MA4L7 Algebraic curves

Miles Reid

Chapter 3. RR and the geometry of curves

7 Introduction

Part III discusses at some length what RR means and what it can do for us, taking on trust the theorem and some of the characterisations of g. The main overall application of RR is the following: ensuring that C has enough global functions with given poles allows us to study the possible ways of embedding C into projective space. In good cases, this allows us to go from abstract notions such as a curve of genus g or a curve with a linear system g^r_d (see below) to a subvariety $C \subset \mathbb{P}^n$ embedded in a definite space and defined by equations that can be studied in explicit ways. For example, a curve of genus 1 is isomorphic to a plane cubic $C_3 \subset \mathbb{P}^3$. Or a nonhyperelliptic curve of genus 4 is embedded in $\mathbb{P}^3$ as a $(2,3)$ complete intersection $C = Q_2 \cap F_3$.

A particularly important general use of RR in complex analysis is to prove that every compact Riemann surface is actually a projective algebraic curve, so an object of algebraic geometry. These ideas have many applications, and open up several branches of research.

7.1 Linear systems and projective embeddings

The RR spaces $\mathcal{L}(C,D)$ provide ways of mapping C to projective space: a basis $f_{1\dots l}$ of $\mathcal{L}(C,D)$ gives the rational map $\varphi_D\colon C \dashrightarrow \mathbb{P}^{l-1}$ that takes $P \mapsto \big(f_1(P) : \cdots : f_l(P)\big)$. Here I study how to establish whether φ_D is an embedding (an isomorphism of C to its image), and if so, what the divisor D has to do with the geometry of $C \subset \mathbb{P}^{l-1}$.

First, some traditional terminology that goes back to antiquity. For C a nonsingular projective curve and $D = \sum d_P P$ a divisor, write

$$|D| = \big\{\operatorname{div} f + D \;\big|\; f \in \mathcal{L}(C,D)\big\} \qquad (7.1)$$

for the *linear system* of D (or *linear series* of D in the Italian tradition). By construction, the divisors $D_f = \operatorname{div} f + D$ for $f \in \mathcal{L}(C,D)$ run through the effective divisors linearly equivalent to D. The set $|D|$ is parametrised by $\mathbb{P}^{l-1} = \big(\mathcal{L}(C,D)\setminus 0\big)/k^\times$, the projective space of 1-dimensional subspaces of the vector space $\mathcal{L}(C,D)$ (with $|D| = \emptyset$ equivalent to $\mathcal{L}(D) = 0$). We picture the linear system $|D|$ as a bunch of points running around C, parametrised by the projective space $\mathbb{P}^{l-1}$, in much the same way as the pencil of plane conics $\lambda Q_1 + \mu Q_2 = 0$ is parametrised by $\mathbb{P}^1_{\langle\lambda:\mu\rangle}$. A common abuse of language is to speak of $D \in |D|$ to mean a divisor $D_f \in |D|$.

It may happen that the effective divisors $D_f \in |D|$ all have a common part $A > 0$. This means that each $f \in \mathcal{L}(C,D)$ satisfies $\operatorname{div} f + D \geq A$, or in other words, $\mathcal{L}(C,D) = \mathcal{L}(C, D-A)$. The biggest such A is the *fixed part* of $|D|$. We write $|D| = A + |D-A|$, where A is the fixed part and $|D-A|$ the *free part*.

I say that $|D|$ is *free* (or *fixed-point free*) if it has no fixed part. Then for every $P \in C$, some $f \in \mathcal{L}(C,D)$ has valuation $v_P(f) = -d_P$. In terms of the sheaf $\mathcal{O}_C(D)$, this means that the global section $f \in \Gamma(C, \mathcal{O}_C(D)) = \mathcal{L}(C,D)$ is $z_P^{-d_P} \times$ unit of $\mathcal{O}_{C,P}$, so that f is a local basis of $\mathcal{O}_C(D)$ as an $\mathcal{O}_C$-module near P. Thus $|D|$ free is synonymous with $\mathcal{O}_C(D)$ *generated by its global sections.*

Remark 7.1 A free linear system $|D|$ of degree d with $\dim \mathcal{L}(C,D) = r+1$ is traditionally called a g^r_d, meaning that $|D|$ consists of effective divisors of degree d moving in an r-dimensional family. For example, the 2-to-1 morphism $C \to \mathbb{P}^1$ from a hyperelliptic curve to $\mathbb{P}^1$ is given by a g^1_2; the hyperplane linear system $|H|$ on a curve of degree $C_a \subset \mathbb{P}^2$ is a g^2_a.

Two traditional sources of confusion: first, $r + 1 = l(C,D)$ is the dimension of $\mathcal{L}(C,D)$ as a vector space, whereas r refers to its projectivisation $\mathbb{P}^r = \big(\mathcal{L}(C,D)\setminus 0\big)/k^\times$, the parameter space of the linear system $|D|$.

Next, the points of $\mathbb{P}^r = |D|$ correspond to $f \in \mathcal{L}(D)$ up to proportionality, that is, to 1-dimensional subspaces of $\mathcal{L}(C,D)$, whereas the target space of $\varphi_D\colon C \to \mathbb{P}^{l-1}$ has $\mathcal{L}(C,D)$ as its linear forms, so its points correspond to codimension 1 subspaces of $\mathcal{L}(C,D)$. If $|D|$ is a free linear system, its divisors $D \in |D|$ correspond to the hyperplanes of $\mathbb{P}^r$, which is the dual projective space to $|D|$.

7.2 Strategy to prove embedding

How do I establish that $\varphi_D\colon C \dashrightarrow \mathbb{P}^{l-1}$ is an isomorphism to its image $\varphi(C) = \Gamma \subset \mathbb{P}^n$? An algebraic variety is a set of points X with locally defined functions $\mathcal{O}_X$ on it. Thus for $\varphi\colon C \to \Gamma$ to be an isomorphism, we need

(1) that it is bijective as a map of point sets, and

(2) that pullback of functions on Γ provide all the functions on C.

Definition 7.2 A divisor D is *very ample* if $\varphi_D\colon C \to \mathbb{P}^{l-1}$ is an isomorphism to its image $\varphi_D(C) = \Gamma \subset \mathbb{P}^{l-1}$, and the hyperplanes of $\mathbb{P}^{l-1}$ cut out the linear system $|D|$ on C.

First of all, if $|D|$ has a fixed part A then D and $D - A$ define the same morphism $\varphi_D = \varphi_{D-A}\colon C \dashrightarrow \mathbb{P}^{l-1}$. This follows as in the resolution of indeterminacies of Proposition 4.1: for rational functions $f_1, \dots, f_l, g \in k(C)$ the two expressions $f_1 : \cdots : f_l$ and $gf_1 : \cdots : gf_l$ define the same rational map to $\mathbb{P}^{l-1}$. Removing a removable singularity by cancelling a common factor $(z-c)f_1/(z-c)f_2 \mapsto f_1/f_2$ does nothing to a rational function (it is just part of the equivalence relation defining it). However, the hyperplane sections of $\varphi_D(C)$ see only the free part of D, and not the fixed part that is removed.

The main result is the following theorem.

Theorem 7.3 *Let D be a divisor on a nonsingular projective curve C. Then D is very ample if and only if the RR spaces of D on C satisfy the conditions:*

(1) $l(D-P) = l(D) - 1$ for every $P \in C$; equivalently, $\mathcal{L}(D-P) \subsetneq \mathcal{L}(D)$. That is, $|D|$ is free.

