Projective nonsingular model of a curve C and the canonical class K_C – a simpler treatment

1 Nonsingular model as Proj of a graded ring

A function field in 1 variable is $K = k(\Gamma)$ for an algebraic curve Γ up to birational equivalence. For $x \in K$ a transcendence basis, $K/k(x)$ is a finite field extension of degree d.

As in Part 1, the integral closure A_x of $k[x]$ in K is a finite $k[x]$-algebra. It is the affine coordinate ring $A_x = k[C_x]$ of a nonsingular affine curve $C_x \to \mathbb{A}^1_x$. Equivalently, I can write $C_x = \operatorname{Spec} A_x$. Ditto for $y = x^{-1}$, giving C_y.

When eventually defined, the whole nonsingular projective curve is the union $C = C_x \cup C_y$ glued along their common open set

$$C_x \setminus (x = 0) \cong C_y \setminus (y = 0). \tag{1.1}$$

Rather than affine pieces over $\mathbb{A}^1_x \cup \mathbb{A}^1_y$, I want to work directly with a projective curve $\pi\colon C \to \mathbb{P}^1$: for this, introduce the graded ring $S = k[s_1, s_2]$ with s_1, s_2 independent variables of degree 1, and set $x = s_1/s_2$, $y = s_2/s_1$. As a graded ring, S has an action of $\mathbb{G}_m = k^\times$ given by

$$(s_1, s_2) \mapsto (\lambda s_1, \lambda s_2) \quad \text{for } \lambda \in k^\times. \tag{1.2}$$

This action extends to the field $k(s_1, s_2)$, and its invariant subfield is $k(x)$, where $x = s_1/s_2$. In common sense terms, I can view $k[x]$ as starting from the graded ring S and setting $s_2 = 1$. This is the simplest case of the familiar homogeneous-inhomogeneous trick of projective geometry.

Conversely, starting from $k[x]$, I get the graded ring S by adjoining a variable t in degree 1 and setting $s_1 = xt$ and $s_2 = t$. Or after setting $s_1 = 1$ in $k[s_1, s_2]$, it provides $t = y = x^{-1}$. In other words, $S = \bigoplus_{n \geq 0} k[x]t^n$, with t a place marker distinguishing 1 in degree 0 from $t^n \cdot 1 = y^n$ in degree $n \geq 0$.

Corresponding to the algebraic extension of function fields $K/k(x)$, I now construct the field extension $L/k(s_1, s_2)$ as the *composite*

$$\begin{array}{ccc} K & — & L \\ | & & | \\ k(x) & — & k(s_1, s_2) = \operatorname{Frac} S \end{array} \tag{1.3}$$

Just as S is obtained from $k[x]$, the composite extension L comes from $K = k(\Gamma)$ by introducing a variable t as place marker for the degree, writing

$s_1 = xt$ and $s_2 = t$, and setting $L = K(t)$. It has transcendence degree 2 over k, but is just a simple transcendental extension of K, with K recovered inside it as the invariant subfield of a $k^\times$ action.

Now $\mathbb{A}^2_{\langle s_1,s_2\rangle} = \operatorname{Spec} S$ is the usual affine cone over $\mathbb{P}^1$. The two affine pieces C_x and C_y of the curve C are given by the integral closures of $k[x]$ and $k[y]$ in K. In the following construction, B is the affine coordinate ring of the affine cone over C. It has a $k^\times$ action, so is a graded ring. As discussed below, its generators have *different degrees* ≥ 1, which is possibly an unfamiliar feature of the construction.

Proposition 1.1 *(i) Write $B \subset L$ for the integral closure of S. Then B is a finite graded S-algebra. The rings A_x and A_y are obtained as*

$$(B[1/s_2])^0 = A_x \quad \textit{and} \quad (B[1/s_1])^0 = A_y. \tag{1.4}$$

The superscript means homogeneous of degree 0, that is, $k^\times$-invariant.

(ii) The affine variety Spec B *is the affine cone over the projective curve C. It has a $\mathbb{G}_m = k^\times$ action with $C = \operatorname{Proj} B = (\operatorname{Spec} B \setminus \{0\})/k^\times$.*

(iii) The integral closure B coincides with the sections ring

$$R(C,D) = \bigoplus_{n\geq 0} \mathcal{L}(C,nD) \tag{1.5}$$

(discussed in [Part 4, 9.2]*), with D the divisor of poles of x.*

(iv) The generators s_1, s_2 of S define a free pencil $|D|$ on C, and the free pencil trick of [Part 4, 9.4–9.5] *implies that B is a free graded S-module*

$$B = S\cdot 1 \oplus S\cdot w_2 \oplus \cdots \oplus S\cdot w_d \cong \bigoplus S(-a_l) \tag{1.6}$$

with $w_1 = 1$ of degree 0, and w_l of degree

$$a_1 = 0 < a_2 \leq \cdots \leq a_d. \tag{1.7}$$

Here, of course, $d = [K : k(x)]$ is the degree of $\pi\colon C \to \mathbb{P}^1$.

