MA4L7 Algebraic curves

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Chapter 2. Divisors and RR spaces – RR theorem assuming Main Propositions I–III

5 Introduction

Chapter 1 set out the main object of study, a nonsingular projective algebraic curve C. For C to be nonsingular at a point $P \in C$ is the condition that the local ring $\mathcal{O}_{C,P}$ is a discrete valuation ring (DVR). Alternatively, an affine curve $C \subset \mathbb{A}^n$ is nonsingular if and only its coordinate ring $k[C]$ is normal (that is, integrally closed). A basic initial circle of ideas is called *resolution of singularities*: this replaces any irreducible algebraic curve Γ with a nonsingular projective curve C having a morphism $C \to \Gamma$ that is finite and birational, and establishes that the nonsingular projective model C is unique up to isomorphism. Over $\mathbb{C}$, these curves can also be identified with compact Riemann surfaces.

Chapter 2 assumes the notion of nonsingular projective curve $C \subset \mathbb{P}^n$ (over an algebraically closed field k), and its field of rational functions $k(C)$. Nonsingular means the local ring $\mathcal{O}_{C,P}$ at every point $P \in C$ is a DVR. For $f \in k(C)^\times$ and $P \in C$, the valuation $v_P(f)$ describes the zeros or poles of f.

The Riemann–Roch theorem controls the vector space $\mathcal{L}(C, D)$ of meromorphic functions with specified poles on a compact Riemann surface or a nonsingular projective algebraic curve – if you allow more poles, you get more functions. Chapter 2 discusses the statement of the Riemann-Roch theorem:

$$\dim \mathcal{L}(C, D) \geq 1 - g + \deg D \tag{5.1}$$

(together with accompanying reasonable conditions that guarantee equality). Here the *divisor* D is a formal sum $D = \sum d_i P_i$ of points $P_i \in C$ with multiplicity d_i. The *Riemann–Roch space* $\mathcal{L}(C, D)$ is the vector space of *rational* or *global meromorphic functions* on C having only poles at P_i of order $\le d_i$ (I assume here for simplicity that $d_i > 0$, the main case). The number $g = g(C)$ is the *genus*, the most important numerical invariant of C. It can be described intuitively as the "number of holes" in topology, but it has many quite different characterisations in analysis and in algebraic geometry, and can be calculated in many different ways.

The proof of RR in algebraic geometry is deduced here from three Main Propositions I–III that I state below, but only prove in Chapter 4.

5.1 Preliminary explanations

The introductory sections have set the scene, and you probably already know a lot of what I am going to say about RR spaces. Let me discuss a small point of language that is very convenient and used throughout. Let $P \in C$ be a nonsingular point of a projective curve, with local parameter $z = z_P$. For $d \in \mathbb{Z}$, consider the condition

$$v_P(f) + d \ge 0. \tag{5.2}$$

This is equivalent to any of the following:

(I) $f \in \mathcal{O}_{C,P} \cdot \frac{1}{z^d} \subset k(C)$, that is, $f = \frac{1}{z^d} f_0$ for some $f_0 \in \mathcal{O}_{C,P}$.

(II) Divided into cases: If $d \ge 0$ then f has a pole of order $\le d$; if $d \le 0$ then f has a zero of order $\ge d$.

(III) $\operatorname{div} f + dP \ge 0$ at P (see below for $\operatorname{div} f$).

(IV) $f \in \mathcal{O}_{C,P}(dP)$ at P.

All of (I–IV) are popular expressions appearing throughout the literature, but the expression (5.2) is concise, and gets around the longwinded case division of (II), and the frequent tedious errors and confusion associated with inequalities involving negative numbers.

The substantive point is that for $d \ge 0$ a function f with pole of order up to d is allowed a *principal part*

$$f = a_{-d}\frac{1}{z^d} + \cdots + a_{-1}\frac{1}{z} + \text{regular function at } P \tag{5.3}$$

that depends on d paramenters $a_{-d}, \dots, a_{-1}$.

Example 5.1 (Weierstrass elliptic curve) In complex analysis, a curve of genus $g = 1$ is the complex manifold $E = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z} \cdot \tau$ (here $1, \tau \in \mathbb{C}$ form an $\mathbb{R}$-basis of $\mathbb{C}$, conventionally with $\operatorname{im} \tau \ge 0$).

The Weierstrass P-function $\wp(z)$ is a global meromorphic function of z that is *doubly periodic*. This means it is invariant under translation by Λ. It is given as the sum $\wp(z) = \sum_{c\in\Lambda} \frac{1}{(z-c)^2}$, modulo a little issue of convergence.[1] The formula shows that $\wp$ is invariant under Λ, so defines a global meromorphic function on $E = \mathbb{C}/\Lambda$. It is an even function of z, and has pole of order 2 at 0. Its derivative $\wp'$ is an odd function of z with pole of order 3 at 0.

