  0  0 -1 -1 -1  0 ]
[ 0  0  0  1  0  0  0  0 -1 -2 -1  0 ]
```

```
[ 0  0  0  0  1  0  0  0 -1 -2 -2  0 ]
[ 0  0  0  0  0  1  0  0  0 -1 -1 -1 ]
[ 0  0  0  0  0  0  1  0 -1 -2 -2 -1 ]
[ 0  0  0  0  0  0  0  1  1  0 -1 -1 ]

As above, read the last 4 lines of RelMatrix as
x^2 = a*y
y^2 = b*t^2
t^4 = d*x*y*z
x*z^3 = l*t^3
x*y*t = ga*z^3

with a*b = l*ga (initially under birational assumptions).
Note that a is not explicitly involved in the other equations.

y*t^2 = b*d*x*z
z^2*t = d*l*y    (Pf. r*x*y*z = x*z^3*t = l*t^4 = d*l*x*y*z)
y*z^2 = b*d*l*t
z*t^3 = b*d^2*l*x
z^4 = b*d^2*l^2
x*t^2 = d*l*ga*z
x*y*z*t = b*d^2*l^2*ga (not a defining relation)

Proof that a*b = l*ga:
  l*ga*t^3 = g*x*z^3 = x^2*y*t = a*y^2*t = a*b*t^3

How are Y15 and Y17 glued together?

Y15 is
a = x^2/y, ga = x*y*t/z^3
de = x*y*z/t^4
la = y*t^2/x*z
ka = z^2*t/y
   with a*la = ka*ga

Y17 is
a = x^2/y, ga = x*y*t/z^3
d = t^4/x*y*z
```

```
b = y^2/t^2
l = x*z^3/t^3
    with a*b = l*ga

Obviously a, ga coincide. Also d = de^-1 and b = de*la,
l = de*ka. And conversely, la = b*d, ka = l*d.
```

5 Counter-examples

33 cases of $\frac{1}{30}(1,a,b,c)$ satisfy JunSuff. The number of affine pieces of their A-Hilb is given by

```
> Sol30 := [A : A in Solutions(30) | A[1] eq 1];
> ASets30 := [ASets(30,B) : B in Sol30];
> [#S : S in ASets30];
[ 30, 30, 38, 30, 30, 34, 30, 30, 30, 30, 30, 30, 45, 36,
37, 34, 37, 38, 73, 38, 70, 38, 34, 63, 46, 58, 56, 50,
38, 59, 158, 56, 42 ]
```

Notice that 10 of these have 30 A-sets, and I expect that for these, A-Hilb is a crepant resolution. A further 15 have ≤ 50 A-sets, and 7 have between 56 and 73, so are likely to be somewhat discrepant or singular or both.[1]

5.1 Reducible example

The exuberant 31st case $A = \frac{1}{30}(1,6,10,13)$ stands out clearly with its 158 A-sets; it turns out to be reducible. Its more interesting affine pieces correspond to those with most equations:

