10 4B Graded Rings and the Main Theorem

10.1 The graded ring $B = R(C, D)$

Recall that Chapter 1 defined the rings A_x and A_y as the integral closure of $k[x]$ and $k[y]$ in the function field K, constructed the affine pieces C_x and C_y with coordinate rings $A_x = k[C_x]$ and $A_y = k[C_y]$, and 3.3 finally glued C_x and C_y together in an ad hoc way over the identification $y = x^{-1}$ to construct the nonsingular projective curve $C = C_x \cup C_y$ over $\mathbb{P}^1$.

The two affine coordinate rings A_x and A_y are naturally accommodated inside a single graded ring B. Rather than x and $y = x^{-1}$, think of the ratio $(s_1 : s_2)$, where s_1, s_2 are coordinates on the $\mathbb{A}^2 \setminus \{0, 0\}$ overlying $\mathbb{P}^1$.

Given C and a transcendental generator $x \in k(C)$ defining a morphism $f \colon C \to \mathbb{P}^1$ as in Chapter 1, write $\operatorname{div} x = D_0 - D$, with D_0 the divisor of zeroes of x and D (or D_∞) its divisor of poles. The new object I introduce is the graded ring made up of the RR spaces of nD

$$B = \bigoplus\nolimits_{n=0}^{\infty} B_n, \quad \text{where} \quad B_n = \mathcal{L}(C, nD). \tag{10.1}$$

The multiplication comes from $\mathcal{L}(C, nD) \subset k(C)$. The elements $(x, 1)$ are in $\mathcal{L}(C, D) \subset k(C)$. Multiplying by an element of $k(C)^\times$ changes the divisor by linear equivalence, but does not change the morphism $C \to \mathbb{P}^1_{\langle s_1, s_2 \rangle}$, given by $(s_1 : s_2) = (x : 1) = (1 : y)$.

I write $s_1 = x$ and $s_2 = 1 \in B_1$ for the two degree 1 elements that give homogeneous coordinates of $\mathbb{P}^1$. The degree marks the character of $\mathbb{G}_m = k^\times$ acting by $(s_1, s_2) \mapsto (\lambda s_1, \lambda s_2)$ in the definition of $\mathbb{P}^1$. The element $s_2 \in B_1$ equals $1 \in k(C)$ but has degree 1. It has divisor D, meaning that it does not spend its allowed pole D. The coordinate $x = s_1/s_2 \in k(C)$ has divisor $D_0 - D$, and the graded ring B attributes the zeros and poles separately as $\operatorname{div} s_1 = D_0$ and $\operatorname{div} s_2 = D$. Later in the chapter I will sometimes modify the notation, writing $B_n = \mathcal{L}(C, nD) \cdot s_2^n$ where I view $\mathcal{L}(C, nD)$ as a vector subspace of $k(C)$ and $s_2^n \in B_n$ as a basis element with unambiguous degree n and divisor nD.

The construction of C in Chapter 1, 3.3 uses the B of (10.1) implicitly: since x and the generators u_i of $A_x = k[C_x]$ become regular on C_y on multiplying by a power y^n, they are in $\mathcal{L}(C, nD)$. Each u_i is a homogeneous fraction

$$u_i = U_i/s_2^n \quad \text{with} \quad U_i \in \mathcal{L}(C, nD) = B_n. \tag{10.2}$$

The same holds for the generators v_j of $A_y = k[C_y]$: each v_j is regular on C_y, so also $V_j = v_j x^n \in \mathcal{L}(C, nD)$, and $v_j = V_j/s_1^n$.