It is known that $\dim \mathcal{L}(E, dO) = d$ for every $d > 0$. (Compare Ex. 5.13; here I write O for the image of $0 \in \mathbb{C}$). For $d = 3$ the space $\mathcal{L}(E, 3O)$ is based by the functions $1, \wp, \wp'$. The map $z \mapsto (1, \wp(z), \wp'(z))$ defines an embedding of $E = \mathbb{C}/\Lambda$ into $\mathbb{P}^2_{\mathbb{C}}$. Its image is the plane cubic curve given by the famous Weierstrass equation $(\wp')^2 = (\wp)^3 + a\wp + b$.

5.2 Divisors D on C and the RR space $\mathcal{L}(C, D)$

I work over an algebraically closed field k. A *nonsingular projective curve* is an irreducible variety $C \subset \mathbb{P}^N$ such that the local ring $\mathcal{O}_{C,P}$ at each $P \in C$ is a DVR. This means $\mathcal{O}_{C,P} \subset k(C)$ is a subring of the function field of C, with maximal ideal $m_P = (z_P)$ the principal ideal generated by a *local parameter* z_P. Every nonzero function $f \in k(C)^\times$ is then of the form $f = z_P^v \cdot f_0$ with $f_0 \in \mathcal{O}_{C,P}^\times$ a unit at P. Here $v = v_P(f)$ is the *valuation* of f at P. We say f has *zero of order* $v_P(f)$ if it is positive, or *pole of order* $-v_P(f)$ if it is negative.

Definitions A *divisor* on C is a finite sum

$$D = \sum d_i P_i \quad \text{with } P_i \in C \text{ and } d_i \in \mathbb{Z}. \tag{5.4}$$

We also write $D = \sum_{P\in C} d_P P$, where the expression assumes $d_P = 0$ for all but finitely many P. A divisor $D = \sum d_P P$ is *effective* (written $D \geq 0$) if $d_P \geq 0$ for every P. The *degree* of D is $\sum_{P\in C} d_P$.

The divisor of a rational function $f \in k(C)^\times$ on C is

$$\operatorname{div} f = \sum v_P(f) P = \text{zeros of } f - \text{poles of } f. \tag{5.5}$$

Here both f and f^{-1} are regular outside a finite set, so (5.5) is a finite sum.

The *RR space* of D on C is defined as the vector subspace

$$\mathcal{L}(C, D) = \left\{ f \in k(C) \;\middle|\; \operatorname{div} f + D \geq 0 \right\} \subset k(C). \tag{5.6}$$

[1]The series does not converge absolutely, but Weierstrass specifies a sensible order of summation that makes it converge for $z \notin \Lambda$ – see the literature or the Wikipedia page.

The condition $\operatorname{div} f + D \geq 0$ applies separately at each point P, and as discussed in 5.1, where $d_P \geq 0$, it allows f to have poles of order $\leq d_P$ (adding the positive part of D cancels its poles), and imposes P as a zero of order $-d_P$ on f at points where $d_P < 0$.

While not strictly necessary, it is informative to use the same condition to define the structure sheaf $\mathcal{O}_C$ and the divisorial sheaf $\mathcal{O}_C(D)$. The constant sheaf $k(C)$ has the fixed pool $k(C)$ of rational functions on every nonempty Zariski open set of C. Inside $k(C)$, the regular functions at P are characterised as $\mathcal{O}_{C,P} = \{f \in k(C) \mid v_P(f) \geq 0\}$. At each P, the divisorial sheaf condition for $\mathcal{O}_C(D)$ replaces this with $\operatorname{div}(f) + d_P \geq 0$; this means that $\mathcal{O}_C(D)$ is generated locally at P by $z_P^{-d_P}$, or $\mathcal{O}_C(D)_P = \mathcal{O}_{C,P} \cdot \frac{1}{z_P^{d_P}}$. (If you know the ideas of algebraic number theory, this is essentially the same as a *fractional ideal*.)

Then $\mathcal{O}_C(D)$ is the subsheaf of the constant sheaf $k(C)$ obtained by imposing the condition $\operatorname{div} f + D \geq 0$ over each Zariski open subset $U \subset C$. In other words the sections of $\mathcal{O}_C(D)$ over U are

$$\begin{aligned}\mathcal{O}_C(D)(U) &= \Gamma(U, \mathcal{O}_C(D)) \\ &= \bigl\{f \in k(C) \bigm| v_P(f) + d_P \geq 0 \text{ for all } P \in U\bigr\} \subset k(C).\end{aligned}$$

This definition is local near each $P \in U$, making the sheaf condition automatic. The definitions make $\mathcal{O}_C(D)$ a locally free sheaf of $\mathcal{O}_C$-modules of rank 1, based by $z_P^{-d_P}$ in a Zariski neighbourhood of P. The global sections $\Gamma(C, \mathcal{O}_C(D))$ is the same thing as the RR space $\mathcal{L}(C, D)$.