Remark 1.2 (a) The *Serre shift* notation $S(-a_l)$ means $S = k[s_1, s_2]$ as a graded S-module with grading shifted by $-a_l$: its degree n part is $k[s_1, s_2]_{n-a_l}$ so that in (1.6), degrees match up after multiplying by w_l of degree a_l.

(b) The proofs are not hard. A little care when taking integral closure ensures that the new integral elements w_l are homogeneous of degree $a_l \geq 1$ for $l = 2, \dots, d$.

(c) The curve C has genus $g = \sum_{l\geq 2}(a_l - 1)$. The range of summation $l \geq 2$ excludes the single negative summand $a_1 - 1 = -1$. The only case the generators $w_2, \dots, w_d$ all have degree 1 is when $g(C) = 0$.

(d) The graded ring B gives rise to the standard affine pieces C_x and C_y of C and their coordinate rings $A_x = k[C_x]$ and $A_y = k[C_y]$. The free graded result (iv) implies that A_x is a free $k[x]$-algebra with basis $u_1 = 1, u_l = w_l/s_2^{a_l}$, and, likewise, A_y a free $k[y]$-algebra with basis $v_1 = 1, v_l = w_l/s_1^{a_l}$. The coordinate change is, of course,

$$y = 1/x, \quad v_l = u_l/x^{a_l}. \tag{1.8}$$

The hyperelliptic curve of [Part 1, Example 3.5] provides a simple and convincing example. See also Examples 2.7–2.8.

The generators x and u_l of A_x are homogeneous elements of L of degree 0 under its $k^\times$ action (that is, $k^\times$-invariant). When the degree m is understood, I use the same letter for a homogeneous form $f \in B$ and the corresponding polynomial $f/s_2^m \in A_x$; informally, set $s_2 = 1$ on the affine curve C_x.

(e) Inseparable OK. Assuming the extension $K/k(x)$ is separable allows convenient reference to textbooks for the proof of finiteness of normalisation. However, a couple of little tricks also cover the inseparable case: see 3.1 in my 2019 notes. This is easy stuff. The essential point is to use the result from Kaplansky that (after $K/k(x)$ is replaced by a normal extension $K^\nu/k(x)$), the field extension is a composite of a separable extension and a purely inseparable extension of $k(x)$. The purely inseparable part of the extension is then $k(x^{1/p^n})$, and in this case finiteness of integral closure is elementary.

2 Canonical module $\mathcal{K}_B$ and canonical class K_C

I define the *canonical module* of $B = R(C, L)$ to be $\mathcal{K}_B = \mathrm{Hom}_S(B, S(-2))$. As a graded S-module, it is isomorphic to $\bigoplus_{l=1}^d S(a_l - 2)$ in view of (1.6). The first summand has shift -2, and the following summands shift ≥ -1.

It is $\mathbb{Z}$-graded, with graded piece

$$\mathcal{K}_{B,m} = \operatorname{Hom}_S(B, S(-2+m))_0 = \bigoplus_{l=1}^{d} k[s_1, s_2]_{a_l-2+m} \tag{2.1}$$

for $m \in \mathbb{Z}$. (The subscript 0 means or graded of degree 0.)

This space is nonzero for $m \geq 1 - a_d$, as required to provide the irregularity $l(K_C - nD)$ in the RR formula for $l(nD)$ when $n = -m \leq a_d - 1$. The genus $g(C) = \sum_{l=2}^{d}(a_l - 1)$ is the dimension of the degree 0 graded piece $(\mathcal{K}_B)_0$. The first summand with shift -2 provides the irregularity of K_C; that is, the 1 term of $1 - g$ in RR.