10.2 B is a free graded module

Write $S = k[s_1, s_2]$ for the subring of B, and view B as a module over S. One sees that it is a finite S-module.[1]

The elements $s_1, s_2 \in \mathcal{L}(C, D)$ base a free pencil in $|D|$, so that the map $x\colon C_x \to \mathbb{A}^1_x$ extends to the morphism $\varphi_D\colon C \to \mathbb{P}^1$ given by $(s_1 : s_2)$. The coordinates s_1, s_2 form a regular sequence for B. This follows by Chapter 3, Proposition 9.1 on coprime divisors, or by the Castelnuovo free pencil trick. As a reminder, the fact that the divisors of s_1 and s_2 have no common support implies that, under multiplication in B, the two subspaces

$$s_1 B_n \text{ and } s_2 B_n \subset B_{n+1} \tag{10.3}$$

have intersection $s_1 s_2 B_{n-1}$. In other words,

$$0 \to B \xrightarrow{s_1, s_2} B^{\oplus 2} \xrightarrow{\binom{-s_2}{s_1}} B \tag{10.4}$$

is an exact sequence of graded S-modules.

It follows that the graded ring B is a free graded module over $S = k[s_1, s_2]$, of the form

$$B = S \oplus \bigoplus_{i=2}^{d} S(-a_i) \quad \text{with } 0 = a_1 < a_2 \le \cdots \le a_d. \tag{10.5}$$

The *Serre twist* notation $S(-a_i)$ means $S = k[s_1, s_2]$ as a graded S-module with grading shifted by $-a_i$: its degree n part is $k[s_1, s_2]_{n-a_i}$. The direct sum decomposition (10.5) gives B a basis $\{U_i\}_{i=1,\dots,d}$ over S with $\deg U_i = a_i$, so to get degree n we must multiply U_i by $f_{n-a_i} \in S$.

That B is a free S-module follows more-or-less trivially by Nakayama's lemma. It is an exercise to see that it is a free *graded* S-module.

Digression Let M be a finite graded module over S, and $x \in S$ a regular generator of M that is homogeneous of degree > 0. Work with the short exact sequence of graded S-modules.

$$0 \to M \xrightarrow{x} M \to \overline{M} \to 0. \tag{10.6}$$

Lemma 10.1 *Suppose that* $\overline{M} = \bigoplus \overline{S}(-d_i)$ *is a free graded module over* $\overline{S} = S/(x)$*, with homogeneous generators* $\overline{m}_i$ *of degree* d_i*. Lift each generator*

[1] Exercise: Using the results of Chapter 3, prove that $s_1\mathcal{L}(nD) + s_2\mathcal{L}(nD) \twoheadrightarrow \mathcal{L}((n+1)D)$ as soon as $(n-1)D$ has degree $\ge 2g - 1$.

to $m_i \in M$ with $m_i \mapsto \overline{m}_i$. The m_i can be chosen homogeneous of the same degree.

Then the m_i base M, and $M = \sum S(-d_i)$ is a free graded module over S.

For a local ring, this would follow by Nakayama's lemma. The proof illustrates the general informal slogan "graded is a special case of local".

Proof of the lemma First, any lift m_i of $\overline{m}_i$ may be a sum of homogeneous bits of different degrees, but replacing it by the degree d_i piece does not change anything.

Take any homogeneneous element $m \in M$. By assumption, its image in $\overline{M}$ can be written as $\sum a_i \overline{m}_i$. Hence $m - \sum a_i m_i$ is in the kernel of $M \to \overline{M}$, which is xM. Hence $m - \sum a_i m_i = xn$ with $n \in M$ a homogeneous element of degree $\deg m - \deg x$. Now the same argument applies to n. Since M is finitely generated, the degree of its elements is bounded below, so that induction on degree proves that the m_i generate M.