If $D \geq 0$ then $\mathcal{O}_C \subset \mathcal{O}_C(D)$. Also, $\mathcal{O}_C(-D) = \mathcal{I}_D \subset \mathcal{O}_C$ is the sheaf of ideals of D (regular functions with zeros on D).

5.3 Principal divisors and linear equivalence

A divisor of the form $\operatorname{div} f$ for $f \in k(C)^\times$ is *principal*. The term arises from principal ideals in the ring of integers of a number field. Two divisors D_1 and D_2 are *linearly equivalent* (written $D_1 \overset{\text{lin}}{\sim} D_2$) if they differ by a principal divisor, that is, $D_1 - D_2 = \operatorname{div} g$ for some $g \in k(C)^\times$.

Remark 5.2 On $\mathbb{P}^1$, any two points are linearly equivalent: the rational function $\frac{z-a}{z-b}$ has divisor $P_a - P_b$. This property characterises $\mathbb{P}^1$ or curves of genus 0. It is *comprehensively false* for any curve of genus $g \geq 1$.

If D_1 and D_2 are effective and disjoint, $D_1 \overset{\text{lin}}{\sim} D_2$ means there is a morphism $g\colon C \to \mathbb{P}^1$ such that $D_1 = g^*(0)$ and $D_2 = g^*(\infty)$. Linear equivalence thus works as a kind of algebraically defined homology between the two divisors. (But beware that algebraic geometry introduces many other kinds of equivalence.)

It is interesting to rework the group law on a nonsingular plane cubic curve [UAG, Chap. 2] in terms of the linear equivalences corresponding to $\operatorname{div}(L_1/L_2)$ with L_1, L_2 lines of $\mathbb{P}^2$. See Ex. 5.13

5.4 Standard tricks

The following points come directly from the definitions. They recur time and again as standard computational devices in the proof and applications of RR.

Proposition 5.3 *(a) For D a divisor and $P \in C$ a point, consider the inclusion $\mathcal{L}(C, D-P) \subset \mathcal{L}(C, D)$. Then any $f \in \mathcal{L}(D) \setminus \mathcal{L}(D-P)$ is a complementary basis element. In other words, we have the dichotomy:*

(i) either $\mathcal{L}(D) = \mathcal{L}(D-P)$;

(ii) or $\mathcal{L}(D) = k \cdot f \oplus \mathcal{L}(D-P)$ for some $f \in \mathcal{L}(D) \setminus \mathcal{L}(D-P)$.

More crudely, $\mathcal{L}(D-P) \subset \mathcal{L}(D)$ has codimension 0 *or* 1.

(b) Moreover, if $\mathcal{L}(D) \neq 0$, case (i) holds for at most finitely many $P \in C$.

(c) $\operatorname{div}(f_1 f_2) = \operatorname{div} f_1 + \operatorname{div} f_2$ *for all $f_1, f_2 \in k(C)^\times$.*

(d) Suppose divisors D_1, D_2 are linearly equivalent, so $D_1 - D_2 = \operatorname{div} g$ for $g \in k(C)^\times$. Then for $f \in k(C)^\times$,

$$f \in \mathcal{L}(D_1) \iff fg \in \mathcal{L}(D_2). \tag{5.7}$$

That is, multiplication by g in $k(C)$ is a change of basis $k(C) \xrightarrow{\sim} k(C)$, and it takes the subspace $\mathcal{L}(D_1)$ to $\mathcal{L}(D_2)$. In particular $l(D_1) = l(D_2)$.

(e) For divisors A and B and some point $P \in C$, suppose both inclusions

$$\mathcal{L}(A-P) \subsetneq \mathcal{L}(A) \quad \text{and} \quad \mathcal{L}(B-P) \subsetneq \mathcal{L}(B) \tag{5.8}$$

are strict. Then also $\mathcal{L}(A+B-P) \subsetneq \mathcal{L}(A+B)$.

The proofs are formal.

(a) Let z_P be a local parameter at P and write $d_P \in \mathbb{Z}$ for the multiplicity of P in D. The condition $\operatorname{div} f + D \geq 0$ at P is equivalent to $z_P^{d_P} f$ regular at P, so $z_P^{d_P} f \in \mathcal{O}_{C,P}$. For $f \in \mathcal{L}(D)$, if $z_P^{d_P} f$ is a unit at P, subtracting a multiple of f cancels the leading term of any $g \in \mathcal{L}(D)$, so that $g - \lambda f \in \mathcal{L}(D - P)$, proving (ii).