(2) $l(D-P-Q) = l(D) - 2$ for every pair of distinct point $P, Q \in C$; that is, $\mathcal{L}(D-P-Q) \subsetneq \mathcal{L}(D-P) \subsetneq \mathcal{L}(D)$. We say that $|D|$ is free and separates points.

(3) $l(D-2P) = l(D) - 2$ for every $P \in C$; equivalently, $\mathcal{L}(D-2P) \subsetneq \mathcal{L}(D-P) \subsetneq \mathcal{L}(D)$. We say that D separates tangent directions *or* separates infinitely near points *in traditional language.*

I start by relating the assumptions of the theorem to the above discussion. (1) is the statement that $|D|$ has no fixed part.

(2) is the condition that $\mathcal{L}(D-P-Q) \subset \mathcal{L}(D)$ has codimension 2, so that there is an $f \in \mathcal{L}(D)$ that vanishes at P and not at Q. In other words, there is a hyperplane of $\mathbb{P}^{l-1}$ through $\varphi_D(P)$ and not through $\varphi_D(Q)$. Thus (2) gives directly that φ_D is bijective on point sets.

To discuss (3), suppose that $P \in C$ appears in D with coefficient d_P, and that z_P is a local parameter of the DVR $\mathcal{O}_{C,P}$. Then by (1) we know that some $f_1 \in \mathcal{L}(D)$ has valuation $v_P(f_1) = -d_P$, so is a basis of $\mathcal{O}_C(D)$ on an affine neighbourhood U of P. Assumption (3) asserts that there is some $f_2 \in \mathcal{L}(D)$ with $v_P(f_2) = -(d_P - 1)$. Then f_2/f_1 is a regular function on U, and is a regular parameter of the local ring $\mathcal{O}_{C,P}$.

In complex analysis, this would complete the proof – we have a injective regular map, and functions on the image include a local analytic parameter at each point P, so the map is an embedding by the implicit function theorem. Also, $\varphi_D(C) \subset \mathbb{P}^{l-1}$ is the image of a compact set in a metric space, so that φ_D is surjective onto its closure.

Proof of the theorem In algebraic geometry, write $\Gamma \subset \mathbb{P}^{l-1}$ for the Zariski closure of the image $\Gamma_0 = \varphi_D(C)$. It is an irreducible subvariety, and by (2), the morphism $\varphi_D \colon C \to \Gamma$ is injective on points. I have to prove that φ_D is surjective to Γ, and that pullback defines an isomorphism of local rings $\varphi_D^* \colon \mathcal{O}_{\Gamma,Q} \cong \mathcal{O}_{C,P}$ for every $P \in C$, where $Q = \varphi(P)$.

The proof consists of three steps: (1) Reduction to a finite morphism $\varphi_x \colon C_x \to \Gamma_x$ on affine pieces $C_x \subset C$ and $\Gamma_x \subset \Gamma$, with the induced homomorphism on the coordinate rings $\varphi_x^* \colon k[\Gamma_x] \subset k[C_x]$ making $k[C_x]$ into a finite module over $k[\Gamma_x]$. (2) Reduction to local commutative algebra with $\varphi_Q^* \colon \mathcal{O}_{\Gamma,Q} \cong \mathcal{O}_{C,P}$ a finite morphism of local rings. (3) Conclusion of the argument by Nakayama's lemma.

Remark 7.4 My treatment fits $\varphi_D \colon C \to \Gamma$ into a diagram $C \to \Gamma \to \mathbb{P}^1$. Then, as in the resolution of singularities of Chapter I, I reinpret C in terms of the integral closure of the affine rings $k[x]$ and $k[x^{-1}]$ of $\mathbb{P}^1$ in the field extension $k(\mathbb{P}^1) \subset k(C)$.

Reduction to affine Write $\Gamma_0 = \varphi_D(C) \subset \mathbb{P}^{l-1}$ and let $\Gamma \subset \mathbb{P}^{l-1}$ be its Zariski closure. Then $\Gamma_0 = \varphi_D(C)$ is an irreducible curve, and Γ adds at most finitely many points $Q \in \Gamma$ (actually none, but that is still to prove). The RR space $\mathcal{L}(C, D)$ gives the linear forms on $\mathbb{P}^{l-1}$, so a choice of homogeneous coordinates $t_{1\dots l}$ for $\mathbb{P}^{l-1}$ gives a basis $f_{1\dots l}$ of $\mathcal{L}(C, D)$ and vice-versa.

Since Γ is a curve, for general coordinates on $\mathbb{P}^{l-1}$, it is disjoint from the codimension 2 subspace $t_1 = t_2 = 0$. The first two elements f_1, f_2 of the corresponding basis of $\mathcal{L}(C, D)$ give effective divisors $\operatorname{div} f_i + D$ with disjoint support.

Write $x = t_1/t_2$ for an affine coordinate on $\mathbb{P}^1$.

Given t_1, t_2 chosen as above, for any $Q \in \Gamma$, I can replace them with appropriate linear combinations so that Q is in the hyperplane $t_1 = 0$ and not in $t_2 = 0$, so that $x = t_1/t_2$ is regular and 0 at Q, that is $x \in \mathcal{O}_{\Gamma,Q}$.

Or, for any given point $P \in C$, I can replace the corresponding f_1, f_2 with appropriate linear combinations so that $f_2 \in \mathcal{L}(C, D) \setminus \mathcal{L}(C, D - P)$ and $f_1 \in \mathcal{L}(C, D - P)$ and $x = f_1/f_2 \in \mathcal{O}_{C,P}$.

Now consider the commutative triangle

$$\begin{array}{ccc} C & \xrightarrow{\varphi_D} & \Gamma \\ & \searrow & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

with $C \to \mathbb{P}^1$ the morphism defined by the ratio $(f_1 : f_2)$, and $\Gamma \to \mathbb{P}^1$ the morphism induced by the linear projection $\mathbb{P}^{l-1} \dashrightarrow \mathbb{P}^1_{\langle t_1, t_2 \rangle}$.

I now reduce to the construction of Part 1. Set $x = f_1/f_2 \in k(C)$. It is a nonconstant rational function on C, so that $k(x) \subset k(C)$ is a finite field extension. As in Part 1, write A_x for the integral closure of $k[x]$ in $k(C)$ and $C_x = \operatorname{Spec} A_x$ for the corresponding affine curve. I can do the same for $y = x^{-1} = f_2/f_1$, and identify C with the union $C_x \cup C_y$.

Since $\Gamma \subset \mathbb{P}^{l-1}$ is disjoint from $t_1 = t_2 = 0$, it is the union of two standard affine pieces Γ_{t_1} and Γ_{t_2} (with $t_i \neq 0$). The affine curve Γ_{t_2} having a finite morphism to $\mathbb{A}^1_x$ with parameter $x = t_1/t_2$ (respectively Γ_{t_1} to $\mathbb{A}^1_y$ with $y = x^{-1} = t_2/t_1$).

This gives affine varieties and morphisms $C_x \to \Gamma_x \to \mathbb{A}^1_x$, with coordinate rings $k[x] \subset k[\Gamma_x] \subset k[C_x]$. What I gain is that $k[C_x]$ is finite as a module over $k[x]$, so a fortiori over $k[\Gamma_x]$.

At this point it clarifies the argument to separate the commutative algebra from the geometry.

Proposition 7.5 *Let $A \subset B$ be finitely generated k-algebras that are integral domains and $m \subset A$ a maximal ideal. Assume the following:*

(i) B is finite as A-module.

(ii) The ideal $I = mB$ is contained in a unique maximal ideal $n \subset B$ and $k = A/m = B/n$.

(iii) $m \to n/n^2$ is surjective.

Then on localising, the morphism of local rings $A_m \to B_n$ is surjective.