Write $\pi\colon \bigoplus S(-d_i) \to M$ for the surjective homomorphism taking the ith basis element to m_i and set $N = \ker \pi$. Since $\overline{M}$ is free, the reduction of $\pi \bmod x$ is an isomorphism. The fact that $N = 0$ so that π is an isomorphism follows from the Snake Lemma applied to the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & \bigoplus S(-a_i) & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & \bigoplus S(-a_i) & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \bigoplus \overline{S}(-a_i) & \to & \overline{M} & \to & 0
\end{array} \tag{10.7}$$

where the down arrows $M \to M$ and so on are multiplication by x. QED

10.3 Numerology of RR

10.3.1 Sharp form of Main Proposition II

Write B_n for the graded piece of B of degree n. (10.5) gives B_n as the direct sum of S_{n-a_i} for $i = 1, \dots, d$. For $n \geq a_d - 1$, this gives

$$\begin{aligned}
\dim B_n = l(nD) &= \sum\nolimits_{i=1}^{d} (n - a_i + 1) \\
&= 1 - g + nd \quad \text{where } g = \sum\nolimits_2^d (a_i - 1).
\end{aligned} \tag{10.8}$$

10.3.2 Book-keeping

For smaller values of n, some of the $n - a_i + d$ are negative, so I don't want to add them into the dimension of a vector space. If $m \geq -1$, the space S_m of homogeneous forms of degree m has dimension $m + 1$. The correct formula for all m is $(m+1)_+ = \max\{m+1, 0\}$, and for every m I have

$$\begin{aligned} m + 1 = (m+1)_+ - (m+1)_-,& \\ \text{where } (m+1)_- = -m - 1 \text{ for } m \leq -2.& \end{aligned} \tag{10.9}$$

10.3.3 Irregularity of nD

For every $n \in \mathbb{Z}$, the negative terms of the sum (10.8) add to the *irregularity* of nD, that is, the difference between $l(nD)$ and $1 - g + \deg(nD)$:

$$\operatorname{irreg}(nD) = \sum\nolimits_{i=1}^{d} (-n + a_i - 1)_+ \tag{10.10}$$

In particular, for $n = 0$, the divisor $0 = 0D$ has $\mathcal{L}(C, 0) = k$ and irregularity

$$\sum\nolimits_{1}^{d} (a_i - 1)_+ = \sum\nolimits_{2}^{d} (a_i - 1) = g. \tag{10.11}$$

10.4 Canonical module $\mathcal{K}_B$

I define the *canonical module* as $\mathcal{K}_B = \operatorname{Hom}_S(B, S(-2))$, which is again a graded S-module. By (10.5) it is

$$\mathcal{K}_B = S(-2) + \sum\nolimits_{i=2}^{d} S(a_i - 2). \tag{10.12}$$

10.4.1 $\mathcal{K}_{B,n}$ measures the irregularity of nD

The dimension of $\mathcal{K}_{B,n}$ in each degree n equals

$$\dim \mathcal{K}_{B,n} = \sum\nolimits_{2}^{d} (-n + a_i - 1)_+, \tag{10.13}$$

which is the irregularity of nD calculated in (10.10). Thus

$$\dim \mathcal{L}(nD) - \dim \mathcal{K}_{B,n} = \sum\nolimits_{1}^{d} (n - a_i + 1) = 1 - g + nd. \tag{10.14}$$

This gives the complete numerology of RR for the divisors nD and $K_C - nD$.

10.4.2 Towards the main result

The canonical module $\mathcal{K}_B = \mathrm{Hom}_S(B, S(-2))$ is so far only defined as a graded S-module. You might reasonably complain that it does not seem to have too much to do with divisors on C.

To answer this, I show in (10.20) how to define the divisor $\operatorname{div}\varphi$ of a nonzero homogeneous element $\varphi \in \mathcal{K}_{B,n}$ of degree n. The *canonical divisor of C associated with φ* is $K_\varphi = \operatorname{div}\varphi - nD$. It turns out eventually that

$$\mathcal{K}_B = \bigoplus_{m\in\mathbb{Z}} \mathcal{L}(C, K_\varphi + mD) \cdot \varphi. \tag{10.15}$$

The numerology of (10.14) then gives Main Proposition III with $K_C = K_\varphi$.

In (10.15), φ acts as a choice of basis of a rank 1 module: I use it to fix the canonical divisor K_φ in its divisor class, then use it to identify the vector subspace $\mathcal{L}(C, K_\varphi + mD)$ of $k(C)$ with $\mathrm{Hom}(B, S(m-n))$ (more precisely, its rational version $\mathrm{Hom}(B, S(m-n)) \otimes k(\mathbb{P}^1)$ as discussed below). Linear equivalence is all about change of basis in a 1-dimensional vector space.