The alternative is that $z_P^{d_P} f$ vanishes at P, so that $z_P^{d_P - 1} f$ is also regular at P, and $f \in \mathcal{L}(D - P)$. If this holds for every $f \in \mathcal{L}(D)$ then (i) holds.

Assertion (a) reflects the fact that the powers of the maximal ideal of a DVR are principal $m^d = (z^d)$, with successive quotients m^{d-1}/m^d isomorphic to the residue field $k = A/m$.

(b) For any nonzero $f \in \mathcal{L}(D)$, the effective divisor $\operatorname{div} f + D = \sum n_P P$ has support a finite set, and (ii) holds for any P not in this.

(c) This follows from the basic property $v(fg) = v(f) + v(g)$ of a discrete valuation: at any $P \in C$, suppose $f_1 = z_P^{d_1} \cdot u_1$ and $f_2 = z_P^{d_2} \cdot u_2$, with units $u_1, u_2 \in \mathcal{O}_{C,P}^{\times}$ and $v_P(f_i) = d_i$. Then $f_1 f_2 = z_P^{d_1 + d_2} u_1 u_2$ with $u_1 u_2$ a unit, so that $v_P(f_1 f_2) = d_1 + d_2$.

(d) This holds because $\operatorname{div}(fg) = \operatorname{div} f + \operatorname{div} g = \operatorname{div} f + D_1 - D_2$. Thus $\operatorname{div} f \geq -D_1$ if and only if $\operatorname{div}(fg) \geq -D_2$.

According to (d), linear equivalence thus concerns the minor matter of the choice of basis in the 1-dimensional vector space $k(C)$. This means that one can usually replace D by a linearly equivalent divisor. Algebraic geometers frequently do this without saying so, using divisor and divisor class interchangeably by abuse of terminology.

(e) This follows from (c): $f \in \mathcal{L}(A) \setminus \mathcal{L}(A - P)$ and $g \in \mathcal{L}(B) \setminus \mathcal{L}(B - P)$ give $fg \in \mathcal{L}(A + B)$ and $v_P(fg) = a_P + b_P$, where a_P and b_P are the coefficients of A and B at P, so $fg \notin \mathcal{L}(A + B - P)$. Q.E.D.

5.5 Main Proposition I

My proof of RR is based on the following Main Propositions I–III. The first two can be treated fairly easily in various ways. However, rather than knocking them off piecemeal, I prove all three by a single comprehensive argument in Chapter 4. Here is Proposition I.

A principal divisor has degree 0*:* $\deg(\operatorname{div} f) = 0$ *for* $f \in k(C)^\times$. Since we interpret the divisor of f as $\operatorname{div} f = \text{zeros of } f - \text{poles of } f$, this says any rational function has the same number of zeros and poles.

Corollary 5.4 *(1) If* $\deg D < 0$ *then* $\mathcal{L}(C, D) = 0$.

(2) For any divisor $l(D) = \dim \mathcal{L}(C, D) \le 1 + \deg D$.

Proof If $0 \ne f \in \mathcal{L}(C, D)$ then $\operatorname{div} f + D$ is an effective divisor, so has degree ≥ 0, hence $\deg D \ge 0$. This proves (1).

(2) follows from (1) by induction on $\deg D$ and Trick (a). Suppose $\deg D \ge 0$ and let $P \in C$ be any point. Then $\deg(D - P) = \deg D - 1$ so by induction $l(D - P) \le \deg D$, and (a) gives $l(D) \le 1 + \deg D$. □

Corollary 5.5 *If* $A = \sum P_i$ *is an effective divisor (where I allow repeated points) then* $l(D - A) \ge l(D) - \deg A$.

This follows by repeated use of Trick (a): in passing from D to $D - A$, the dimension of $\mathcal{L}(D - P_1 - \cdots - P_i)$ either decreases by 1 or is unchanged at each step. Q.E.D.

Motivation for (I) On a compact Riemann surface, we can prove Main Proposition I by contour integration and the Cauchy integral theorem. In fact, let f be a global meromorphic function and write

$$\mathrm{d}\log f = \frac{\mathrm{d}f}{f} \quad \text{or locally} \quad \frac{\mathrm{d}f/\mathrm{d}z}{f}\mathrm{d}z$$

for its logarithmic derivative. This has pole of order 1 with residue $v_P(f)$ at every zero or pole of f: for where $f = z^n \cdot f_0$ with f_0 a unit, we get $\mathrm{d}\log f = \frac{n}{z} + \text{regular}$. (Check that this works in all three cases $n > 0$, $n = 0$ and $n < 0$.) The integral $\frac{1}{2\pi i}\oint \mathrm{d}\log f$ around a contour thus counts the zeros and poles in the interior of the contour.