In the current case, $A = k[\Gamma_x]$ and $B = k[C_x]$. I have arranged that B is finite over A. Next $m = m_Q$ is the maximal ideal of a point $Q \in \Gamma_x$. The variety $V(I)$ of the ideal $I = mB$ consists of the points of C_x that map to Q. This consists of at most one point of C by (2), with $A/m = \mathcal{O}_{C,P}/m_p = k$. It is nonempty by the following lemma.

Lemma 7.6 *$mB \neq B$, so mB is contained in a maximal ideal of B.*

By contradiction, assume $B = mB$ and suppose b_i generate B. Then $b_i = \sum a_{ij} b_j$ with $a_{ij} \in m$, and the usual determinant trick gives $\Delta B = 0$ where $\Delta = \det(\delta_{ij} - a_{ij})$. Then $\Delta = 0$ because $1_A \in B$, but $\Delta \cong 1 \bmod m$, which is a contradiction.

So $C_x \to \Gamma_x$ is surjective, and since φ_D is injective then $Q = \varphi_D(P)$ for a unique P; this implies (b). Finally, (c) holds since (3) implies that some $f \in \mathcal{L}(C, D - P)$ has $v_P(f) = -(d_P - 1)$ which gives $v_P(f/f_2) = 1$.

Reduction to local Replace $A \subset B$ by their localisations $A_m \subset B_n$. One checks that the following still hold.

(i) B_n is still finite as A_m module.

(ii) The ideal $I_n = mB_n$ is contained in nB_n and we still have $k = A/m = A_m/mA_m$, $k = B/n = B_n/nB_n$.

(iii) $nB_n/n^2B_n = n/n^2$, so that $mA_m \to nB_n/n^2B_n$ remains surjective.

Proof of the local statement We have $I_n \subset n$, and by (3), and the image of I_n generates n/n^2. This means that $n = I_n + n^2$, so that Nakayama's lemma (applied to the B-module n) implies that $I_n = n$.

Now B is a finitely generated k-algebra and n a maximal ideal, it follows by the weak NSS that $B/n = k$ (the same k). Therefore 1 generates $B/I = B/mB$, so that Nakayama's lemma (appplied to the A-module B) implies that 1 generates B.

8 Traditional applications of RR

8.1 Many characterisations of $g = 0$

I have already treated the statement of RR for $C = \mathbb{P}^1$ several times as remarks or exercises. There is a lot to say about it, in much the same way that there is a lot to say about the elements of the empty set.

Proposition 8.1 *Let C be a curve. Equivalent conditions*

(1) There exists a divisor D of degree ≥ 1 such that $l(D) = 1 + \deg D$; or

(1a) the same for every divisor D of degree ≥ 1.

(2) *There exist* $P \neq Q \in C$ *such that* $P \overset{\text{lin}}{\sim} Q$*; or*

(2a) *the same for every* $P, Q \in C$.

(3) $g(C) = 0$.

(4) $C \cong \mathbb{P}^1$.

This is all easy. If $l(D) = 1 + \deg D$ with $\deg D > 1$, the same continues to hold for $D - P$, and by induction we get a divisor of degree 1 with $l(D) = 2$. Then the linear system $|D|$ contains every $P \in C$ as a divisor, proving 2. The map $\varphi_D \colon C \to \mathbb{P}^1$ is an isomorphism by Theorem 7.3.

8.2 Treatment of $g = 1$

The ideas around RR provides practically the whole of the geometric theory and function theory of elliptic curves. First, to restate RR in the special case $g = 1$, it says that $l(D) = \deg D$ for every divisor D of degree ≥ 1. For D of degree 0, either $D \overset{\text{lin}}{\sim} 0 \overset{\text{lin}}{\sim} K_C$ or $l(D) = 0$.

A curve of genus 1 is isomorphic to a plane cubic $C \cong C_3 \subset \mathbb{P}^2$. Just choose any divisor D of degree 3. The $l(D) = 3$, whereas $l(D - P) = 2$ and $l(D - P - Q) = 1$ for every $P, Q \in C$, so that $\varphi_D \colon C \to \mathbb{P}^2$ is an isomorphism to its image by Theorem 7.3. The linear system of lines of $\mathbb{P}^2$ pull back to the set $|D|$ of effective divisors linearly equivalent to D, so that the image $\varphi_D(C)$ is a nonsingular cubic curve.

Next, for the group law, the basic point is that a divisor D of degree 1 on C has $l(D) = 1$, so is linearly equivalent to a uniquely specified effective divisor of degree 1, necessarily a point $P \in C$. This makes the set of points of C into a coset of the group $\operatorname{Pic}^0 C$ of divisor classes of degree 0. We need to specify a point $O \in C$ as the neutral element to get out of the coset and into the subgroup.

This construction is important, so I spell it out: write $\operatorname{Div} C$ for the group of all divisors of C (the free Abelian group generated by the points $\{P \in C\}$), and $\deg \colon \operatorname{Div} C \to \mathbb{Z}$ for the degree map. Its kernel is the group $\operatorname{Div}^0 C$ of divisors of degree 0.

The principal divisors

$$\operatorname{PDiv} C = \{\operatorname{div} f \mid f \in k(C)^\times\} \tag{8.1}$$

also form a group, isomorphic to $k(C)^\times / k^\times$. This is a subgroup of $\operatorname{Div}^0 C$, because by Main Proposition (I) a principal divisor has degree 0.

Now define $\operatorname{Pic}^0 C$ as the quotient group

$$\operatorname{Pic}^0 C = \operatorname{Div}^0 C / \operatorname{PDiv} C = \operatorname{Div}^0 C / \overset{\text{lin}}{\sim}. \tag{8.2}$$

The group law on this is just addition of divisors mod linear equivalence, and the zero element is the class of the zero divisor.

Along with $\operatorname{Pic}^0 C$, consider its coset $\operatorname{Pic}^1 C$ formed by divisors of degree 1 up to linear equivalence. As we have seen, this is in bijection with C itself. Now choosing any point $O \in C$ provides a bijective map $\operatorname{Pic}^0 C \to \operatorname{Pic}^1 C \to C$ by $[D] \mapsto [D+O]$. That is, a divisor class D of degree 0 maps to the divisor class $D + O$, which is linearly equivalent to a unique $P \in C$; the inverse bijection $C \to \operatorname{Pic}^0 C$ takes P to the class of $P - O$. In conclusion, the group law on C is

$$(P, Q) \mapsto (P - O, Q - O) \mapsto (P + Q - 2O) \mapsto (P +_C Q),$$

where the middle step is addition in Pic^0, and $P +_C Q$ is the unique effective divisor linearly equivalent to $P + Q - O$.

There are a couple of exercises concerned with interpreting the traditional geometric $P + Q + R$ form of the group law on a nonsingular plane cubic curve (otherwise known as the secant-tangent construction) [UAG, Chap. 2] within the current treatment.

8.3 $g \geq 2$: canonical embedding versus hyperelliptic

A curve C of genus g has a canonical divisor K with $\deg K = 2g - 2$ and $l(K) = g$. In the main case $g \geq 2$, we have the following dichotomy.

Theorem 8.2 *Write $\varphi_K \colon C \to \mathbb{P}^{g-1}$ for the canonical map of C, defined by $|K_C|$. Then* either *φ_K is an isomorphism to its image $C \subset \mathbb{P}^{g-1}$ and the hyperplanes of $\mathbb{P}^{g-1}$ cut out the canonical system $|K|$ on C.* Or *C has a linear system g^1_2, and φ_K is obtained as the composite of the double covering $C \to \mathbb{P}^1$ given by the g^1_2, followed by the embedding $\mathbb{P}^1 \cong \Gamma_{g-1} \subset \mathbb{P}^{g-1}$ as the rational normal curve of degree $g - 1$.*

Every curve of genus $g = 2$ is hyperelliptic: the canonical system $|K_C|$ is itself a g^1_2.