```
> r:=30; A := Sol30[31]; A // Answer: [ 1, 6, 10, 13 ]
> AS := ASets(r,A); [i : i in [1..158] | #AS[i] ge 17];
```

This says that A-sets numbered $25, 29, 30, 31, 32, 33, 51, 52, 84, 89, 90, 99, 107$ have ≥ 17 defining equations. `AS[25]` gives

$$x^6, x^5t, x^4t^2, x^3y, x^2z, xy^2, xyt, xzt, xt^3, y^5, y^3t, yz, yt^2, z^3, z^2t, zt^2, t^5$$

[1] I have computed lots of these cases, and am beginning to understand some of the mechanisms.

and `MonomialBasis(quo<RR|AS[25]>)` gives:

$$
\begin{array}{llllllcccccc}
1 & x & x^2 & x^3 & x^4 & x^5 & & y & xy & x^2y & & \\
t & xt & x^2t & x^3t & x^4t & & & yt & & & & \\
t^2 & xt^2 & x^2t^2 & x^3t^2 & & & & & y^2 & & & \\
t^3 & & & & & & & & y^2t & & & \\
t^4 & & & & & & & & & y^3 & & \\
 & & & & & & & & & & y^4 & \\
 & z & xz & z^2 & xz^2 & & & & & & & \\
 & zt & & & & & & & & & &
\end{array}
\tag{4}
$$

The function `RelMatrix` gives the monomial relations; I always include the relation for $xyzt$, although it is redundant in many cases, because it computes the discrepancy.

$$
\begin{array}{rrrr}
1 & 1 & 1 & 1 \\
6 & -1 & 0 & 0 \\
5 & -3 & 0 & 1 \\
4 & 0 & 0 & 2 \\
3 & 1 & 0 & -3 \\
2 & -2 & 1 & 0 \\
1 & 2 & 0 & -1 \\
1 & 1 & -2 & 1 \\
1 & -4 & 1 & 1
\end{array}
\qquad\qquad
\begin{array}{rrrr}
1 & 0 & -1 & 3 \\
0 & 5 & 0 & 0 \\
-1 & 3 & 0 & 1 \\
-3 & 1 & 1 & -1 \\
-2 & 1 & 0 & 2 \\
0 & 0 & 3 & 0 \\
-3 & 0 & 2 & 1 \\
0 & -1 & 1 & 2 \\
-5 & 0 & 0 & 5
\end{array}
$$

It turns out that all the functions on this affine piece of A-Hilb are generated by the five coefficients $\delta, d, \beta, e, \gamma$, that correspond to rows $5, 18, 9, 13, 8$ of the above matrix of relations:

$$
x^3y = \delta t^3, \quad t^5 = dx^5, \quad xzt = \beta y^4, \quad yz = ex^3t, \quad xyt = \gamma z^2 \tag{5}
$$

The distinguishing feature that makes this affine piece a reducible component of A-Hilb (stuck at the origin) is that the monomial cone they generate is not proper: here $\delta^2 d\beta e\gamma = 1$, or in terms of rows of the matrix of relations:

$$
\begin{array}{rrrrrl}
 & 2\times(3 & 1 & 0 & -3) & \\
+ & (-5 & 0 & 0 & 5) & \\
+ & (1 & -4 & 1 & 1) & = 0, \\
+ & (-3 & 1 & 1 & -1) & \\
+ & (1 & 1 & -2 & 1) &
\end{array}
$$

so these monomials span a whole vector space.

5.2 Lemma

Let Z be an A-cluster with monomial basis (4) parametrized by $\delta, d, \beta, e, \gamma$ of (5). Then

$$\begin{array}{rclcrclcrcl} yt^2 &=& \delta dx^2 && x^2z &=& \delta^2 d\beta y^2 && xt^3 &=& \delta^3 d^2\beta\gamma z \\ xy^2 &=& \delta^2 dt && zt^2 &=& \delta^3 d^2\beta y && x^6 &=& \delta^6 d^3\beta^2\gamma y \\ y^3t &=& \delta^3 d^2 x && z^2t &=& \delta^3 d^2\beta e x^3 && x^5t &=& \delta^4 d^2\beta^2\gamma y^3 \\ y^5 &=& \delta^5 d^3 && z^3 &=& \delta^7 d^4\beta^2 e && x^4t^2 &=& \delta^7 d^4\beta^2\gamma \end{array}$$

Moreover,

$$\delta = \delta^3 d\beta e\gamma, \quad \textit{that is,} \quad \delta \times (1 - \delta^2 d\beta e\gamma) = 0.$$

Near the monomial cluster (given by $\delta = d = \beta = e = \gamma = 0$), necessarily $\delta = 0$. This is a copy of $\mathbb{A}^4$ in A-Hilb, consisting of clusters supported at the origin. If $\delta^2 d\beta e\gamma = 1$, the same relations define a cluster that can be free, but that cannot approach the origin.

5.3 Proof

To clarify the statement: over any k algebra generated by $\delta, d, \beta, e, \gamma$, any A-cluster with this monomial basis has the stated relations.

This is similar to the simple example given on p. 17. For example, there must be a relation $yt^2 = \xi x^2$, and ξ is found by observing that x^5 is basic, so $\xi x^5 = x^3yt^2 = \delta t^5 = \delta dx^5$.

Finally, note that t^4 is basic, so the final relation involving δ comes from

$$\delta t^4 = x^3yt = \gamma x^2z^2 = \delta^2 d\beta\gamma y^2 z = \delta^3 d\beta\gamma t^4$$

(the third equality is the relation $x^2z = \delta^2 d\beta y^2$.)