The arguments proving (10.15) establish at the same time that the linear equivalence class of the divisor K_φ is independent of the choice of n and the nonzero $\varphi \in \mathcal{K}_{B,n}$. This justifies writing it K_C. The class of K_C is in fact also independent of the choice of transcendental generator $x \in k(C)$.

I still have two small issues to explain before the formal statement of the Main Theorem 10.2. Section 10.5 treats *premultiplication*, a basic idea in duality theory. Here it gives the graded S-module $\mathcal{K}_B = \mathrm{Hom}_S(B, S)$ the structure of a graded B-module. Multiplying elements of $\mathcal{K}_B$ by functions in $k(C)$ is a prerequisite for discussing rational equivalence of their divisors.

Next, (10.20) expresses the definition of $\operatorname{div}\varphi$ in terms of the local parameters z_{Q_i} at the different points $\{Q_1, \dots, Q_k\}$ of the fibre $f^{-1}P \subset C$ over $P \in \mathbb{P}^1$. For $P \in \mathbb{A}^1_x$, Section 10.6 treats[2] the localisation $A_x \otimes_{k[x]} \mathcal{O}_{\mathbb{P}^1,P}$ as the semilocal ring $\bigcap_{Q_i \in f^{-1}(P)} \mathcal{O}_{C,Q_i}$ and chooses the local parameters z_{Q_i} so that each z_{Q_i} is a unit at the other points $Q_{i'}$.

10.5 Premultiplication makes $\mathcal{K}_B$ into a graded B-module

Premultiplication gives the S-module $\mathcal{K}_B = \mathrm{Hom}_S(B, S(-2))$ the structure of a graded B-module. The formula to multiply $\varphi\colon B \to S$ by b is

$$(b \cdot \varphi)(x) = \varphi(bx). \tag{10.16}$$

[2] This will eventually move to Chapter 1.

This deceptively simple equation in abstract algebra merits some emphasis as a key issue in all work on duality: S is of course not a B-module, so it does not make sense to multiply elements of S by functions in B. However, you can make the space of homomorphisms to S into a B-module, provided you look before you leap: first do the multiplication in B, then apply φ.

10.5.1 Premultiplication and the dual of a field extension

Let $F_0 \subset F_1$ be a field extension of finite degree $d = [F_1 : F_0]$. Then F_1 is a d-dimensional vector space over F_0, so has a vector space dual $F_1^\vee = \operatorname{Hom}_{F_0}(F_1, F_0)$. Premultiplication makes $F_1^\vee$ into a 1-dimensional vector space over F_1.

Does $F_1^\vee$ have a "natural" basis element? The trace map $\operatorname{Tr}_{F_1/F_0}$ is an element of $F_1^\vee$, and is nonzero if F_1/F_0 is separable,[3] so you can use it as a basis. However, trace is zero for an inseparable extension F_1/F_0. The dual vector space $F_1^\vee$ is still there, but without a preferred basis.

10.6 Affine coordinates and local parameters from B

For a nonconstant morphism $f\colon C \to \mathbb{P}^1$, write $f^{-1}(P) = \{Q_i\}_{i=1,\dots,k}$ for the set theoretic fibre over $P \in \mathbb{P}^1$. I use generators of B to put affine

Figure 10.1: Covering $\varphi\colon C \to \mathbb{P}^1$ with $\varphi^{-1}(P) = \{Q_1, \dots, Q_k\}$

and local coordinates on C, and treat the semilocal ring $\mathcal{O}_{C,P} = \bigcap \mathcal{O}_{C,Q_i}$ systematically. In particular, I show how to choose local parameters z_{Q_i} for each $\mathcal{O}_{C,Q_i}$ such that z_{Q_i} is a unit at Q_j for all $j \neq i$.

