MA4L7 Algebraic curves

Part 2. The RR theorem (assuming (I–III))

5 Introduction

Part 1 set out the main object of study, a nonsingular projective algebraic curve C. For C to be nonsingular at a point $P \in C$ is the condition that the local ring $\mathcal{O}_{C,P}$ is a discrete valuation ring (DVR). Alternatively, an affine curve $C \subset \mathbb{A}^n$ is nonsingular if and only its coordinate ring $k[C]$ is normal (that is, integrally closed). A basic initial circle of ideas is called *resolution of singularities*: this replaces any irreducible algebraic curve Γ with a nonsingular projective curve C having a morphism $C \to \Gamma$ that is finite and birational, and establishes that the nonsingular projective model C is unique up to isomorphism. Over $\mathbb{C}$, these curves can also be identified with compact Riemann surfaces.

Part 2 assumes the notion of nonsingular projective curve $C \subset \mathbb{P}^n$ (over an algebraically closed field k), and its field of rational functions $k(C)$. Nonsingular means the local ring $\mathcal{O}_{C,P}$ at every point $P \in C$ is a DVR. For $f \in k(C)^\times$ and $P \in C$, the valuation $v_P(f)$ describes the zeros or poles of f.

The Riemann–Roch theorem controls the vector space $\mathcal{L}(C,D)$ of meromorphic functions with specified poles on a compact Riemann surface or a nonsingular projective algebraic curve – if you allow more poles, you get more functions. Part 2 discusses the statement of the Riemann-Roch theorem:

$$\dim \mathcal{L}(C,D) \geq 1 - g + \deg D \tag{5.1}$$

(together with accompanying reasonable conditions that guarantee equality). Here the *divisor* D is a formal sum $D = \sum d_i P_i$ of points $P_i \in C$ with multiplicity d_i. The *Riemann–Roch space* $\mathcal{L}(C,D)$ is the vector space of *rational* or *global meromorphic functions* on C having only poles at P_i of order $\leq d_i$ (I assume here for simplicity that $d_i > 0$, the main case). The number $g = g(C)$ is the *genus*, the most important numerical invariant of

C. It can be described intuitively as the "number of holes" in topology, but it has many quite different characterisations in analysis and in algebraic geometry, and can be calculated in many different ways.

The proof of RR in algebraic geometry is deduced here from three Main Propositions (I–III) that I state below, but only prove in Part IV.

6 Divisors and the RR space

Definitions I work over an algebraically closed field k. A *nonsingular projective curve* is an irreducible variety $C \subset \mathbb{P}^N$ such that the local ring $\mathcal{O}_{C,P}$ at each $P \in C$ is a DVR. This means $\mathcal{O}_{C,P} \subset k(C)$ is a subring of the function field of C, with maximal ideal $m_P = (z_P)$ the principal ideal generated by a *local parameter* z_P. Every nonzero function $f \in k(C)^\times$ is then of the form $f = z_P^v \cdot f_0$ with $f_0 \in \mathcal{O}_{C,P}^\times$ a unit at P. Here $v = v_P(f)$ is the *valuation* of f at P. We say f has *zero of order* $v_P(f)$ if it is positive, or *pole of order* $-v_P(f)$ if it is negative.

A *divisor* is a finite combination of points of C, written $D = \sum d_i P_i$ with $P_i \in C$ and $d_i \in \mathbb{Z}$; or alternatively $D = \sum_{P \in C} d_P P$, where the expression assumes that $d_P = 0$ for all but finitely many P. A divisor $D = \sum d_P P$ is *effective* (written $D \geq 0$) if $d_P \geq 0$ for every P. The *degree* of D is $\sum_{P \in C} d_P$.

The divisor $\operatorname{div} f$ of a rational function $f \in k(C)^\times$ is

$$\operatorname{div} f = \sum v_P(f) P = \text{zeros of } f - \text{poles of } f. \tag{6.1}$$

Both f and f^{-1} are regular outside a finite set, so (6.1) is a finite sum. A divisor of the form $\operatorname{div} f$ for $f \in k(C)^\times$ is *principal*. Two divisors D_1 and D_2 are *linearly equivalent* if they differ by a principal divisor, that is, $D_1 - D_2 = \operatorname{div} g$ for some $g \in k(C)^\times$.

