Normal characterises DVR

Let A be a 1-dim Noetherian local integral domain

Then A is normal if and only if A is a DVR.

Note: 1-dim local integral domain
<=> A has exactly 2 prime ideals 0 in m.
Normal = A is integrally closed in K = Frac A.

Step 1. 1-dim Noetherian integral domain.
For any nonzero x in m, there exists y in A, y notin (x)
such that m = { z in A | z*y in (x) }.

Consider the module A/(x). Any nonzero ybar in A/(x), has
an annihilator ideal Ann ybar. In terms of a representative
y in A this means that y is not in (x), and Ann is the ideal
of A
Ann ybar = { z in A | z*y in (x) }.

If Ann ybar is maximal among all the ideals Ann ybar then it
is prime. Given 1-dim local, the only possibility is
Ann ybar = m.

Work with y in A. Suppose z1, z2 notin Ann ybar. Then
z2*y notin (x), so that Ann (z2*ybar) contains Ann ybar,
and the maximal assumption implies they are equal. Then
z1 notin Ann ybar implies z1*z2*y notin (x), and also the
product z1*z2 notin (x). Therefore the ideal Ann ybar is
prime.

Note: The submodule A.ybar in A/(x) is then isomorphic to
the residue field A/m. This means that m is an _associated
prime_ of A/(x), so that the result is part of primary
decomposition: quite generally any finite module M has a
Jordan-Holder sequence with successive quotients of the
form A/P with P a in Spec A an associated prime of M.

[Step 1 did not use integrally closed.]

Step 2. If x, y as in Step 1, then element y/x in Frac A
is not in A, but m(y/x) in A. Therefore, dichotomy: either

(I) (y/x)*m subset m. This implies that y/x is integral over A
by the determinant trick. This is impossible if A is integral,
because y/x in A, contradicting y notin (x).

(II) (y/x)*m = A. This implies there exists some t in m such that t = x/y. Then m = (t) is principal: in fact for z in m, (y/x)*z in A, so that z = t*((y/x)*z) in (t).