The module $\mathcal{K}_B$ provides the beautiful numerical properties of the canonical class (for more on this, redo Ex. 5.11 and 5.12). The small drawback is that as currently described, it does not seem to have much to do with C itself: where is the canonical divisor K_C of C? The answer is as follows:

Theorem 2.1 *(1) A nonzero graded homomorphism $\varphi \in \mathcal{K}_{B,m}$ has a well defined effective divisor* $\operatorname{div}\varphi$ *on C. See 2.3, Definition 2.5.*

(2) The divisors $\operatorname{div}\varphi - mD$ *are linearly equivalent for all m and all nonzero $\varphi \in \operatorname{Hom}(B, S(-2+m))$.*

(3) Define the canonical class K_C as the divisor class $\operatorname{div}\varphi - mD$*. Then*

$$\mathcal{K}_{B,m} = \mathcal{L}(C, K_C + mD) \quad \textit{for } m \in \mathbb{Z}. \tag{2.2}$$

Addendum 2.6 adds several items to this main result: (4) independence of the transcendental generator x. Then, under the additional assumption that $\pi\colon C \to \mathbb{P}^1$ is separable: (5) Hurwitz's formula

$$K_C = -2D + \text{Ramification divisor}, \tag{2.3}$$

and (6) the trace homomorphism $B \to S$ defines a preferred element $s \in \mathcal{L}(C, K_C + mD)$ with divisor R. Finally, (7) a rational Kähler 1-form $\Omega^1_{k(C)/k}$ has divisor class in K_C; up to this point, my treatment of K_C in Theorem 2.1 makes no use of differential forms, and does not require π to be separable.

2.1 Premultiplication

To relate the graded module $\mathcal{K}_B$ to the curve C, the first step is to transform it from a free S-module of rank d into a module of rank 1 over the S-algebra

B. This step is a basic ingredient in any treatment of duality, and is called *premultiplication.*

For $E \subset F$ a finite field extension of degree d, view F as a d-dimensional vector space over E. The dual $F^\vee = \operatorname{Hom}_E(F, E)$ is again a d-dimensional vector space over E. However, $F^\vee$ also has a *premultiplication* action by F that makes it into a 1-dimensional vector space over F: multiplication by x takes $\varphi \in \operatorname{Hom}_E(F, E)$ into the new map

$$x \cdot \varphi \in \operatorname{Hom}_E(F, E) \quad \text{given by} \quad (x \cdot \varphi)(y) = \varphi(xy) \quad \text{for } y \in F. \tag{2.4}$$

Please think this through: of course, F does not act on the target E, but it acts on the Hom space $\operatorname{Hom}_E(F, E)$ by multiplying in the domain before applying the map. Prove the following as an exercise.

Lemma 2.2 *Choose a basis $w_1, \dots, w_d$ for F over E. Multiplication by x is given by a $d \times d$ matrix $M \in \mathrm{GL}(d, E)$. In the dual basis $(w_1^\vee, \dots, w_d^\vee)$, the element $x \in F$ acts on $F^\vee = \operatorname{Hom}_E(F, E)$ by the transpose matrix ${}^{\mathrm{tr}}M$.*

Suppose the basis has $w_1 = 1_E$. Then the dual basis element $w_1^\vee \in F^\vee$ bases $F^\vee$ over F. [Hint: $x \cdot w_1^\vee$ takes $1 \mapsto x$, and any $\varphi \in F^\vee$ is determined by $\varphi(1)$.]

The dual $F^\vee$ is a 1-dimensional F vector space, but with no fixed basis. Denoting it by $F_E^\vee$ would indicate its dependence on E. For a curve C the divisor *class* K_C is independent of the transcendental generator x or the morphism $\pi\colon C \to \mathbb{P}^1$ (cf. Addendum 2.6(5–6)), but there is no well defined canonical divisor. *Duality is a relative notion.*

Duality applies equally to modules over a ring. Proposition 1.1(iv) says that the S-algebra B is a free graded S-module. Its dual is the canonical module $\mathcal{K}_B = \operatorname{Hom}_S(B, S(-2))$; the shift -2 only modifies the grading. Premultiplication means $g \in B$ takes $\varphi \in \mathcal{K}_B$ into

$$g \cdot \varphi \quad \text{given by} \quad g \cdot \varphi\colon p \mapsto \varphi(g \cdot p). \tag{2.5}$$

It makes $\mathcal{K}_B$ into a torsion-free graded B-module of rank 1. This is an analog of a fractional ideal in a number field.

The case when $\mathcal{K}_B$ is a free B-module is important, but is a special case. We say that B is *projectively Gorenstein.* See the discussion around Example 2.8.