```
d:=Random(KK); be:=Random(KK); e:=Random(KK); ga:=Random(KK);
Dimension(quo<RR|[x^3*y,t^5-d*x^5,x*z*t-be*y^4,y*z-e*x^3*t,
x*y*t-ga*z^2,y*t^2,x*y^2,y^3*t,y^5,x^2*z,z*t^2,z^2*t,z^3,x*t^3,
x^6,x^5*t,x^4*t^2]>);

de:=Random(KK); be:=Random(KK); e:=Random(KK); ga:=Random(KK);
d := (de^2*be*ga*e)^-1;
```

```
Dimension(quo<RR|[x^3*y-de*t^3, t^5-d*x^5, x*z*t-be*y^4,
y*z-e*x^3*t, x*y*t-ga*z^2, y*t^2-de*d*x^2, x*y^2-de^2*d*t,
y^3*t-de^3*d^2*x, y^5-de^5*d^3, x^2*z-de^2*d*be*y^2,
z*t^2-de^3*d^2*be*y, z^2*t-de^3*d^2*be*e*x^3,
z^3-de^7*d^4*be^2*e, x*t^3-de^3*d^2*be*ga*z,
x^6-de^6*d^3*be^2*ga*y, x^5*t-de^4*d^2*be^2*ga*y^3,
x^4*t^2-de^7*d^4*be^2*ga]>);
```

5.4 First reducible example

$\frac{1}{30}(1, 6, 10, 13)$ is the very interesting case with 158 affine pieces. No 16 is the first reducible case. Note that it is reducible although the cone of invariant ratios is a proper cone, defining the birational component. It has monomial ideal generated by `AS30[16]`

$$x^6, x^4t^2, x^3y, x^3z, x^2z^2, xyt, xt^3, y^2, yz, yt^2, z^3, z^2t, zt^2, t^5$$

and complementary monomial basis `MonomialBasis(quo<RR|AS30[16]>)`

$$\begin{aligned}1, t, t^2, t^3, t^4, z, zt, z^2, y, yt,\\ x, xt, xt^2, xz, xzt, xz^2, xy, x^2, x^2t,\\ x^2t^2, x^2z, x^2zt, x^2y, x^3, x^3t,\\ x^3t^2, x^4, x^4t, x^5, x^5t\end{aligned}$$

I reorder the rows of its `RelMatrix` (by blundering about, in a process that is still time-consuming and uncertain).

```
> M := RelMatrix(30,[1,6,10,13],AS30[16],Eig);
> M0 := Submatrix(M,[4,8,5,13,11,12,1,3,9,10,2,6,14,7,15],[1..4]);

> M0;
[ 3  1  0 -3]
[ 1  0 -1  3]
[ 3  0  1 -1]
[-3  0  2  1]
[-2  1  0  2]
[ 0  0  3  0]
[ 1  1  1  1]
```

```
[ 4  0  0  2]
[-2  2 -1  0]  y^2 = rho*x^2*z
[-3  1  1 -1]  y*z = la*x^3*t
[ 6 -1  0  0]  x^6 = a*y
[ 2  0  2 -4]  x^2*z^2 = l*t^4
[ 0 -1  1  2]  z*t^2 = ka*y
[ 1  1 -2  1]  x*y*t = ga*z^2
[-5  0  0  5]  t^5 = d*x^5

> KernelMatrix(M0);
[ 1  0  0  0  0  0  0  0  0  0  0 -1  0 -1  0]  x^3*y = l*ga*t^3    OK by t^4
[ 0  1  0  0  0  0  0  0  0  0  0  0 -1 -1  0]  x*t^3 = ka*ga*z     OK by z^2
[ 0  0  1  0  0  0  0  0  0  0 -1 -1 -1  0]  x^3*z = l*ka*ga*t   OK by z*t
[ 0  0  0  1  0  0  0  0  0  0  0 -1  0  0 -1]  z^2*t = l*d*x^3     OK by x^5
[ 0  0  0  0  1  0  0  0  0  0  0 -1  0 -1 -1]  y*t^2 = l*ga*d*x^2  OK by x^3
[ 0  0  0  0  0  1  0  0  0  0  0 -2 -1 -1 -1]  z^3 = l^2*ka*ga*d   OK by t
[ 0  0  0  0  0  0  1  0  0  0  0 -2 -1 -2 -1]  x*y*z*t = l^2*ka*ga^2*d  OK by
[ 0  0  0  0  0  0  0  1  0  0  0 -1 -2 -2  0]  x^4*t^2 = l*ga*d*a  OK by x^2
[ 0  0  0  0  0  0  0  0  1  0  0 -1  1 -1 -1]
[ 0  0  0  0  0  0  0  0  0  1  0 -1  1  0 -1]
[ 0  0  0  0  0  0  0  0  0  0  1  0 -2 -1  1]
```

The last 4 rows of M_0 correspond to the relations

$$x^2z^2 = lt^4, \;\; zt^2 = \kappa y, \;\; xyt = \gamma z^2, \;\; t^5 = dx^5.$$

The kernel matrix expresses the remaining 11 rows of M_0 as integral linear combinations of the last 4. For the first 8, these are positive,

ratios l, κ, γ, d base the invariant lattice M, and express as of the last 4, so the ratios monomials in them.

$$y^2 = \rho * x^2 * z, \;\; y * z = \lambda * x^3 * t, \;\; x^6 = a * y.$$

The linear relations give the first 8 rows of M_0 as last 4, with l, κ, γ, d as close to a basis as one can come. The above statements

$$x^3y = l\gamma t^3, \ldots, x^4t^2 = l\gamma da$$

are proved by the standard argument, involving cancelling the stated basic monomial. On the other hand, the kernel matrix says that Row9 of M_0

equals Row10 plus Row14, predicting the equality $\rho = \lambda\gamma$ on the birational component, but this cannot be proved without further assumptions, and is not true on the whole of A-Hilb. The reason this can't be proved: there is no deduction for the value of ρ, because every monomial except t kills x^2z, whereas

$$\rho x^2 zt = y^2 t = \text{so what?}.$$

The way out is provided by the following lemma.

5.5 Lemma.

(I) 1. $\lambda\kappa = ld$. 2. $ad = \kappa^2\gamma$. 3. $a\lambda = l\kappa\gamma$.

(II) $(\rho - \lambda\gamma) \times \text{various} = 0$. More specifically, 4. $a\rho = l\kappa\gamma^2 = a\lambda\gamma$. 5. $\kappa\rho = l\gamma d = \kappa\lambda\gamma$. 6. $l\rho = l\lambda\gamma$.

Corollary: *either $\rho = \lambda\gamma$, or $a = \kappa = l = 0$.*

Proof.