Take a countour Γ that divides the surface up into an interior containing all the zeros and poles and an exterior containining no zeros and poles. Then $\frac{1}{2\pi i}\oint \mathrm{d}\log f = \deg(\operatorname{div} f)$ if we view Γ as surrouning its interior, and $= 0$ if we view it as surrounding its exterior. Equating the two gives $\deg(\operatorname{div} f) = 0$.

On a compact Riemann surface, Corollary 5.4, (1) includes the statement that a global holomorphic function is constant. In complex analysis, this follows from the Maximum Modulus principle: the modulus $|f|$ of a global holomorphic function f would be a continuous function, and on a compact space this takes a maximum value at some point P. But then the modulus would be constant, and hence also f is constant.

5.6 Main Proposition II

There exist a family D_n of divisors on C for which $\deg D_n \to \infty$ *while the difference* $1+\deg D_n - l(D_n)$ *remains bounded, say* $1+\deg D_n - l(D_n) \le N$.

Corollary 5.6 *Assume this. Then the same bound* $1 + \deg D - l(D) \le N$ *holds for every divisor D on C.*

Proof The first step is to show that for every D, there is some n such that $\mathcal{L}(D_n - D) \neq 0$. In fact if $D = A - B$ with A, B effective divisors, choose n such that $l(D_n) > \deg A$. Corollary 5.5 implies that $\mathcal{L}(D_n - A) \neq 0$.

Thus replacing $D_n - D = D_n - A + B$ by a linearly equivalent divisor, I can assume it is effective, say $D_n - D \sim \Delta > 0$; I can turn that around to $D \sim D_n - \Delta$. Now Corollary 5.5 again implies that $l(D) \ge l(D_n) - \deg \Delta$, so that

$$\begin{aligned} 1 + \deg D - l(D) &\le 1 + \deg D + \deg \Delta - l(D_n) \\ &\le 1 + \deg D_n - l(D_n) \le N. \qquad \square \end{aligned}$$

5.7 Discussion of II

In complex analysis, this is the hard part of Riemann–Roch, that requires partial differential equations to prove the existence of harmonic functions (satisfying the Laplace equation $\Delta f = 0$, with singularities at the poles interpreted in terms of boundary value problems), and then the Cauchy–Riemann equations to link harmonic functions and holomorphic functions. Riemann's own motivation for the statement (that he never proved correctly) involved the ideas of electrostatics: a point electric charge must defines a harmonic potential "for physical reasons".

In algebraic geometry, II is straightforward. A projective curve is birational to a plane curve, say via a morphism $f\colon C \to \overline{C}_a \subset \mathbb{P}^2$, where $\overline{C}_a$ is a plane curve of degree a (usually singular) defined by $F_a(x,y,z) = 0$. Choose coordinates x, y, z so that the line $z = 0$ meets $\overline{C}$ only transversally at nonsingular points, and set $H = \operatorname{div} z$ for the divisor "at infinity". It is the effective divisor defined by z/x away from the line $x = 0$ and by z/y away from $y = 0$, and it has degree a because the homogeneous polynomial F_a cuts out a points with multiplicity 1 on the line $z = 0$.

Now any homogeneous form $G_n(x,y,z)$ of degree n defines a rational function G_n/z^n on C with poles at most nH. It is an exercise to see that this restriction provides a subspace of $\mathcal{L}(C, nH)$ of dimension $1 - \binom{a-1}{2} + an$.

If $f\colon C \to \overline{C}_a$ is not an isomorphism, there are more functions of C than on $\overline{C}_a$, so that this does not give the whole of $\mathcal{L}(nH)$, but it is enough to prove II.

5.8 Definition of genus $g(C)$ and immediate consequences

In view of Corollary 5.6, it makes sense to define

$$g(C) = \max_D\{1 + \deg D - l(D)\}$$

taken over every divisor D on C. It then follows by definition that

$$l(D) \geq 1 - g + \deg D \quad \text{for every } D, \tag{5.9}$$

and equality holds for some D.

Say that D is *regular* if $l(D) = 1 - g + \deg D$. Otherwise, define the *irregularity* of D as the difference $l(D) - (1 - g + \deg D)$ is. The full form of RR in 5.11 includes a formula for the irregularity of D.

This definition is the most appropriate for the logical purpose of proving the RR theorem. I use it in the following sections, and in algebraic geometric applications. It relates to several other definitions in algebraic geometry, topology, analysis, and different flavours of homology or cohomology, as discussed later

Proposition 5.7 *(a) Every divisor D of degree $\geq g$ has $\mathcal{L}(D) \neq 0$.*

(b) There exists a divisor D_0 with $\deg D_0 = g - 1$ and $\mathcal{L}(D_0) = 0$.