I assume that $P \in \mathbb{A}^1_x$ is the point $x = \alpha$. (Working with $P \in \mathbb{A}^1_y$ is similar, and I ignore it here.) Then $x - \alpha$ is a local parameter for the DVR $\mathcal{O}_{\mathbb{P}^1,P}$, and it has valuation $v_{Q_i}(x - \alpha) = m_i \geq 1$ at each Q_i.

Write the basis of B in (10.5) as $U_1 = 1, U_2, \dots U_d$ with U_j of degree a_j. They provide affine coordinates $u_1 = x = s_1/s_2$ and $u_j = U_j/s_2^{a_j}$ for

[3] TO DO: link to Appendix of Chap. 3 on separability. The trace map is nonzero for a separable extension of degree p. Exc: work this out for an Artin–Schreier extension with min poly $x^p - x - \alpha$. Hint: x itself has $\operatorname{Tr} x = 1$ in char 2, but 0 in char > 2. x^2 has $\operatorname{Tr} x = 1$ in char 3, but 0 in char > 3.

$j = 2, \dots, d$ on the affine space $\mathbb{A}^1_x \times \mathbb{A}^{d-1}$ containing C_x. Compared to the material of Chapter 1, there is a small extra precision here. Namely, A_x and A_y have the same rank, and their generators $U_j/s_1^{a_j}$ and $U_j/s_2^{a_j}$ correspond to the summands $S(-a_j)$ of (10.5), and are related simply by the powers $(s_1/s_2)^{a_j}$.

Linear forms in these coordinates also provide parameters for the local ring $\mathcal{O}_{C,Q}$ at each $Q \in C$. Indeed, if $Q \in C$ is the point $(\alpha_1, \dots, \alpha_d)$, the corresponding point of $\mathbb{A}^1 \times \mathbb{A}^{d-1}$ has local coordinates $z_j = u_j - \alpha_j$ for $j = 1, \dots, d$, and since C is nonsingular, one of these cuts C transversally at Q, so generates the maximal ideal m_Q of $\mathcal{O}_{C,Q}$. For a finite set of points $\{Q_i\} \subset C$, it is clear that I can choose the local parameter z_{Q_i} at Q_i (as a k-linear combination of the above local coordinates z_j) so that it is a nonzero at the other points $Q_{i'}$, so that it is a unit of $\mathcal{O}_{C,Q_{i'}}$.

The local ring $\mathcal{O}_{\mathbb{P}^1,P}$ of $P \in \mathbb{P}^1$ is the subring of the rational function field $k(x)$ of functions regular at P. The affine coordinate ring A_x of C_x is a free algebra over $k[x]$; write $\mathcal{O}_{C,P} = A_x \otimes_{k[x]} \mathcal{O}_{\mathbb{P}^1,P}$ for its localisation at P as $k[x]$-module. This is the *semilocal ring* $\bigcap_{Q_i} \mathcal{O}_{C,Q_i} \subset k(C)$.

It has finitely many maximal ideals $m_{Q_i} \cap \mathcal{O}_{C,P}$, with localisation at Q_i the local ring $\mathcal{O}_{Q_i}$. It is a free $\mathcal{O}_{\mathbb{P}^1,P}$-algebra of rank d, an integral domain, and a principal ideal domain: if the parameters z_{Q_i} are chosen as above with z_{Q_i} a unit at Q_j for $i \neq j$ then they are the only prime elements of $\mathcal{O}_{C,P}$.

10.6.1 Semilocal ring $\bigcap_{Q_i} \mathcal{O}_{C,Q_i}$

More generally, if $Q_i \in C$ is a finite set of points of a nonsingular curve, the subring $\bigcap_{Q_i} \mathcal{O}_{C,Q_i} \subset k(C)$ is a semilocal ring. It is an integral domain and is a principal ideal domain: choose the local parameters $z_i \subset \mathcal{O}_{C,Q_i}$ so that z_i is a unit at every Q_j. Then every nonzero ideal if principal, based by $\prod z_i^{m_i}$ for some $m_i \geq 0$. Also every fractional ideal is principal, based by $\prod z_i^{m_i}$ for some $m_i \in \mathbb{Z}$.