From a highbrow point of view, standard arguments in scheme theory turn a graded B-module M into an *associated sheaf* $\widetilde{M}$ on $C = \operatorname{Proj} B$. Here $\widetilde{\mathcal{K}_B}$ is a torsion-free sheaf of rank 1, and hence (over the nonsingular curve C), a locally free sheaf of rank 1 (also known as *line bundle* or *invertible sheaf*). The divisor of a section or rational section of the sheaf $\widetilde{\mathcal{K}_B}$ provides a divisor class K_C with the properties required for Theorem 2.1. In other words $\widetilde{\mathcal{K}_B} = \mathcal{O}_C(K_C)$. Here, rather than appeal to the highbrow theory, I make an effort to untangle the definitions in elementary language; the highbrow theory is not especially difficult for the reader with a modest background in schemes.

2.2 Localising $\pi\colon C \to \mathbb{P}^1$ over $P \in \mathbb{P}^1$

We know B and $\mathcal{K}_B$ as dual modules over S or over $\mathbb{P}^1$. I need to know how a form $f \in B$ or a graded homomorphism $\varphi \in \mathcal{K}_B$ behaves at the different points of the cover $\pi\colon C \to \mathbb{P}^1$. I treat the localisation of C over $P \in \mathbb{P}^1$ in some detail, for use in the proof of Theorem 2.1 and Addendum 2.6. In what follows, I mostly take for granted the "homogeneous to inhomogeneous" passage between B (an algebra over $S = k[s_1, s_2]$) and A_x (an algebra over $k[x]$) discussed in Proposition 1.1 and Remark 1.2(d): informally, set $s_2 = 1$, and don't worry too much about the homogeneous degree.

A point $Q \in C_x$ has image $P = \pi(Q) \in \mathbb{A}^1_x$, with P the point $x = a$. Set $\{Q_i\} = \pi^{-1}P$ for $i = 1, \dots, \#\pi^{-1}P$. The local parameter $z_P = x - a$ at P generates the maximal ideal of $\mathcal{O}_{\mathbb{P}^1,P}$. Viewed as the polynomial function $z_P \in A_x$, its zeros define the effective divisor on C

$$\pi^*P = \operatorname{div}_0 z_P = \sum d_i Q_i \quad \text{where } v_{Q_i}(z_P) = d_i. \tag{2.6}$$

This has degree $d = \sum d_i$ and is linearly equivalent to D, the divisor of poles of x or of $z_P = x - a$.

The local ring $\mathcal{O}_{\mathbb{P}^1,P}$ of $P \in \mathbb{P}^1$ is the subring of the rational function field $k(x)$ of functions regular at P. The affine coordinate ring A_x of C_x is a free algebra over $k[x]$; write $\mathcal{O}_{C,P} = A_x \otimes_{k[x]} \mathcal{O}_{\mathbb{P}^1,P}$ for its localisation at P. This is the *semilocal ring* $\bigcap_{Q_i} \mathcal{O}_{C,Q_i} \subset k(C)$, with finitely many maximal ideals $m_{Q_i} \cap \mathcal{O}_{C,P}$. It is a free $\mathcal{O}_{\mathbb{P}^1,P}$-algebra of rank d, an integral domain, and in fact a unique factorisation domain, as I now discuss.

The quotient $\mathcal{O}_{C,P}/(z_P \cdot \mathcal{O}_{C,P})$ is a d-dimensional algebra over k, clearly never an integral domain for $d > 1$. It decomposes as the direct sum

$$\mathcal{O}_{C,P}/(z_P \cdot \mathcal{O}_{C,P}) = \bigoplus \mathcal{O}_{C,Q_i}/(z_P \cdot \mathcal{O}_{C,Q_i}) \tag{2.7}$$

corresponding to the divisor π^*P of (2.6), with summands

$$\mathcal{O}_{C,Q_i}/(z_P \cdot \mathcal{O}_{C,Q_i}) \cong \sum k[z_{Q_i}]/(z_{Q_i}^{d_i}) \quad \text{where } v_{Q_i}(z_P) = d_i. \tag{2.8}$$

The factors in (2.7) and (2.8) only depend on the local rings $\mathcal{O}_{C,Q_i}$ at Q_i and the d_i, so that (2.8) holds whatever local parameters z_{Q_i} I choose.

If $d_i = 1$, the local parameter z_P at $P \in \mathbb{P}^1$ is also a local parameter at $Q_i \in C$; the morphism $\pi\colon C \to \mathbb{P}^1$ is then *unramified* or *etale* locally at Q_i. However, z_P is a function on $\mathbb{P}^1$, so unable to distinguish the different points Q_i over P. To see the factors in (2.7), I must do something different.