(c) Equality holds in (5.9) for every D with $\deg D \geq 2g - 1$.

Proof (a) is clear. I prove (b) by induction on $l(D)$. Equality holds in (5.9) for some divisor D. If $\mathcal{L}(D) = 0$ and equality holds, then $\deg D = g - 1$, as required. However, if $\mathcal{L}(D) \neq 0$, Trick (b) applies: for all but finitely many points, $l(D - P) = l(D) - 1$. Since degree and dimension both drop by 1, it follows that $D - P$ also has equality in (5.9), with $l(D - P) < l(D)$. Now induction on $l(D)$ takes us down to $\mathcal{L}(D) = 0$ with equality still holding in (5.9), and this proves (b).

Now for (c), suppose that $\deg D \geq 2g - 1$ and choose some D_0 as in (b). Then $\deg(D - D_0) \geq g$ and so $\mathcal{L}(D - D_0) \neq 0$ by (a). That is, $D - D_0$ is linearly equivalent to an effective divisor A with $\deg A = \deg(D - D_0) = 1 - g + \deg D$. Turn this around to $D_0 \sim D - A$. Then Corollary 5.5 gives

$$l(D) \leq l(D_0) + \deg A = 1 - g + \deg D. \tag{5.10}$$

This is the opposite inequality to (5.9), and proves (c). Q.E.D.

5.9 The critical range $[0, 2g-2]$

The treatment so far determines the value of $l(D)$ in the cases

- $\deg D < 0$: Then Corollary 5.4 gives $\mathcal{L}(D) = 0$.
- $\deg D > 2g-2$: Then Proposition 5.7, (c) gives $l(D) = 1 - g + \deg D$.

In the range $[0, 2g-2]$ however, we shouldn't expect a definite answer: $l(D)$ really depends on the individual divisor D.

The cases $g = 0$ and $g = 1$ are rather simple, and separate from the main development. I treat them as worked exercises.[2]

For $g \geq 2$, a notable point is that there are $2g$ steps between -1 and $2g-1$. If D_{-1} is a divisor of degree -1, and D_{2g-1} a divisor of degree $2g-1$,

- the difference in degrees is

$$\deg D_{2g-1} - \deg D_{-1} = 2g - 1 - (-1) = 2g, \tag{5.11}$$

- whereas the difference in the dimension of their RR spaces is

$$l(D_{2g-1}) - l(D_{-1}) = g - 0 = g \tag{5.12}$$

The next section works with chains of $2g$ steps

$$D_{-1} < D_0 < \cdots < D_{i-1} < D_i < \cdots < D_{2g-2} < D_{2g-1}, \tag{5.13}$$

with $\deg D_i = i$. Each adds one point P_i. By Trick (a), each step has $l(D_{i+1}) - l(D_i) = 0$ or 1s. Hence, exactly g steps must go up, and g remain unchanged.

5.10 Main Proposition III

With g defined as above, there exists a divisor K on C with $\deg K = 2g-2$ and $l(K) = g$. The key point in numerology is that

$$l(K) = g \ > \ 1 - g + \deg K, \tag{5.14}$$

so that K is irregular.

[2]For $g = 0$, the range is empty, and $l(D) = 1 + \deg D$ for every divisor with $\deg D \geq -1$. For $g = 1$, the range is $[0, 0]$. For a divisor D of degree 0, either $D \overset{\text{lin}}{\sim} 0$ and $l(D) = 1$; or not, and then $l(D) = 0$. The linear equivalence classes of degree 0 on a curve C of genus $g = 1$ are closely related to the group law on an elliptic curve (see Ex. 5.13 for a first introduction).

This proposition is considerably more subtle than I–II, and it is the crucial point that will occupy us in the final stages of the proof.

This $K = K_C$ is called a *canonical divisor* of C. We see that it is unique up to linear equivalence, and its divisor class is the *canonical class* of C. It is irregular, the biggest irregular divisor, and contains every irregular divisor class. I prove in the next section that it control the irregularity of every divisor, via the following property.

Lemma 5.8 *$\mathcal{L}(C, K_C) = \mathcal{L}(C, K_C + P)$ for every $P \in C$.*

Proof Main Proposition III says that $l(K) = g$. However, $K_C + P$ has degree $2g - 1$, so it is in the regular range of Proposition 5.7, (c), so that $l(K + P) = 1 - g + 2g - 1 = g$. In passing from K_C to $K_C + P$, the degree goes up by 1, but the irregularity of K_C highlighted in (5.14) is lost.