Proof Equality $\mathcal{L}(K - P) = \mathcal{L}(K)$ holds only for $g = 0$ (when both spaces are zero). For RR would give $l(P) - g = 1 - g + \deg P$, that is, $l(P) = 2$, one of the characterisations of $\mathbb{P}^1$ of Proposition 8.1.

Next if $\mathcal{L}(K - P - Q) = g - 2$ for every $P, Q \in C$ then Theorem 7.3 guarantees that φ_K is an embedding, which is one leg of the dichotomy. It remains to analyse the other leg, when $\mathcal{L}(K - P - Q) = g - 1$ for some $P + Q$. In this case RR gives

$$l(P + Q) - (g - 1) = 1 - g + 2, \quad \text{that is,} \quad l(P + Q) = 2. \tag{8.3}$$

Thus $D = P + Q$ has $l(D) = 2$, so that $|D|$ is a g^1_2. It forms a pencil $|D|$, made up of moving pairs $P + Q \in |D|$ parametrised by $\mathbb{P}^1$, each of which also has $\mathcal{L}(K_C - P - Q) = g - 1$. When P, Q are distinct, they go to the same point under φ_{K_C}. When they coincide $\mathcal{L}(K_C - 2P) = \mathcal{L}(K_C - P)$ so that ever $f \in \mathcal{L}(K_C - P)$ vanishes twice at P, so cannot provide a local parameter at P. The g^1_2 defines a 2-to-1 morphism $\varphi_D \colon C \to \mathbb{P}^1$, so that C is hyperelliptic. Q.E.D.

8.4 Hyperelliptic special linear systems

The hyperelliptic curves $y^2 = f_{2g+2}(x_1, x_2)$ provide the valuable portfolio of introductory examples discussed in introductory Lecture 3. They provide in particular curves of every genus g, and the topological picture of a Riemann surface of genus g. They also play a structural role in the theory of linear systems, starting with their role as counterexamples to canonical embedding as above, and at several points in what follows.

Every special linear system on a hyperelliptic curve C comes from its special pencil $|A| = g^1_2$. I discuss this in more detail: write t_1, t_2 for homogeneous coordinates on $\mathbb{P}^1$, corresponding to a basis $f_1, f_2 \in \mathcal{L}(A)$. For any $b \geq 1$, the homogeneous forms of degree b in t_1, t_2 form a vector space of dimension $b + 1$ based by

$$S^b(t_1, t_2) = \left\{t_1^b, t_1^{b-1}t_2, \ldots, t_2^b\right\}. \tag{8.4}$$

These forms are linearly independent on $\mathbb{P}^1$, as are their pullbacks $S^b(f_1, f_2)$ in $k(C)$ (because the ratio $f_1/f_2 = t_1/t_2$ is a nonconstant function on C, and a transcendental generator of its function field $k(C)$). They base a $(b + 1)$-dimensional vector subspace of $\mathcal{L}(bA)$. In the case $b = g - 1$, this means that $(g - 1)A$ has degree $2g - 2$ and $l((g - 1)A) = g$, and therefore $(g - 1)A \overset{\text{lin}}{\sim} K_C$ is a canonical divisor. It follows that $S^b(f_1, f_2)$ base $\mathcal{L}(bA)$ for $b = 1, \ldots, g - 1$.

The divisor $gA = K_C + A$ has degree $2g$, so is regular and has $l(gA) = g + 1$ by Proposition 5.7, (c). Thus $S^g(f_1, f_2)$ also base the whole of $\mathcal{L}(gA)$, so that the morphism φ_{K_C+A} is also composed of $C \to \mathbb{P}^1$. You have to go

Figure 8.1: Geometric Riemann–Roch: the hyperplanes of $\mathbb{P}^{g-1}$ through the linear span $\langle D\rangle$ of D cut out the complete linear system $|K_C - D|$, and vice-versa. In particular, for $d \le g-1$, a divisor $D = P_1 + \cdots + P_d$ moves in a linear system g^r_d if and only if the points of D in $C \subset \mathbb{P}^{g-1}$ span a projective linear subspace of dimension $d-r-1$.

to $(g+1)A$ before you find a function y on C that is not a rational function of f_1/f_2, so is capable of distinguishing the pairs of conjugate points of $|A|$ and generating $k(C)$ as a quadratic extension of $k(\mathbb{P}^1)$.

Recall that a special linear system $|D|$ on a curve C is one for which $l(D) > 1-g+\deg D$, so that $|K_C-D| \ne \emptyset$. The moving part of $|D|$ is a linear subsystem of $|K_C|$. For a hyperelliptic curve $|K_C| = |(g-1)A| = (g-1)|A|$. It follows that every special linear system on C has a multiple of g^1_2 as its moving part.

8.5 Geometric form of RR

A main feature of the RR formula

$$l(D) - l(K_C - D) = 1 - g + \deg D \tag{8.5}$$

is that, for a given C and given degree $\deg D$ in the range $[0, \dots, 2g-2]$, if $\mathcal{L}(D)$ is bigger than expected, then so is $\mathcal{L}(K_C - D)$. A geometer feels the desire to draw this as the picture of Figure 8.5. If linearly independent, a set of d points in projective space would span a linear subspace $\mathbb{P}^{d-1}$. Linear dependences between the points of D correspond to the dimension of the linear system $g^r_d = |D|$. For example, three points map to collinear point of $\varphi_K(C) \subset \mathbb{P}^{g-1}$ if and only if $D = |P_1 + P_2 + P_3|$ moves in a g^1_3.

9 Clifford's theorem and the free pencil trick

9.1 Multiplying RR spaces and the linear-bilinear problem

So far, I have discussed RR spaces mainly as k-vector subspaces of the function field $k(C)$. This section adds the multiplication in $k(C)$, that defines a bilinear map $\mathcal{L}(D_1) \times \mathcal{L}(D_2) \to \mathcal{L}(D_1 + D_2)$ for any two divisors D_1, D_2.

Lemma 9.4 and Proposition 9.5 give a first introduction to the *linear-bilinear problem*: if a bilinear map $V_1 \times V_2 \to W$ is nondegenerate in some sense, does that imply a lower bound on the rank of the associated linear map $V_1 \otimes V_2 \to W$?

The following argument is quite elementary, but plays a significant role at several points in what follows. I say that two effective divisors A, B on C are *coprime* if they have no points in common. That is $A = \sum a_P P$ and $B = \sum b_P P$ have $a_P, b_P \geq 0$, but no P is in the support of both. Coprime means that vanishing at A and B are independent conditions, so vanishing at both A and B is equivalent to vanishing at $A + B$. This implies the following result.

Proposition 9.1 (coprime divisors) *Let A, B be effective divisors that are coprime, and D any divisor. Then the vector subspaces $\mathcal{L}(D - A)$ and $\mathcal{L}(D - B)$ of $\mathcal{L}(D)$ intersect in $\mathcal{L}(D - A - B)$.*

Write $i_A \colon \mathcal{L}(D - A) \hookrightarrow \mathcal{L}(D)$ for the inclusion map, and similarly for B. These are just inclusion maps (the identity of $k(C)$). Then the sequence

$$0 \to \mathcal{L}(D - A - B) \xrightarrow{\left(-i_B, i_A\right)} \mathcal{L}(D - A) \oplus \mathcal{L}(D - B) \xrightarrow{\binom{i_A}{i_B}} \mathcal{L}(D). \quad (9.1)$$

is exact. Therefore $l(D) \geq l(D - A) + l(D - B) - l(D - A - B)$. □

(9.1) is formally the same shape as a codimension 2 Koszul complex.

