## MA4L7 Algebraic curves. Example sheet 2

## Assessment deadline: Thu 24th Jan 2:00 pm

Week 2 of lectures were on integral extensions, finite $A$-modules, normalisation, characterisation of DVR. The material is standard, covered in many commutative algebra textbooks. I mostly follow [UCA, esp. Chap. 8].

Recall that *finite $A$-module* means finitely generated as $A$-module: every element can be written as a *linear combination* of finitely many generators $e_1, \dots, e_n$. (As opposed to a finitely generated $A$-algebra $A \subset B$, when every $b \in B$ is a *polynomial* combination of generators $x_1, \dots, x_n$.)

**1. Tower law.** Let $A \subset B_1 \subset B_2$ are integral domains. If $B_1$ is finite as $A$-module and $B_2$ is finite as $B_1$-module prove that $B_2$ is finite as $A$-module.

Given the determinant trick [UCA, 2.7], modify the argument to prove the same statement for integral extensions.

**2. Standard open sets $X_g$.** If $X$ is an affine algebraic variety with coordinate ring $k[X]$ and $g \in k[X]$, it is known that the open subvariety $X_g = \{P \in X \mid g(P) \neq 0\}$ is also affine, and has coordinate ring $k[X_g] = k[X][\frac{1}{g}]$. The $X_g$, called *standard open sets*, form a basis of the Zariski topology of $X$.

Prove that $k[X_g]$ is a finite $k[X]$-module if and only if $1/g$ is integral over $k[X]$. If $k[X]$ is already normal (integrally closed in $k(X)$), this happens only if $g$ is a unit of $k[X]$, so that $X_g = X$. Thus the inclusion $X_g \subset X$ is usually not a finite morphism.

**3. Finite and nonfinite extension.** The nodal cubic $C \subset \mathbb{A}^2$ given by $y^2 = x^2(x+1)$ has the usual parametrisation $f \colon \mathbb{A}^1 \to C \subset \mathbb{A}^2$ given by $x = t^2 - 1$, $y = t(t^2 - 1)$. Show that $f$ is finite, that is, $k[\mathbb{A}^1]$ is a finite $k[C]$-module. [Hint: $k[C] \cdot 1_{k[t]}$ contains $x, y$; what more do you need to get $k[\mathbb{A}^1]$? You might start by finding a basis of the vector space $k[t]/k[x, y]$.]

Now replace $\mathbb{A}^1$ by the hyperbola $H : s(t-1) = 1 \subset \mathbb{A}^2_{\langle t,s\rangle}$ and consider the polynomial map $f : H \to C$ given by $x = t^2 - 1$, $y = t(t^2 - 1)$. Show that $f$ is a bijective map. Show that it is not finite (that is, $k[H]$ is not a finite $k[C]$-module).

**4. Similar exercise.** The cuspidal cubic $\Gamma : y^2 = x^3$ has the parametrisation $\mathbb{A}^1 \to \Gamma$ given by $x = t^2$, $y = t^3$. Show that it is finite. On the other hand $H = \mathbb{A}^1 \setminus 0$ defined by $st = 1$ is a nonsingular curve, and $x = t^2$,

$y = t^3$ maps $H$ isomorphically to $\Gamma \setminus (0,0)$. Show that $H \to \Gamma$ is not finite. (Since it misses the singular point, we don't want to allow it as a resolution of singularities.)

**5. Explicit normalisation.** Let $A$ be a UFD with field of fractions $K = \operatorname{Frac} A$, and assume 2 is invertible in $A$. For $a \in A$, consider the quadratic field extension $K(\alpha)/K$ where $\alpha^2 = a$ (that is, $\alpha = \sqrt{a}$). If $a$ is square-free, show that $A[\alpha] \subset K(\alpha)$ is integrally closed. [Hint: write out the minimal polynomial of $c + d\alpha$.]

Let $A$ be a UFD with $K = \operatorname{Frac} A$, and assume that 6 is invertible. Let $a, b \in A$ be square-free coprime elements. Consider the cubic extension field $L = K(\sqrt[3]{a^2 b})$. Prove that the integral closure of $A$ in $L$ is generated by $y, z$ satisfying $y^3 = a^2 b$, $z^3 = ab^2$. The ideal of all relations holding between $y, z \in L$ needs a third generator. Find it.

If $a = (x-1)(x-2)$ and $b = x(x+1)$, determine the normalisation of the affine plane curve $y^3 = ab^2$.

**6. Normalisation of monomial curve.** Following on from the cuspidal cubic $y^2 = x^3$, determine the normalisation of $k[x,y]/(y^2 - x^5)$. Same question for $k[x,y]/(y^3 - x^7)$. More generally, if $a, b$ are coprime, find the normalisation of $x^a = y^b$. [Hint: If you want to write $x = t^b$ and $y = t^a$ you are on the right track. However, for this to be a normalisation, you still have to establish that $t \in \operatorname{Frac}(A)$ where $A = k[x,y]/(x^a - y^b)$. In other words, express $t$ in terms of $x$ and $y$.]

**7. Trace in a finite field extension.** Let $K \subset L$ be a finite field extension. Recall from Galois theory that any $y \in L$ has a *minimal polynomial*, an irreducible polynomial

$$p(T) = T^d + c_{d-1}T^{d-1} + \cdots + c_1 T + c_0 \in K[T]$$

such that $p(y) = 0$, so that $K[y] = K[T]/(p(T))$; it follows that $K[y] = K(y)$ is a field, since $(p(T))$ is a maximal ideal. We say that $L/K$ is a *primitive extension* with generator $y$ if $L = K(y)$.

Consider the multiplication map $\mu_y \colon L \to L$ consisting of multiplication by $y$. If $L/K$ is a primitive extension, write out the matrix of $\mu_y$ in the basis $1, y, \dots, y^{d-1}$, and deduce that its trace is $\operatorname{Tr}_{L/K} \mu_y = -c_{d-1}$.

In general, prove that the trace of $\mu_y$ equals $-c_{d-1}[L : K(y)]$. [Hint: let $b_j$ for $j = 1, \dots, [L : K(y)]$ be any basis of $L/K(y)$, and calculate the trace of $\mu_y$ in the basis $y^i b_j$ of $L/K$.