5.11 The RR theorem assuming Main Propositions I–III

Theorem 5.9 *Let C, $g = g(C)$ and K_C be as above. For every divisor D on C*

$$l(D) - l(K - D) = 1 - g + \deg D \tag{5.15}$$

Proof, Step 1 Equality holds in (5.15) if $\deg D \geq 2g-1$ or $\deg D < 0$. In the first case, $K - D$ has degree < 0 so $l(K - D) = 0$ by Corollary 5.4, and $l(D) = 1 - g + \deg D$ by Proposition 5.7, (c). In the same way, if $\deg D < 0$ then $l(D) = 0$ and $l(K - D) = 1 - g + \deg(K - D) = g - 1 - \deg D$ so (5.15) also holds.

Thus there is nothing to prove unless $\deg D$ is in the range $[0, \dots, 2g-2]$. For the cases $g = 0$ and $g = 1$, see 5.13.

Step 2 Consider any increasing chain of divisors $D_{-1} < \cdots < D_{2g-1}$ as in (5.13) with $\deg D_i = i$ for $i = -1, \dots, 2g-1$. Each of the $2g$ step $D_{i-1} < D_i$ adds one point: $D_i = D_{i-1} + P_i$. Then Trick (a) gives

$$\text{either} \quad l(D_i) = l(D_{i-1}) \quad \text{or} \quad l(D_i) = l(D_{i-1}) + 1. \tag{5.16}$$

The chain starts at D_{-1} with $l(D_{-1}) = 0$ and ends at D_{2g-1} with $l(D_{2g-1}) = g$. Thus in the dichotomy of Trick (a), exactly g steps go up by 1, and g steps remain unchanged.

Step 3 The same conclusion applies to the increasing chain $K_C - D_i$ as i decreases from $2g-1$ down to -1. This also has $2g$ steps, starting from degree -1 and going up to degree $2g-1$; each step

$$K_C - D_{i+1} = K_C - D_i - P_i < K_C - D_i$$

adds the point P_i. By the same argument as in Step 2, exactly g steps go up by 1, and g steps remain unchanged.

Now *not both inclusions*

$$\mathcal{L}(D_i) \subset \mathcal{L}(D_{i+1}) \quad \text{and} \quad \mathcal{L}(K - D_{i+1}) \subset \mathcal{L}(K - D_i) \tag{5.17}$$

can be strict. In fact $D_{i+1} = D_i + P_i$, so if both inclusions were strict, Trick (e) of Proposition 5.3 would imply that $\mathcal{L}(K) \subsetneq \mathcal{L}(K + P_i)$ is strict, which would contradict Lemma 5.8.

Step 4 For every D with $0 \le \deg D \le 2g-1$ and every $P \in C$, the pair $D - P < D$ is contained in a chain as in Step 2 (in many ways). In fact, subtract off any $\deg D$ points from $D - P$ to get down to degree -1, and add any $2g - 1 - \deg D$ points to D to get up to degree $2g-1$.

By Step 3 only one of the inclusions $\mathcal{L}(D-P) \subset \mathcal{L}(D)$ and $\mathcal{L}(K-D) \subset \mathcal{L}(K - D + P)$ can be strict at each step. However, the two chains up and down each have $2g$ steps, of which g go up and g remain unchanged. It follows that exactly one of the inclusions is strict. That is,

$$\begin{aligned} &\textit{either} \;\; l(D) - l(D-P) = 1 \;\text{ and }\; l(K-D+P) - l(K-D) = 0, \\ &\textit{or} \qquad l(D) - l(D-P) = 0 \;\text{ and }\; l(K-D+P) - l(K-D) = 1. \end{aligned} \tag{5.18}$$

Step 5 Theorem 5.9 now follows by induction, starting from $\deg D = -1$. In fact, if (5.15) holds for $D - P$, it follows for D by (5.18). Q.E.D.

5.12 Motivation for III

On a compact Riemann surface S, the canonical class corresponds to the space of holomorphic 1-forms Ω^1_S. A holomorphic 1-form is locally of the form $s = g(z)\mathrm{d}z$ with g a holomorphic function. We also have meromorphic 1-forms obtained by allowing $g(z)$ to be a meromorphic function, and $K = \operatorname{div} s$ is formed from the zeros and poles of the $g(z)$.

Now Lemma 5.8 is the statement that a meromorphic 1-form s on a Riemann surface cannot have a simple pole at P as its only pole. In fact, the integral of s on a contour around P gives $\frac{1}{2\pi i}\oint s =$ residue of S of P, but the same contour can be viewed as bounding the exterior of S on which s is holomorphic. Therefore the residue would be zero, so that s does not actually have a pole.

In analysis, a meromorphic function f with pole of order d at P has the local form

$$f(z) = \frac{a_d}{z^d} + \frac{a_{d-1}}{z^{d-1}} + \cdots + \frac{a_1}{z} + \text{regular}$$

with the *principal part* having d free parameters $a_1, \dots, a_d$. Corollary 5.4 corresponds to the idea that allowing poles on an effective divisor D allows principal parts depending on a vector space of dimension $\deg D = \sum d_P$.