9.2 Clifford's theorem

The divisors D with degree in the range $(0, 2g - 2)$ may be irregular. In this range, the maximum value of $l(D)$ is given by the hyperelliptic linear systems $|rA| = r|A|$ discussed in 8.4.

Theorem 9.2 (Clifford's theorem) *Let D be a divisor having degree*

$$d = \deg D \quad \textit{in the range } 0 < d < 2g - 2, \textit{ and } \quad l(D) = r + 1. \quad (9.2)$$

Then $d \geq 2r$. Moreover equality holds only for $|D| = |rA|$ where A is a g^1_2 on a hyperelliptic curve, as in 8.4

Addendum 9.3 *The inequality $d \geq 2r$ holds in the range $-2 \leq d \leq 2g$, and strict inequality holds except for the following cases:*

(1) C is hyperelliptic with $A = g^1_2$, and $D = rA$ for $r = 1, \dots, g - 2$, so that $d = 2r$ and $l(D) = r + 1$.

(2) $D \overset{\text{lin}}{\sim} 0$ *and* $r = d = 0$*, or* $D \overset{\text{lin}}{\sim} K_C$ *and* $r = g - 1$*,* $d = 2g - 2$*.*

(3) $d = -2$ *and* $r = -1$*, or* $d = 2g$ *and* $l(D) = g + 1$*, so* $r = g$*.*

Only case (1) has any content, and is specific to hyperelliptic curves. The rest is formal tidying up. Strict inequality holds for every divisor D of degree -1 (when $l(D) = 0$, so $r = -1$) or $2g - 1$ (when $l(D) = g$, so $r = g - 1$).

Proof Consider the multiplication map $\mathcal{L}(D) \times \mathcal{L}(K_C - D) \to \mathcal{L}(K_C)$. The two RR spaces $\mathcal{L}(D)$ and $\mathcal{L}(K_C - D)$ are k-vector subspaces of $k(C)$ and the map is multiplication in $k(C)$. It is clearly bilinear over k and nondegenerate (this means that $f \neq 0$, $g \neq 0$ implies that $fg \neq 0$).

Lemma 9.4 *If* $V_1 \times V_2 \to W$ *is a nondegenerate bilinear map, the induced linear map*

$$\psi \colon V_1 \otimes V_2 \to W \tag{9.3}$$

has $\operatorname{rank} \psi \geq \dim V_1 + \dim V_2 - 1$.

Proof of Lemma Write $n = \dim V_1$ and $m = \dim V_2$. Recall that $V_1 \otimes V_2$ contains *primitive tensors* $v_1 \otimes v_2$ for $v_1 \in V_1$ and $v_2 \in V_2$. If we write $V_1 \otimes V_2$ as the space $\operatorname{Mat}(n \times m)$ of $n \times m$ matrices, the primitive tensors are the tensors of rank 1. These form the affine subvariety defined by the 2×2 minors of a matrix. This variety clearly has dimension $n + m - 1$.

The kernel of ψ in (9.3) is a vector subspace $V_1 \otimes V_2$ intersecting the primitive tensors only in 0. It follows that its codimension in $V_1 \otimes V_2$ is at least $n + m - 1$.

An algebraic geometer considers it clearer to express the same argument in projective space: $\mathbb{P}(\ker \psi)$ is a linear subspace in the projective space $\mathbb{P}^N = \mathbb{P}(V_1 \otimes V_2)$, and is disjoint from the Segre embedding of $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$. This is the projectivisation of the above subvariety of $\operatorname{Mat}(n \times m)$ of matrices of rank 1. It is a projective subvariety of dimension $\dim V_1 + \dim V_2 - 2$, and a projective linear subspace of smaller codimension must intersect it. Therefore our kernel must have codimension $\geq \dim V_1 + \dim V_2 - 1$. This gives $\operatorname{rank} \psi \geq \dim V_1 + \dim V_2 - 1$ as required. □

For the proof of Clifford's theorem, the map ψ of the lemma has rank $\geq l(D) + l(K_C - D) - 1$ and maps to the g-dimensional space $\mathcal{L}(K_C)$. Putting this together with the RR formula give

$$l(D) + l(K_C - D) - 1 \leq g \tag{9.4}$$

$$l(D) - l(K_C - D) = 1 - g + d. \tag{9.5}$$

Adding the two gives $2l(D) \leq d+2$, that is, $d \geq 2r$. This proves the inequality.

I turn now to the case of equality. Suppose that C has genus g, and that D is a divisor of degree $2r$ with $l(D) = r+1$. Write $E = K_C - D$, so that the RR formula says that $\deg E = 2g-2-2r$ and $l(E) = g-r$. The two divisors D and E appear symmetrically with $r \leftrightarrow g-1-r$. There is nothing to prove if $r \leq 1$ or $r \geq g-2$: if $r=1$ then $|D|$ is a g_2^1, whereas if $r = g-2$ then $|E|$ is a g_2^1, so that C is hyperelliptic and $|D| = |(g-2)E|$.

In the contrary case, both $|D|$ and $|E|$ are linear systems of (projective) dimension ≥ 2. For a point $P \in C$, both $|D-P|$ and $|E-P|$ are still positive dimensional linear systems, so there are $D \in |D|$ and $E \in |E|$ with at least the point P in common, but neither contained in the other (they move indepently in nontrivial linear systems); fix such a D and E.

Write $D' = \gcd(D, E)$ for the greatest common divisor of D and E. By this I mean the greatest divisor that is $\leq$ both D and E. The two divisors $A = D - D'$ and $B = E - D'$ are effective and coprime, and $D'' = D' + A + B = D + E - D' = \operatorname{lcm}(D, E)$ is the least divisor $\geq$ both D and E.

Now $D' + D'' = D + E \overset{\text{lin}}{\sim} K_C$. Proposition 9.1 on coprime pairs applies to A and B and $\mathcal{L}(D'')$, and (9.1) gives

$$l(D') + l(D'') \geq l(D) + l(E). \tag{9.6}$$

Now by assumption D and $E = K_C - D$ are linear systems g_{2r}^r and g_{2s}^s (for $s = g-1-r$), with the biggest possible RR spaces for their degree. The same must apply to $l(D')$ and $l(D'')$.

To spell this out, $D + E = K_C$, and $l(D) + l(E) = g+1$. Also $D' + D'' = D + E \overset{\text{lin}}{\sim} K_C$, and as just proved,

$$l(D') + l(D'') \geq l(D) + l(E). \tag{9.7}$$

Now the Clifford inequality applied to D' and D'' gives

$$\deg D' \geq 2(l(D') - 1), \quad \text{and} \quad \deg D'' \geq 2(l(D'') - 1), \tag{9.8}$$

hence

$$\begin{aligned} 2g-2 = \deg D' + \deg D'' &\geq 2(l(D') + l(D'')) - 4 \\ &\geq 2(l(D) + l(E)) - 4 = 2g-2, \end{aligned}$$

which does not leave much room for argument.

This proves that D' also has the equality $\deg D' = 2(l(D') - 1)$, so is a $g^{r'}_{2r'}$ but with $r' < r$. By induction on r, it follows that C is hyperelliptic. By the discussion of 8.4, all the divisors in the argument are multiples of $A = g^1_2$. Q.E.D.