Lemma 2.3 *Write $A_x = \bigoplus_{l=1}^d k[x]\cdot u_l$ as in Remark 1.2(d). The affine curve $C_x = \operatorname{Spec} A_x$ is then contained in $\mathbb{A}^d = \mathbb{A}^1_x \times \mathbb{A}^{d-1}$, with $u_2, \dots, u_d$ coordinates on the second factor. The u_l distinguish the different points $Q_i \in \pi^{-1}P$ over each $P : (x = a)$.*

In more detail, for each Q_i, take a general linear combination $\sum_{l=2}^d \alpha_{il} u_l$ that is zero at Q_i (with constants $\alpha_{il} \in k$), and set $z_{Q_i} = x - a + \sum_{l=2}^d \alpha_{il} u_l$. Then

(i) $z_{Q_i} \in A_x \subset \mathcal{O}_{C,P}$ is a local parameter at Q_i and a unit at Q_j for $j \neq i$.

(ii) The semilocal ring $\mathcal{O}_{C,P}$ is a UFD. It has prime elements z_{Q_i} and units $\mathcal{O}^\times_{C,P} = \bigcap_{Q_i} \mathcal{O}^\times_{C,Q_i}$. Any $f \in k(C)^\times$ is of the form $f = f_0 \cdot \prod_i z_{Q_i}^{a_i}$ with $a_i = v_{Q_i}(f)$ and $f_0 \in \mathcal{O}^\times_{C,P}$ a unit. In particular,

$$z_P = g \cdot \prod\nolimits_i z_{Q_i}^{d_i} \quad \text{with } \pi^*P = \sum d_i P_i \text{ as in (2.6).} \tag{2.9}$$

(iii) Write $g_i = \prod_{j\neq i} z_{Q_j}^{d_j} \in A_x$. Each g_i is in the semilocal ring $\mathcal{O}_{C,P}$, is a unit at Q_i, and maps to zero in the other summands of (2.8). Thus the g_i base the summands in (2.8) as Artinian modules over $\mathcal{O}_{\mathbb{P}^1,P}$ or over the semilocal ring $\mathcal{O}_{C,P}$.

(iv) For each i, set $\lambda_i = g_i + z_{Q_i}^{d_i}$. Then $\lambda_i \in \mathcal{O}^\times_{C,P}$ and the elements $e_i = g_i/\lambda_i \in \mathcal{O}_{C,P}$ are idempotents for the direct sum decomposition (2.7) of Artinian rings.

Proof The point $Q_i \in C \subset \mathbb{A}^d$ is a nonsingular point of a curve, and z_{Q_i} defines a hyperplane of $\mathbb{A}^d$ through Q_i. For a general choice, it does not contain the tangent line T_{C,Q_i} nor any of the other Q_j, so (i) is clear. I can write any nonzero $f \in k(C)$ as $f = f_0 \cdot \prod z_{Q_i}^{a_i}$ with $a_i = v_{Q_i}(f)$ and f_0 a unit at each Q_i, so $f_0 \in \mathcal{O}^\times_{C,P}$. In particular, the rational function $\prod_i z_{Q_i}^{d_i}/z_P$ has valuation 0 at each Q_i, so is a unit of $\mathcal{O}_{C,P}$; this gives (ii). The product in (iii) is a unit at Q_i and maps to zero in each of the other factors, so the statement is clear. For (iv), λ_i and g_i have the same image in $\mathcal{O}_{C,Q_i}/(z_P)$, with λ_i a unit at every Q_j. Thus $e_i = g_i/\lambda_i$ maps to $1 \in \mathcal{O}_{C,Q_i}/(z_P \cdot \mathcal{O}_{C,Q_i})$ and to 0 in the other factors, as required. $\square$

Remark 2.4 The $z_{Q_i} \in A_x$ are defined on the affine curve C_x, with the good "local parameter" properties of Lemma 2.3 after localisation. Results stated in terms of the semilocal ring $\mathcal{O}_{C,P}$ hold on a Zariski open set of C including all the Q_i, so on a neighbourhood of the fibre $\pi^{-1}P$ that I don't need to specify.