Corresponding to the g-dimensional RR space $\mathcal{L}(C, K_C)$ of Main Proposition (III), an analytic definition of the genus $g(S)$ is as the dimension of the space of global holomorphic 1-forms. Now given any global holomorphic 1-form s, contour integration provides a linear relation on the possible principal parts of f. Indeed, if we take a contour going around all the poles of f then $\frac{1}{2\pi i} \oint fs$ equals the sum of the residues of fs. Viewing the contour as going around its exterior, we see the integral is zero. In other words, the g holomorphic 1-form of S provide g linear conditions on the possible principal parts, which explains the right-hand side $1 - g + \deg D$ of the RR formula. The irregularity of D covers the possibility that these conditions are not linearly independent, and this also explains the formula $l(K - D)$ for the irregularity of D.

5.13 Curves of genus 0 and 1

The results Proposition I–II–III all hold for $g = 0$ and $g = 1$ (as does, of course, the RR theorem itself). For example, the critical range of degrees $[0, 2g-2]$ is empty for $g = 0$, or is just the single value $[0]$ for $g = 1$. However, worrying about the initial cases may be a bit of a distraction in studying the main body of theory, so I state everything here for you to think over separately as exercises.

For $g = 0$, I prove in the next section that $C \cong \mathbb{P}^1$. Let t_1, t_2 be homogeneous coordinates, and set $x = t_2/t1$ for an affine parameter. Write P for the point $t_1 = 0$, corresponding to $x = \infty$; any other point $Q \in \mathbb{P}^1$ is linearly equivalent to P so would do equally well. For $n \geq 0$ the RR space $\mathcal{L}(nP)$ corresponds to polynomials of degree $\leq n$ in x, which is a vector space of dimension $n+1$, or to the space $S^n(t_1, t_2)$ of degree n homogeneous polynomials.

The formula $\dim \mathcal{L}(nP) = n + 1$ continues to hold for $n = 0$ (giving the constant functions as polynomials of degree 0). It also holds when $n = -1$, giving the vector space 0. The canonical divisor is given by $K_C - 2P$. It is irregular because it has degree -2 compared to $1 + 0 + -2 = -1$ You can check that it satisfies all that is required of it in the above.

The case $g = 1$ over the complex numbers was discussed in Example 5.1 as the quotient E of $\mathbb{C}$ by the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. As there, write O for the image of 0 in E. In this case, the methods of complex analysis give alternative methods: the holomorphic differential $\mathrm{d}z$ is invariant under translation by Λ, and for a meromorphic function f, the contour integration $\oint f \mathrm{d}z$ around the unit parallelogram gives the single linear relation on the principle part of f at its poles, and proves $l(nP) \leq n$. The Weierstrass P-function and its derivative provide functions $x = \wp$ and $y = \wp'$ with respective poles of order 2 and 3 at O.

Exercise 5.10 Show that the functions

$$\begin{aligned} &1, x, \dots, x^m, y, xy, \dots, x^{m-2}y \in \mathcal{L}(2mO), \\ &\qquad \text{respectively} \quad 1, x, \dots, x^m, y, xy, \dots, x^{m-1}y \in \mathcal{L}((2m+1)O) \end{aligned} \tag{5.19}$$

base the RR spaces. Show also that $y^2 \in \mathcal{L}(6O)$ satisfies the Weierstrass equation $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{C}$. [Hint: Argue on their leading term at O.]

Since $\Lambda \subset \mathbb{C}$ is a subgroup of the additive group $\mathbb{C}^+$, the quotient $E = \mathbb{C}/\Lambda$ is itself a group with $O \in E$ the additive unit and the quotient map $\mathbb{C} \to E$ a group homomorphism.

Exercise 5.11 Take the logarithmic form $\mathrm{d}\log f = \frac{\mathrm{d}f}{f}$ and its contour integration around the unit parallelogram. This proves that for a meromorphic function $f \in E$ the divisor $\operatorname{div} f$ has degree 0.

Moreover, if $\operatorname{div} f = \sum n_i P_i$ with P_i the image of $z_i = a_i \in \mathbb{C}$, then also $\sum n_i a_i \in \Lambda$, so that $\operatorname{div} f$ adds to 0 in the group law of $E = \mathbb{C}/\Lambda$. To prove this take the contour integral of $\frac{zf'}{f}\mathrm{d}z$. You first calculate its residues. Next (since the integrand is not invariant under translation by Λ) you verify that the integrals along opposite sides of the unit parallelogram cancels modulo integer multiplies of Λ.