9.3 The Castelnuovo free pencil trick

The Castelnuovo free pencil trick applies Proposition 9.1 on coprime divisors to give a lower bound on the rank of multiplication maps

$$\mathcal{L}(E_1) \otimes \mathcal{L}(E_2) \to \mathcal{L}(E_1 + E_2) \tag{9.9}$$

when we can find a suitable free pencil D inside $|E_1|$.

Let D be a divisor and $U \subset \mathcal{L}(C, D)$ a 2-dimensional vector subspace such that the linear subsystem $\mathbb{P}^1_U \subset |D|$, made up of the effective divisors

$$\operatorname{div} s + D = D_s \quad \text{for } s \in U, \tag{9.10}$$

is a free pencil. This just means that a basis s_1, s_2 of U gives divisors $D_1 = \operatorname{div} s_1 + D$ and $D_2 = \operatorname{div} s_2 + D$ that are coprime in the sense of Proposition 9.1. This is usually a free g^1_d (see Remark 7.1), but the logic of the argument allows $U \subset \mathcal{L}(C, D)$ to be a strict subspace.

Proposition 9.5 *Let D and U be as above, and E any divisor. Consider the multiplication map $\mu_U \colon U \otimes \mathcal{L}(E) \to \mathcal{L}(D + E)$. Then*

$$\operatorname{rank} \mu = \dim(U \otimes \mathcal{L}(E)) - l(D - E) = 2l(E) - l(D - E). \tag{9.11}$$

Proof The assertion is a particular case of Proposition 9.1. In fact under μ_U, the two summands of

$$U \otimes \mathcal{L}(E) = s_1 \otimes \mathcal{L}(E) \oplus s_2 \otimes \mathcal{L}(E) \tag{9.12}$$

map to

$$s_1 \cdot \mathcal{L}(E) \quad \text{and} \quad s_2 \cdot \mathcal{L}(E) \subset \mathcal{L}(D + E), \tag{9.13}$$

which intersect in $\mathcal{L}(E - D_1 - D_2)$.

One traditionally expresses this as an exact sequence

$$0 \to \mathcal{L}(E - D) \to \mathcal{L}(E)^{\oplus 2} \to \mathcal{L}(D + E). \tag{9.14}$$

The argument for (9.14) can also be written intrinsically (with a small additional headache).

Many of the interesting consequences of the Castelnuovo free pencil trick related to special divisors. However, if all the divisors in (9.14) are in the regular range (that is, $\deg(E - D) \geq 2g - 1$), an easy calculation with the RR formula shows that the final map is surjective.

9.4 Max Noether's theorem

This is the typical application of the Castelnuovo free pencil trick. Let C be a nonhyperelliptic curve of genus g. Recall from Theorem 8.2 that K_C is very ample, and identify C with its canonical image $C = \varphi_{K_C}(C) \subset \mathbb{P}^{g-1}$.

Theorem 9.6 (Max Noether's theorem) *For $d \geq 1$, the forms of degree d on $\mathbb{P}^{g-1}$ map surjectively to $\mathcal{L}(C, dK_C)$.*

By construction, saying that $C = \varphi_{K_C}(C) \subset \mathbb{P}^{g-1}$ means that the linear forms on $\mathbb{P}^{g-1}$ are the RR space $\mathcal{L}(C, K_C)$. In other words, the hyperplanes of $\mathbb{P}^{g-1}$ cut out the complete canonical system $|K_C|$. The theorem states that, in the same way, the hypersurfaces of $\mathbb{P}^{g-1}$ of degree d cut out the complete linear system $|dK_C|$.

The basic case to work with is $d = 2$. The product $s_i s_j \in \mathcal{L}(C, 2K_C)$ because $\operatorname{div}(s_i s_j) = \operatorname{div} s_1 + \operatorname{div} s_2$. It is required to prove that the products $s_i s_j$ include $3g - 3$ elements that are linearly independent in $\mathcal{L}(C, 2K_C)$. The key to this is the Castelnuovo free pencil trick.

Linearly general position For $C \subset \mathbb{P}^n$ a nonsingular curve that spans $\mathbb{P}^n$, it is known that *a sufficiently general hyperplane $H \subset \mathbb{P}^n$ cuts C in d points that are linearly in general position.* This result is a curious backwater of the algebraic geometry literature, and I leave the proof to the Appendix below.

Choose g general points $P_1, \dots, P_g$ of $C \subset \mathbb{P}^{g-1}$, and assume the P_i map to the coordinate points $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{P}^{g-1}$. Applied to the hyperplane $V(s_1)$, linearly general position implies that s_1, s_2 vanish on the divisor $A = P_3 + \cdots + P_g$ but not at any other point of C, so that $s_1, s_2 \in \mathcal{L}(C, K_C - A)$ form a free pencil as in 9.5.

Now $\mathcal{L}(A) = 1$. This follows (say) from Standard Trick (b): the points $P_3, \dots, P_g$ are general, so subtracting them one by one from K_C give irregularity $l(K_C - A) = 2$, therefore $l(A) - 2 = 1 - g + \deg A$.

The conclusion of the free pencil trick Proposition 9.5 is that the two subspaces $s_1\mathcal{L}(K_C)$, $s_2\mathcal{L}(K_C)$ intersect only in the 1 dimensional space $\mathcal{L}(A)$. This means that the $2g - 1$ monomials

$$s_1^2, \dots, s_1 s_g \quad \text{and} \quad s_2^2, s_2 s_3, \dots, s_2 s_g \tag{9.15}$$

are linearly independent in $\mathcal{L}(2K_C - A)$. They vanish at the $g - 2$ points of A by construction.

Now the $g-2$ monomials $s_3^2, \dots, s_g^2$ are linearly independent modulo $\mathcal{L}(2K_C - A)$, because at each P_i, s_i^2 is nonzero, and the others zero. They thus form a complimentary basis of $\mathcal{L}(2K_C)$.

For $d \ge 3$, by the same argument, the two subspaces $s_1\mathcal{L}((d-1)K_C)$ and $s_2\mathcal{L}((d-1)K_C)$ intersect in $\mathcal{L}((d-2)K_C + A)$. Since $d \ge 3$ this is in the regular range, so there sum maps surjectively to $\mathcal{L}(dK_C - A)$, and the monomials $s_3^d, \dots, s_g^d$ again form a complementary basis of $\mathcal{L}(dK_C)$. QED

To do

Worked example for $g = 4$, $g = 5$: use Max Noether's theorem to get $Q_2 \cap F_3$ in $\mathbb{P}^3$ and $Q_1 \cap Q_2 \cap Q_3$ as plausible constructions the canonical models. Add a few hints looking forward to the Petri analysis.

To do The mult map $\mathcal{L}(D) \otimes \mathcal{L}(K_C - D) \to \mathcal{L}(K_C)$ is called the *Petri map.* There are favourable cases in which it is surjective, that has nice consequence.

I also use the Castelnuovo free pencil trick in Chapter 4, (page 6 in the 2020 notes) in the proof that s_1, s_2 in the complete sections ring $R(C, D)$ form a regular sequence.

10 Appendix on inseparability

10.1 Definitions

The material here is not really essential for algebraic curves (except for the easy part of the proof of linearly general position), but I hope eventually to put it all together as an appendix for the reader who needs it. Inseparable extensions are usually only mentioned in passing in a Galois theory course, and mainly to get rid of them. However there is no special mystery or difficulty about what is going on, even if it is not specially familiar.

The first thing to say is the paradoxical geometric property of an inseparable function or map. In analysis, or in geometry in characteristic 0, a function $f(x)$ that has zero derivative everywhere, or a map φ all of whose partial derivatives are identically zero is of course a constant. In characteristic p this does not hold. If a polynomial f has $f' \equiv 0$, the only thing one can say is that f only involves its variables to the pth power.