10.6.2 Fibre over P

This applies[4] to $\mathcal{O}_{C,P}$ with $Q_i = \varphi^{-1}(P)$ the set theoretic inverse image of P. The ideal $m_P\mathcal{O}_{C,P} = (x-a)\mathcal{O}_{C,P}$ is generated by $\prod z_i^{m_i}$, where $m_i = v_{Q_i}(x-a)$. The quotient $\mathcal{O}_{C,P}/(z_P \cdot \mathcal{O}_{C,P})$ is a d-dimensional algebra over k, clearly never an integral domain for $d > 1$. It decomposes as the direct sum

$$\mathcal{O}_{C,P}/(z_P \cdot \mathcal{O}_{C,P}) = \bigoplus \mathcal{O}_{C,Q_i}/(z_P \cdot \mathcal{O}_{C,Q_i}) \tag{10.17}$$

[4] I will put this into Chapter 1 in the next draft, and get rid of the repetition.

corresponding to the divisor of zeros of $x-a$, with summands

$$\mathcal{O}_{C,Q_j}/(z_P\cdot\mathcal{O}_{C,Q_j})\cong\bigoplus k[z_{Q_j}]/(z_{Q_j}^{m_j})\quad\text{where } v_{Q_j}(z_P)=d_j. \tag{10.18}$$

The factors in (10.17) and (10.18) only depend on the local rings $\mathcal{O}_{C,Q_j}$ at Q_j and the d_j. Thus (10.18) holds for any local parameters z_{Q_j} at Q_j, and I can choose these independently.

10.6.3 The canonical module $\mathcal{K}_B$ has rank 1

By (10.5), B is free of rank d as an S-module. It is an integral domain, so contained in its field of fractions $L=\operatorname{Frac}B$, an extension field of $k(s_1,s_2)$ of degree $[L:k(s_1,s_2)]=d$ (this "composite field" L is discussed at more length in 10.8). If U_i bases B over $S=k[s_1,s_2]$, then the same U_i base the field L over $k(s_1,s_2)$. The dual $\mathcal{K}_B$ (10.12) also has rank d as an S-module, with the dual basis $U_i^\vee$. I can view it as submodule of the d-dimensional vector space $L^\vee$ dual to L over $k(s_1,s_2)$. As just described, $L^\vee$ is also a 1-dimensional vector space over L by premultiplication.

Rank 1 says that any two nonzero homogeneous elements of $\mathcal{K}_B$ are proportional: multiplying by a power of s_2 reduces them to homogeneous of degree 0, after which their ratio is a function in $k(C)^\times$. This is the background to the linear equivalence result of Theorem 10.2, (1).

10.7 Divisor of $\varphi\in\mathcal{K}_{B,m}$ and the main result

The valuation $v_Q(\varphi)$ at $Q\in C$ of a nonzero homogeneous element $\varphi\in\mathcal{K}_{B,m}$ is defined by

$$v_Q(\varphi)=\max\bigl\{\alpha\bigm|\varphi(z_Q^{-\alpha})\in\mathcal{O}_{\mathbb{P}^1,P}\bigr\}. \tag{10.19}$$

Here $P=f(Q)$, and the local parameter z_Q at $Q=Q_i$ is chosen as in 10.6 to be a unit at the other point Q_j of the fibre $f^{-1}(P)$. The divisor $\operatorname{div}\varphi$ is then the conventional sum

$$\operatorname{div}\varphi=\sum_{Q\in C}v_Q(\varphi)Q. \tag{10.20}$$

Theorem 10.2 *(1) The class of the divisor $K_\varphi=\operatorname{div}\varphi-mD$ up to linear equivalence is independent of m and the nonzero $\varphi\in\mathcal{K}_{B,m}$.*