```
1. z^2*t^2 = \la*\ka*x^3*t from
 y*z = \la*x^3*t
 z*t^2 = \ka*y

2. Similarly a*d = ka^2*ga; we can prove a*d*y = ka^2*ga*y from
 x^6 = a*y
 t^5 = d*x^5
 a*d*y = d*x^6 = x*t^5 = ka*ga*z*t^2 = ka^2*g*y

3. Similarly for a*la = l*ka*ga; we can prove l*ka*ga*x^3*t = a*la*x^3*t
from
 x^6 = a*y
 y*z = la*x^3*t
 a*la*x^3*t = a*y*z = x^6*z = l*ka*ga*x^3*t    Q.E.D. (Lemma)

4. 5. 6.
rho*x^2*z = y^2
a*rho*x^2*z*t = a*y^2*t = x^6*y*t = ga*x^5*z^2 = l*ga*x^3*t^4 =
```

```
l*ka*ga^2*x^2*z*t,
(rho - la*ga)* various = 0.

proves a*rho = l*ka*ga^2 = a*la*ga

rho*x^2*z = y^2
ka*rho*x^2*z = ka*y^2 = y*z*t^2 = l*ga*d*x^2*z

proves ka*rho = l*ga*d = ka*la*ga

Take 7 variables $\rho,la,a,l,ka,ga,d$. Suppose that
  la*ka=l*d,  a*d=ka^2*ga,  a*la=l*ka*ga
and EITHER $\rho = la*ga$ OR $a = ka = l = 0$. Then
$\rho,la,a,l,ka,ga,d$ parametrise A-clusters with
ideal

I:=[y^2-rho*x^2*z,y*z-la*x^3*t,x^6-a*y,x^2*z^2-l*t^4,
z*t^2-ka*y,x*y*t-ga*z^2,t^5-d*x^5,x^3*y-l*ga*t^3,
x*t^3-ka*ga*z,x^3*z-l*ka*ga*t,z^2*t-l*d*x^3,
y*t^2-l*ga*d*x^2,z^3-l^2*ka*ga*d,
x*y*z*t-l^2*ka*ga^2*d,x^4*t^2-l*ga*d*a];

Computer check:

rho:=Random(KK);
la:=Random(KK);
a:=Random(KK);
l:=Random(KK);
ka:=Random(KK);
ga:=Random(KK);
d:=Random(KK);

// uncomment one of these two lines
// a:=ka^2*ga*d^-1; la:=l*ka*ga*a^-1; rho := la*ga;
// a:=0; l:=0; ka:=0;

I:=[y^2-rho*x^2*z,y*z-la*x^3*t,x^6-a*y,x^2*z^2-l*t^4,
z*t^2-ka*y,x*y*t-ga*z^2,t^5-d*x^5,x^3*y-l*ga*t^3,
```

```
x*t^3-ka*ga*z,x^3*z-l*ka*ga*t,z^2*t-l*d*x^3,
y*t^2-l*ga*d*x^2,z^3-l^2*ka*ga*d,
x*y*z*t-l^2*ka*ga^2*d,x^4*t^2-l*ga*d*a];
Dimension(quo<RR|I>);
```