Separable Let $K \subset L$ be a field extension with $[L : K] < \infty$. The following equivalent conditions define what it means for $x \in L$ to be *separable* over K, or (with a minor change of wording) for the whole extension $K \subset L$ to be separable.

- The minimal polynomial $f_x \in k[X]$ of x splits into distinct factors in any extension of L.
- The minimal polynomial f_x has formal derivative $f' \neq 0$.
- The tensor product $L \otimes_K L$ has no nilpotents.
- The trace homomorphism $\mathrm{Tr}_{L/K} \colon L \to K$ is nonzero. Moreover the trace provides a nondegenerate bilinear pairing

$$\mathrm{Tr}_{L/K}(xy) \colon L \times L \to K. \tag{10.1}$$

Sample argument: if the extension $K \subset L$ is inseparable, then there is an a in K such that $x^p - a$ has a root in L. Then $L \otimes L$ has two such elements $x_1 = x \otimes 1$ and $x_2 = 1 \otimes x$ with $x_1^p = x_2^p$, therefore $(x_1 - x_2)^p = 0$, so it has nilpotents. At the same time, calculating the trace of any element of L by any method must involve sums of p identical terms, so the answer can only add to zero.

```
TO DO. Discussion and proof of that.
```

Purely inseparable The following equivalent conditions define what it means for $x \in L$ or the whole extension $K \subset L$ to be *purely inseparable*.

- The minimal polynomial f_x is $X^{p^n} - a = (X - \alpha)^{p^n})$.
- x has no other conjugates in any extension field $L \subset L'$.
- If $L \subset L'$ is any extension field and $\varphi \colon L \to L'$ a K-linear homomorphism then $\varphi\big|_L = \mathrm{Id}_L$.
- $K(x)/K$ is normal and $\mathrm{Aut}_K(K(x)) = \{\mathrm{Id}\}$.

```
Standard discussion from Galois theory. L/K is a _normal_
extension if L contains all the roots of the min poly f_x
of every x in L
 <=> L = splitting field of some F in K[X]
 <=> all the conjugates in L' of any x in L are still in L
 <=> for L in L' extension, any field homomorphism/ k from
    L -> L' takes L to itself.
```

The following arguments are already in Chapter 1 of the 2022 notes.

```
Theorem [Kaplansky] Assume L/K is normal. Then L is
in a unique way the composite of L^sep and L^insep

                  L

            L^sep    L^insep

                  K

where L^sep = { x in L | x is separable }
and   L^insep = { x in L | x is purely inseparable }.

Then
  L^sep/K is Galois (normal and separable)
  L/L^insep is Galois with the same group
    Gal(L/L^insep) = Gal(L^sep/K) = Aut(L/K)
  L^insep/K is the fixed subfield of Aut(L/K)
  L/L^sep is purely inseparable
The proof is straightforward verification using the
main results of Galois theory.

Composition of field extensions: the 2 field extensions
are disjoint, and as a K-algebra L = L^sep tensor_K L^insep.
```

10.2 Finiteness of integral closure

The result for finiteness of integral closure works for finite field extensions (separable or not), depending on Kaplansky's theorem. This is already in the notes for Chapter 1, but it fits more logically here.

10.3 Frobenius morphism

```
The characteristic p identity
  (x+y)^p = x^p + y^p
means that every ring R of char p has an automorphism
Frob_R = phi_R = phi : R -> R defined by x |-> x^p.

This idea already provides the whole of the Galois
theory of finite fields: write q = p^n. The finite
```

```
field FF_q has Frobenius map phi.

  -> The subset of FF_q of elements fixed under phi
  is the subfield FF_p (the roots of x^p = x).
  -> phi generates Gal(FF_q) \iso ZZ/n, and for
  m | n, the fixed subfield of phi^m is FF_p^m.

For an algebraic variety (or scheme) over a field of
characteristic p, we have to distinguish the
  _absolute_ Frobenius  versus  _geometric_ Frobenius.
The point is that although phi is a ring homomorphism,
it is NOT a k-algebra homomorphism, since it messes
with the ground field k itself.

You can twist the absolute Frobenius (that takes
P = [x1,.. xn] -> [x1^p,.. xn^p]) into a morphism of varieties
by changing the target to be a variety with a different action
of the field k. Although not so much for this course, this is
important in many other areas of algebra and number theory
over a field of characteristic p, and was the key first step
in the Weil--Grothendieck--Deligne treatment of the Riemann
hypothesis over a finite field.
```

10.4 Theorem on linearly general position

Let $\Gamma \subset \mathbb{P}^n$ be an irreducible curve of degree d spanning $\mathbb{P}^n$. Then, at least in characteristic zero, a general hyperplane section of Γ is a set of points in linearly general position. This means that Γ cut with a general hyperplane $H = \mathbb{P}^{n-1}$ is a set of d points such that every subset of n points spans H. For example, if $n \geq 3$, every general hyperplane section of Γ contains a secant line that is not a trisecant.

```
Step 1. Reduction from C in PP^n with degenerate linear
dependencies on its general hypersection
 to C in PP^3 with every secant a multisecant.

Pf. Take linear projection from n-3 general points and follow
your nose. The projection is generic, so if C in PP^n was
nonsingular then C in PP^3 remains so.

Step 2. C in PP^3: every general secant is a multisecant
```

```
    => tangent lines at every two general points
    P, Q are coplanar

Pf. The secant line PQ has a 3rd point R. If you move the
point P infinitesimally to P', the secant line RP' must
contain a point Q' infinitely near to Q. Therefore RPP' QQ'
are all coplanar, and the plane containing them contains the
tangent line to P and the tangent line to Q.

Step 3. The tangent lines to C concurrent in pairs
    => they all pass through some A in PP^3. That is,
    C has an inseparable projection from A in PP^3
    (This is Samuel's strange curve.)

Pf. Baby projective geometry. 3 or more lines in PP^3 that are
pairwise concurrent are either all coplanar or all concurrent.
The tangent lines contain all points of C, so they are not all
coplanar.
```

Hartshorne and Samuel define an irreducible projective curve Γ all of whose tangent lines at nonsingular points are concurrent at A to be *strange*.

In characteristic 0, there are no strange curves (except, arguably, a straight line). Because the projection from A would give a rational map $\Gamma \dashrightarrow \mathbb{P}^{n-1}$ with differential everywhere zero. Then the map would have to be constant to a single point.

In characteristic p, the condition just means that the projection from A is inseparable. There are any number of such curves, but Samuel proved that they are all singular except the plane conic in characteristic 2.

For the proof and discussion, in this edition of the notes, I attach a typeset version of Samuel's appendix.

Remark 10.1 I have a number of current obsessions in this subject.

(I) Modern algebraic geometers have extraordinary difficulties in relating to language of the past - the 19th century, the Italian era, the Zariski and Weil period around WW2, the Serre and Grothendieck period. The problem with the latter is that there are brilliant researchers tackling hard problems, and when they get a result they publish it, warts and all.

(II) I have the distinct memory that Mumford told me around 1980 that the linearly general position statement for a singular irreducible curve is of course false in general, because there are curves whose every hyperplane section is a configuration that has an action of $\mathbb{F}_p^+$ action. I'm not there yet, but I hope to understand his hint eventually. (Or maybe I just misunderstood it.)

(III) Samuel's notion of strange curve may be related to group schemes of order p or p^2. The projection from A is an inseparable morphism $C \to \Gamma$, and the functions on C are generated by a single new coordinate function x_1/x_0.