More precisely, given nonzero elements $\varphi_1\in\mathcal{K}_{B,m_1}$ and $\varphi_2\in\mathcal{K}_{B,m_2}$, one is a multiple of the other by a power of s_2 times a rational function:

$$\frac{\varphi_2}{s_2^{m_2}}=f\cdot\frac{\varphi_1}{s_2^{m_1}}\quad\textit{with } f\in k(C) \tag{10.21}$$

so that

$$\operatorname{div}\varphi_2 - m_2 D = \operatorname{div} f + \operatorname{div}\varphi_1 - m_1 D. \tag{10.22}$$

The linear equivalence class $K_C \overset{\text{lin}}{\sim} K_\varphi$ *is the* canonical class *of* C*.*

(2) *Consider the divisor* $K_\varphi = \operatorname{div}\varphi - mD \overset{\text{lin}}{\sim} K_C$ *for a nonzero element* $\varphi \in \mathcal{K}_{B,m}$*. Then*

$$\mathcal{K}_{B,n} = \mathcal{L}(C, K_\varphi + nD)\cdot\varphi. \tag{10.23}$$

The numerical properties of K_C *follow from this by 10.14:*

$$l(K_C) = g \quad \textit{and} \quad \deg K_C = 2g - 2. \tag{10.24}$$

This completes the proof of Main Proposition III.

(3) *The class of* K_C *is also independent of the choice of transcendental element* x*.*

The set-up before the theorem missed one item needed in the proof. As discussed in 10.1, write $B_m = \mathcal{L}(C, mD)\cdot s_2^m$ with $s_2 = 1 \in B_1$. In the definition of B, I replace the finite dimensional k-vector space $B_m = \mathcal{L}(C, mD) \subset k(C)$ by the whole of $k(C)$, and define the bigger ring

$$B \subset B_{\mathrm{rat}} = \bigoplus_{m\in\mathbb{Z}} B_{\mathrm{rat},m}, \quad \text{where} \quad B_{\mathrm{rat},m} = k(C)\cdot s_2^m \tag{10.25}$$

I think of $B_{\mathrm{rat},m}$ as the rational elements as opposed to the regular elements of $\mathcal{L}(mD)$. The element $b = g\cdot s_2^m \in B_{\mathrm{rat},m}$ has divisor $\operatorname{div} b = \operatorname{div} g + mD$.

In the same way, I write

$$\mathcal{K}_B \subset \mathcal{K}_{B,\mathrm{rat}} = \operatorname{Hom}_{k(s_1,s_2)}(B_{\mathrm{rat}}, k(s_1,s_2)(-2)) \tag{10.26}$$

Since the B-module structure of $\mathcal{K}_B$ is given by premultiplication, I can multiply a homogeneous element $\varphi\colon B \to S(-2)$ by $g \in k(C)$ and by powers s_2^m for $m \in \mathbb{Z}$ to get $g\cdot\varphi \in \mathcal{K}_{B,\mathrm{rat}}$, and it clear from the definition in (10.19) and (10.20) that

$$\operatorname{div}(g\cdot\varphi) = \operatorname{div} g + \operatorname{div}\varphi, \quad \text{and} \quad \operatorname{div}(s_2^m\cdot\varphi) = \operatorname{div}\varphi + mD. \tag{10.27}$$

This proves (10.21).

By definition, an element $g \in k(C)^\times$ belongs to $\mathcal{L}(C, K_\varphi + nD)$ if and only if $\operatorname{div} g + K_\varphi + nD \ge 0$, that is $\operatorname{div} g + \operatorname{div}\varphi + (n-m)D \ge 0$. However, again by the premultiplication rule $\operatorname{div}(g\cdot\varphi) = \operatorname{div} g + \operatorname{div}\varphi$, this happens if and only if $g \in \mathcal{K}_{B,n}$.