2.3 The divisor of a graded homomorphism $\varphi \in \mathcal{K}_{B,m}$

For a polynomial $f \in A_x$, Lemma 2.3(ii) simplifies to give:

$$v_{Q_i}(f) = \max\bigl\{a \bigm| f \in z_{Q_i}^a \cdot \mathcal{O}_{C,P}\bigr\}. \tag{2.10}$$

For a rational function $f \in k(C)^\times$, getting rid of possible poles of f at the other points Q_j over P leads to the somewhat more involved formula:

$$v_{Q_i}(f) = \max\Bigl\{a \in \mathbb{Z} \Bigm| \prod\nolimits_{j\neq i} z_{Q_j}^n \cdot f \in z_{Q_i}^a \cdot \mathcal{O}_{C,P}\Bigr\} \quad \text{for } n \gg 0. \tag{2.11}$$

The valuation $v_{Q_i}(\varphi)$ of a graded homomorphism φ is defined as the dual to (2.10) under the S-bilinear perfect pairing

$$\mathcal{K}_B \times B \to S(-2) \quad \text{that evaluates } (\varphi, f) \mapsto \varphi(f). \tag{2.12}$$

Definition 2.5 Let $\varphi \in \mathcal{K}_{B,m}$ be a graded homomorphism of degree m as in (2.1). Setting $s_2 = 1$ on the affine piece C_x replaces B by A_x and S by $k[x]$, so makes φ into a $k[x]$-homomorphism $A_x \to k[x]$. In the notation of 2.2, Lemma 2.3, the valuation of φ at a point $Q_i \in \pi^{-1}P$ is given by

$$v_{Q_i}(\varphi) = \max\bigl\{a \bigm| \varphi(z_{Q_i}^{-a} \cdot A_x) \subset \mathcal{O}_{\mathbb{P}^1,P}\bigr\}. \tag{2.13}$$

That is, the valuation of $\varphi\colon A_x \to k[x]$ at Q_i is the maximum a for which φ extends to the fractional ideal $z_{Q_i}^{-a} \cdot \mathcal{O}_{C,P}$.

In the rational case, I view a rational graded homomorphism φ as a $k(x)$-linear map on the function fields $k(C) \to k(x)$, and set

$$v_{Q_i}(\varphi) = \max\Bigl\{a \in \mathbb{Z} \Bigm| \varphi\Bigl(\prod\nolimits_{j\neq i} z_{Q_j}^n \cdot z_{Q_i}^{-a}\Bigr) \subset \mathcal{O}_{\mathbb{P}^1,P}\Bigr\} \quad \text{for } n \gg 0. \tag{2.14}$$

The powers of the z_{Q_j} cancel the poles of f at the other points Q_j before applying φ, so this is dual to the procedure of (2.11).

The divisor of a graded homomorphism $\varphi \in \mathcal{K}_{B,m}$ (or rational graded homomorphism) is defined as $\operatorname{div}\varphi = \sum_{Q\in C} v_Q(\varphi)$.

2.4 Proof of Theorem 2.1

Establishing the definition of $\operatorname{div}\varphi$ in Definition 2.5 was the main issue (1). With it in place, the rest follows from the simple points below, together with the fact that rational homomorphisms φ of fixed degree m form a 1-dimensional vector space over $K = k(C)$.

Since $\mathcal{K}_B$ is a graded S-module, multiplying by a form $a_n(s_1, s_2) \in B_n$ takes $\varphi \in \mathcal{K}_{B,m}$ to $a_n \cdot \varphi \in \mathcal{K}_{B,m+n}$. It is clear that

$$\operatorname{div}(a_n \cdot \varphi) = \operatorname{div}(a_n) + \operatorname{div}(\varphi) \quad \text{for } a_n \neq 0. \tag{2.15}$$

This applies in particular to the powers of s_2 with $\operatorname{div} s_2 = D$ that I used tacitly for dehomogenising.

Now $\mathcal{K}_B$ has rank 1 as a B-module, so $\mathcal{K}_B \otimes L$ is a 1-dimensional vector space over the composite field L of (1.3). Each graded component $(\mathcal{K}_B \otimes L)_m$ for $m \in \mathbb{Z}$ is a 1-dimensional vector space over $K = k(C)$.

It follows that for each m the divisors $\operatorname{div}\varphi$ for all $\varphi \in (\mathcal{K}_B \otimes L)_m$ are linearly equivalent. Moreover, $s_2^{-m} \cdot \varphi \in (\mathcal{K}_B \otimes L)_0$, so that the divisors $\operatorname{div}(s_2^{-m} \cdot \varphi) = \operatorname{div}\varphi - mD$ form a single equivalence class for all m and all φ. This proves (2) and defines the divisor class K_C.