This presumably means that it factors via geometric Frobenius $C \to C^{(1)}$ (up to isomorphism: a priori we don't know what projective space $C^{(1)}$ is embedded in). This inseparable may be a μ_p or α_p torsor (or both). Then possibly the singularities of C relates to the zeros of p-closed vector fields. The correct treatment must navigate the counterexamples

$C =$ straight line and $C =$ plane conic in characteristic 2.

(IV) Big challenge: to find grown-up counterexamples to linearly general position for irreducible curve, or to prove they don't exist.

Pierre Samuel
Lectures on old and new results on algebraic curves
Bombay, Tata Institute, 1966

Appendix to Chap. II, p. 76–78
Nonsingular strange curves

For proving the existence of a plane model of a function field with only nodes (Chap. II, Section 1), we had to avoid the *strange* curves of characteristic p, that is, the curves C in projective space all of whose tangents have a common point. A posteori (that is, using facts about divisors of differentials), one can prove that we were fighting against a phantom. More precisely:

Theorem *The only nonsingular projective strange curves are the lines, and in characteristic* 2*, also the plane conics.*

That a plane conic

$$ayz + bzx + cxy + dx^2 + ey^2 + fz^2 = 0$$

is strange in characteristic 2 is well known and easily proved. The equation of the tangent at (x, y, z) is

$$XF'_x + YF'_y + ZF'_z = X(bz + cy) + Y(cx + az) + Z(ay + bx) = 0,$$

and is satisfied by the point (a, b, c) (here $(a, b, c) \neq (0, 0, 0)$ because otherwise our conic is a double line).

Conversely, let $C \subset \mathbb{P}^n$ be a nonsingular strange curve, defined over an algebraically closed field k of characteristic $p > 0$. By a suitable choice of coordinates, we may assume that the point A common to all tangents to C has homogeneous coordinates $(1, 0, \dots, 0)$, and that C has no points at which two coordinates vanish (except perhaps for A).

Let $L = k(C)$ be the function field of C, and

$$(x, x_2, \dots, x_n) \quad \text{with } x \text{ and } x_i \in L.$$

the affine coordinate functions of C outside the hyperplane H (last coordinate $= 0$).[1]

[1] I interpret this to mean that $\mathbb{P}^n$ has homogeneous coordinates $u_1, \dots, u_n, v$ with $v = 0$ the hyperplane at infinity, $x = x_1 = u_1/v$ and $x_i = u_i/v$. The point A is on the hyperplane at infinity. The choice of coordinates gives that $u_1 \neq 0$ at all such points.

Since all tangents to C pass through A, we have

$$D_{x_2} = \cdots = D_{x_n} = 0 \quad \text{for any } k\text{-derivation } D \text{ of } L.$$

That is,

$$x_2, \dots, x_n \in L^p. \tag{1}$$

We are going to compute the divisor $\operatorname{div}(dx)$. At a point $P \in C$ away from $C \cap H$, the curve C is transversal to the hyperplane $x_1 = 0$, whence $x - x(P)$ is a uniformizing parameter at P. Thus

$$v_P(dx) = 0 \quad \text{for } P \in C \setminus (C \cap H). \tag{2}$$

By hypothesis, all points of $C \cap H$ lie in the affine piece with coordinates $(1/x, x_2/x, \dots, x_n/x)$. We set $y = 1/x$ and $y_i = x_i/x$, so that $y \in L^p y_i$ for $i = 2, \dots, n$

Suppose first that $P \neq A$. We have $y(P) = 0$, and $y_i(P) \neq 0$ for $i = 2, \dots, n$. Since the maximal ideal of the local ring $\mathcal{O}_P$ (the valuation ring of v_P) is generated by $y, y_2 - y_2(P), \dots, y_n - y_n(P)$, there exists an index i for which $t = y_i - y_i(P)$ is a uniformizing parameter at P.

Since $y \in L^p y_i$ and since $v_P(y) > 0$, the expansion of y as a power series in t is

$$y = (y_i(P) + t)(\alpha_0 t^{pj_P} + \alpha_1 t^{p(j_P+1)} + \cdots) \quad \text{with } \alpha_0 \neq 0 \text{ and } j_P > 0.$$

This contains terms of degree pj_P and $pj_P + 1$ with nonzero coefficients. Therefore $v_P(y) = pj_P$ and $v_P(dy/dt) = pj_P$. Also, since $dx = -dy/y^2$, it follows that

$$v_P(dx) = -pj_P \quad \text{with } j_P > 0. \tag{3}$$

Finally, if $A \in C$, we have $y(A) = y_2(A) = \cdots = y_n = 0$. As above, one of the y_i is a uniformizing parameter at A, say $t = y_i$. From $y \in L^p y_i$ and $v_A(y) > 0$, we get the power series expansion

$$y = t(\alpha_0 t^{pj_A} + \alpha_1 t^{p(j_A+1)} + \cdots) \quad \text{with } \alpha_0 \neq 0 \text{ and } j_A \geq 0.$$

Hence $v_A(y) = pj_A + 1$, and $v_A(dy/dt) = pj_A$. Since $dx = -dy/y^2$, we get

$$v_A(dx) = -pj_A - 2 \quad \text{with } j_A \geq 0. \tag{4}$$

From (2), (3) and (4), and from the fact that $C \cap H \neq \emptyset$, we see that the degree of $\operatorname{div}(dx)$ is < 0.

Since it is $2g - 2$ (where g denotes the genus of C), it is necessarily -2, and $g = 0$. Looking at (3) and (4), we see that only two cases may happen

(a) $C \cap H$ consists of only one point $P \neq A$. Then

$$v_P(dx) = -2, \quad p = 2, \quad j = 1, \quad \text{and } v_P(y) = 2.$$

This last relation shows that $C \cdot H = 2P$, whence C has degree 2. We get a conic in characteristic 2.

(b) $C \cap H$ contains only the point A. Then

$$v_A(dx) = -2, \quad j_A = 0, \quad \text{and } v_A(y) = 1,$$

so that $C \cdot H = A$; thus C has degree 1 and is a straight line. QED

Remark There exist, of course, many singular strange curves in characteristic p: take a function field L of transcendence degree 1 over k, functions $z_2, \dots, z_n \in L$ which generate L^p over k, and choose $z \in L \setminus L^p$. Then $L = k(z, z_2, \dots, z_n)$. The affine curve D with coordinate functions $(z, z_2, \dots, z_n)$ is a model of L. Take its projective closure $\overline{D}$. It is easily seen that all tangents to $\overline{D}$ at nonsingular points pass through the point $(1, 0, \dots, 0)$.

Bibliographic information

The book Pierre Samuel, *Lectures on old and new results on algebraic curves*, Bombay, Tata Institute, 1966 is in Univ. of Warwick library QA.565.S2 (reserve in Leamington) and in the Math Inst. Library, same QA.565.S2

The material is reworked in [Hartshorne, IV.3, pp. 310–316].

See also [Arbarello, Cornalba, Griffiths and Harris, III.1, pp. 109–113].

The main aim in all of these references is to prove that any curve projects birationally to a plane curve of degree d with δ nodes as the only singularity, so that they can use their favourite characterisation or definition of the genus as $\binom{d-1}{2} - \delta$ and of $\mathcal{L}(K_C)$ as adjoints of degree $d-3$ (forms of degree $d-3$ vanishing at the nodes). My treatment of K_C is quite different.

[ACGH] has an alternative and quite different idea: use Harris' Galois-monodromy argument to prove that the generic hyperplane section of C cannot have both linearly dependent and independent subsets of points. I have not yet understood whether this works in characteristic p.

Most seriously, I still have no idea whether there are any irreducible curves (necessarily in characteristic p, "strange"' and singular) whose general hyperplane section is not in linearly general position.