Provisional draft from here on

10.8 Composite field L

The field of fractions $L = \operatorname{Frac} B$ of the integral domain B is the composite field L is introduced in 10.8. One can view it as the extension $k(C)(s_2)$ of the function field of C. It allows me to relate any two nonzero homogeneous elements of $\mathcal{K}_B$ as a ratio involving a power of s_2 (having divisor mD) times a function $k(C)$. This eventually provides the linear equivalence needed for 10.2, (2).

In algebra, the definition of L is as the composite of the two field extensions $k(x) \subset k(C)$ and $k(x) \subset k(s_1, s_2)$. It is the purely transcendental extension $L = k(C)(s_2)$ obtained by replacing the single transcendental generator x by two independent transcendental generators s_1, s_2 with $x = s_1/s_2$.

In geometry, L is the function field of an affine cone over C in a projective embedding using some $\mathcal{L}(C, nD)$ (say the construction of Chapter 1 of C as a projective curve). It is the field of fractions of $R(C, D)$ or of $A_x[s_2]$, with s_2 the coordinate on the $\mathbb{A}^1$-bundle over C made up by $\mathbb{G}_m$ orbits.

I give the elements s_1, s_2 degree 1, and $k(C)$ degree 0. This gives the rings $k[s_1, s_2]$ and B their usual gradings, and gives both fields $k(s_1, s_2)$ and L an action of $\mathbb{G}_m = k^\times$. The ring $B = R(C, D)$ has the alternative description as the integral closure of S in L.

Caution Although the fields $k(s_1, s_2)$ and L have $\mathbb{G}_m$-actions, *they are not graded rings*: if $K = \operatorname{Frac} A$ is the field of fractions of a graded integral domain A, the only elements $f \in K$ that are sums of homogeneous terms are of the form $f = g/h$ with homogeneous denominator h. For example $1/(1 + s_2) \in k(s_1, s_2)$ has no such expression. Homogeneous of degree $n \in \mathbb{Z}$ means in the λ^n eigenspace of the action of $\lambda \in \mathbb{G}_m$. An element of L that is homogeneous of degree n is of the form $f \cdot s_2^n$ with $f \in k(C)$.

10.9 More results assuming separable

The arguments so far does not make any reference to the separability or otherwise of the transcendental generator x of $k(C)$.

Addendum 10.3 *Suppose in addition that x is a separable transcendence basis of $k(C)$, so that the extension $k(C)/k(x)$ is separable.*

(4) There are only finitely many points $Q \in C$ with $f(Q) = P \in \mathbb{A}^1_x$ and such that the local parameter $z_P = x - \alpha$ of $\mathcal{O}_{\mathbb{P}^1, P}$ has valuation $v_Q(x - \alpha) > 1$, and similarly for y.

This means that the $R = \sum_{Q\in C}(v_Q(x-\alpha)-1)Q$ *is a finite sum, and defines the* ramification divisor *of* $f\colon C\to\mathbb{P}^1$. *It is effective and supported on the points* Q *at which the local parameter* $z_P = x-\alpha$ *downstairs at* $P\in\mathbb{P}^1$ *fails to be a local parameter at* $Q\in C$.

(5) **Hurwitz' theorem:** $K_C \overset{\text{lin}}{\sim} -2D+R$ *In particular, the genus of* C *is determined by* $2g-2 = -2d+\deg R$.

(6) *There is a canonically defined trace homomorphism* $\mathrm{Tr}\colon B\to S$, *which is element of* $K_{B,2}$. *Its divisor is* $\operatorname{div}\mathrm{Tr} = R \overset{\text{lin}}{\sim} K_C+2D$.

10.10 Relation with Kähler 1-forms $\Omega^1_{C/k}$

The traditional description of K_C is in terms of differential 1-forms.

10.10.1 Definition of the space of $\Omega^1_{C/k}$

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It relates to the normal bundle to C in its diagonal
embedding in C x C.
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10.10.2 The identification $\Omega^1_{C/k} = \mathcal{O}_C(K_C)$

Relating 1-forms to the canonical module $K_{B,m}$ defined above