Fix one graded homomorphism $\varphi \in \mathcal{K}_{B,n}$ of degree n, to define a representative $K_C = \operatorname{div}\varphi - nD$ (effective or otherwise) of the linear equivalence class K_C. Then according to the definition of RR space,

$$\begin{aligned} \mathcal{L}(K_C + mD) &= \left\{ g \in k(C) \mid \operatorname{div} g + \operatorname{div}\varphi - nD + mD \geq 0 \right\} \\ &= \left\{ g \in k(C) \mid \operatorname{div}(g\varphi s_2^{-n+m}) \text{ is regular} \right\} \\ &= \mathcal{K}_{B,m}, \end{aligned} \tag{2.16}$$

which proves (3). □

2.5 Main Theorem 2.1 continued

Statements (5–6) below depend on the assumption that the transcendence basis x or the morphism $\pi\colon C \to \mathbb{P}^1$ is *separable*, that is, $k(x) \subset k(C)$ is a finite separable extension. The proof of (7) also uses a separable cover. My arguments up to now have not involved separability.

Addendum 2.6 *(4) The divisor class K_C is independent of the transcendental generator x of $k(C)$.*

(5) Hurwitz's formula: $K_C = -2D + R$, *where the ramification divisor R of $\pi\colon C \to \mathbb{P}^1$ is defined in* (2.17). *In particular, the genus of C is given by* $2g - 2 = -2d + \deg R$.

Thus while K_C itself is a divisor class, the choice of π defines a unique effective divisor R in the class of $K_C + 2D$ or $K_{C/\mathbb{P}^1}$.

(6) *The trace homomorphism* $\mathrm{Tr}\colon B \to S$ *is an element of* $\mathcal{K}_{B,2}$*, Its divisor* $\operatorname{div}\mathrm{Tr}$ *is a well defined effective divisor linearly equivalent to* $K_C + 2D$ *or* $K_{C/\mathbb{P}^1}$*. It is equal to the ramification divisor* $R = \sum(d_Q - 1)Q$*.*

(7) *The notion of Kaehler* 1*-form* $\Omega^1_{C/k}$ *and its divisor is treated in 2.5.4. Then any nonzero rational* 1*-form* $s \in \Omega^1_{k(C)/k}$ *has divisor* $\operatorname{div} s$ *linearly equivalent to* K_C*.*

2.5.1 Notes towards the proof, (4)

This is a kind of tower law. Any two different covers $C \to \mathbb{P}^1$ fit together into a commutative square over a common $\mathbb{P}^1$. Say $p_1, p_2\colon C \to \mathbb{P}^1$ and $q_1, q_2\colon \mathbb{P}^1 \to \mathbb{P}^1$ with $q_1 \circ p_1 = q_2 \circ p_2$. Then calculate K_C via p_1 and via $q_1 \circ p_1$, etc. Of course, the RR formula already implies directly that K_C is unique up to linear equivalence.

2.5.2 Definition of the ramification divisor R, (5)

Assume that x is a separable transcendence basis of $k(C)$. The definition of separable is that the minimal polynomial $p \in k[X]$ of any $y \in k(C)$ has distinct roots, or equivalently, that p and its formal derivative p' are coprime in $k[X]$. Standard arguments (including the hcf property $ap + bp' = 1$ with $a, b \in k[X]$) then imply that $\pi\colon C \to \mathbb{P}^1$ has $d = \deg \pi$ distinct points over $P \in \mathbb{P}^1$ for a dense Zariski open set, that is, over all but finitely many points.

For $Q \in C$ set $\pi(Q) = P$, and view Q as one point $Q = Q_i \in \pi^{-1}P$. Recall the d_i from $\pi^*P = \sum d_iQ_i$ in (2.6) and the properties (2.7)–(2.8). It follows from what I just said that for all but finitely many points $P \in \mathbb{P}^1$, all the $d_i = 1$. Thus

$$R = \sum\nolimits_{Q\in C}(d_Q - 1)Q \tag{2.17}$$

is a finite sum, and defines the *ramification divisor* R of π.

Near Q, the cover π defines a finite extension $\mathcal{O}_{\mathbb{P}^1,P} \subset \mathcal{O}_{C,Q}$ of DVRs, and the local parameter of $\mathcal{O}_{\mathbb{P}^1,P}$ factorises as $z_P = \text{unit} \times z_Q^{d_Q}$ as in (2.8). The coefficient $d_Q - 1$ in R is the conversion factor between valuations measured upstairs in units of z_Q and downstairs in units of z_P.

The valuation of a rational function $f \in k(C)$ or a rational graded homomorphism φ could be expressed in terms of B or $\mathcal{K}_B$, and calculated in terms of z_P. However, the definitions used in 2.3 are in terms of z_Q. The coefficient $d_Q - 1$ in the ramification divisor is the difference between measuring in units of z_Q or $z_P = z_Q^d$.

The formula (2.17) relates to separable extensions of DVR, with $z_P = (z_Q)^d$. The local ramification divisor $(d_Q - 1)Q$ arises when calculating the dual

$$\mathrm{Hom}(O_{C,Q}, O_{\mathbb{P}^1,P}) \tag{2.18}$$

2.5.3 Trace, (6)

Do it yourself.

2.5.4 Relation with Kaehler 1-forms Ω^1_C, (7)

My narrative makes no use of Kaehler differentials, that form the mainstay of traditional treatments of RR.

For a ring A and A-algebra B, the module of Kaehler 1-forms $\Omega^1_{B/A}$ is a B-module defined by the universal mapping property for A-derivations $d\colon B \to \Omega^1$. See for example Matsumura [M], Chapter 9 for a professional treatment of its definition and main properties. The more basic treatment given in Shafarevich [Sh], Chapter 3 is adequate for my current needs.

As usual, let $K = k(C)$ be the function field of a curve C over an algebraically closed field k. Then $\Omega^1_{K/k}$ is a 1-dimensional vector space over K, based by dx for any separable transcendental generator $x \in K$. This is elementary: any $z \in K$ is the root of a monic polynomial $p \in k(x)[T]$ that is separable. We can treat p as an implicit polynomial equation $p(x,z) = 0$ for z, and separable means that $\frac{\partial p}{\partial z} \neq 0 \in K$. From this it follows that $dz = -\frac{\partial p/\partial x}{\partial p/\partial z} dx$.

If x is an affine parameter on $\mathbb{A}^1_x \subset \mathbb{P}^1$ then one calculates that dx has a pole of order 2 at infinity, since $x = 1/y$ give $dx = -(1/y^2)dy$.

The local ring $\mathcal{O}_{C,Q}$ at $Q \in C$ is a DVR with local parameter z_Q. One checks by the same kind of argument that $\Omega^1_{C,Q} = \Omega^1_{\mathcal{O}_{C,Q}/k}$ is a free module of rank 1 based by dz_Q, that is $\mathcal{O}_{C,Q} dz_Q$,

For a separable extension, dz_P evaluated in $\Omega^1_{C,Q} = d_Q z_Q^{d_Q - 1} dz_Q$.

There is a different proof based on relative formula for Ω^1.

The whole of my treatment uses Hom and does not use Kaehler differentials.

Example 2.7 (Nonsingular plane curve $C_a \subset \mathbb{P}^2$) Let $C_a \subset \mathbb{P}^2$ be a nonsingular curve of degree a with defining equation $F_a(s_1, s_2, s_3)$. Assume

$(0,0,1) \notin C_a$ so $F(0,0,1) \neq 0$. Then F_a is a monic equation for s_3 over $S = k[s_1, s_2]$, and the homogeneous coordinate ring $B = k[s_1, s_2, s_3]/(F_a)$ is a free module over S with basis $1, s_3, s_3^2, \ldots, s_3^{a-1}$, so that $B \cong S \oplus S(-1) \oplus \cdots \oplus S(a-1)$.

Example 2.8 (Trigonal curve) The case $d = 3$ corresponds to a trigonal curve or 3-to-1 cover $C \to \mathbb{P}^1$. It leads to $B = S \cdot 1 \oplus S \cdot y \oplus S \cdot z \cong S \oplus S(-a) \oplus S(-b)$ with $0 < a \leq b \leq 2a$. The ideal of relations holding between y, z is typically generated by

$$\begin{aligned} y(y-c) &= dz, \\ (y-c)(z-e) &= df, \\ z(z-e) &= fy, \end{aligned} \quad \text{that is,} \quad \bigwedge^2 \begin{pmatrix} z-e & y & d \\ f & z & y-c \end{pmatrix} = 0, \tag{2.19}$$

where $c, d, e, f \in S$ are homogeneous of degree $a, 2a-b, b, 2b-a$. The curve C is nonsingular for fairly general choice of c, d, e, f, and is a trigonal curve $C \to \mathbb{P}^1$ of genus $a + b - 2$.

References

[M] H. Matsumura, Commutative ring theory, CUP, 1986

[Sh] I.R. Shafarevich, Basic algebraic geometry. I, Springer 2013 (3rd Ed.)

[The final couple of paragraphs still need some editing.]