The *RR space* of D on C is defined as the vector subspace

$$\mathcal{L}(C, D) = \bigl\{ f \in k(C) \bigm| \operatorname{div} f + D \geq 0 \bigr\} \subset k(C). \tag{6.2}$$

If D is effective the condition $\operatorname{div} f + D \geq 0$ allows f to have poles of order $\leq d_P$ at each P (adding the positive part of D cancels its poles). The more general case is a neat way of allowing poles of order $\leq d_P$ where $d_P > 0$, and imposing a zero of order $\geq b$ where $b = -d_P > 0$ (so the zeros of f cancel the negative terms in D).

While not strictly necessary, it is informative to use the same condition to define the structure sheaf $\mathcal{O}_C$ and the divisorial sheaf $\mathcal{O}_C(D)$. The constant sheaf $k(C)$ has the fixed pool $k(C)$ of rational functions on every nonempty

Zariski open set of C. Inside $k(C)$, the regular functions are characterised as $\mathcal{O}_{C,P} = \{f \in k(C) \mid v_P(f) \geq 0\}$. At each P, the divisorial sheaf condition for $\mathcal{O}_C(D)$ replaces this with $\operatorname{div}(f) + d_P \geq 0$; this means that $\mathcal{O}_C(D)$ is generated locally at P by $z_P^{-d_P}$, or $\mathcal{O}_C(D)_P = \mathcal{O}_{C,P} \cdot \frac{1}{z_P^{d_P}}$. (If you know the ideas of algebraic number theory, this is essentially the same as a *fractional ideal.*)

Then $\mathcal{O}_C(D)$ is the subsheaf of the constant sheaf $k(C)$ obtained by imposing the condition $\operatorname{div} f + D \geq 0$ over each Zariski open subset $U \subset C$. In other words the sections of $\mathcal{O}_C(D)$ over U are

$$\begin{aligned}\mathcal{O}_C(D)(U) &= \Gamma(U, \mathcal{O}_C(D)) \\ &= \left\{ f \in k(C) \;\middle|\; v_P(f) + d_P \geq 0 \text{ for all } P \in U \right\} \subset k(C).\end{aligned}$$

This definition is local near each $P \in U$, making the sheaf condition automatic. The definitions make $\mathcal{O}_C(D)$ a locally free sheaf of $\mathcal{O}_C$-modules of rank 1, based by $z_P^{-d_P}$ in a Zariski neighbourhood of P. The global sections $\Gamma(C, \mathcal{O}_C(D))$ is the same thing as the RR space $\mathcal{L}(C, D)$.

If $D \geq 0$ then $\mathcal{O}_C \subset \mathcal{O}_C(D)$. Also, $\mathcal{O}_C(-D) = \mathcal{I}_D \subset \mathcal{O}_C$ is the sheaf of ideals of D (regular functions with zeros on D).

The following points come at once from the definitions; they recur repeatedly as standard computational devices in the proof and all applications of RR.

Proposition 6.1 (Standard tricks) *(a) For D a divisor and $P \in C$ a point, consider the inclusion $\mathcal{L}(C, D - P) \subset \mathcal{L}(C, D)$. Then any $s \in \mathcal{L}(D) \setminus \mathcal{L}(D - P)$ is a complementary basis element. In other words, we have the dichotomy:*

- *(i) either $\mathcal{L}(D) = \mathcal{L}(D - P)$;*
- *(ii) or $\mathcal{L}(D) = k \cdot s \oplus \mathcal{L}(D - P)$ for some nonzero $s \in \mathcal{L}(D)$.*

More crudely, $\mathcal{L}(D - P) \subset \mathcal{L}(D)$ has codimension ≤ 1.

(b) Moreover if $\mathcal{L}(D) \neq 0$, case (i) holds for at most finitely many $P \in C$.

(c) $\operatorname{div}(f_1 f_2) = \operatorname{div} f_1 + \operatorname{div} f_2$ *for all $f_1, f_2 \in k(C)^\times$.*

(d) Suppose D_1 and D_2 are linearly equivalent divisors, with $D_1 - D_2 =$ $\operatorname{div} g$ *for $g \in k(C)^\times$. Let $f \in k(C)^\times$. Then*

$$f \in \mathcal{L}(D_1) \iff fg \in \mathcal{L}(D_2). \tag{6.3}$$

That is, multiplication by g defines an isomorphism $k(C) \to k(C)$ that takes $\mathcal{L}(D_1)$ to $\mathcal{L}(D_2)$. In particular $l(D_1) = l(D_2)$.

(e) Suppose $\mathcal{L}(A+B-P)=\mathcal{L}(A+B)$ *for divisors* A *and* B *and* $P\in C$. *Then not both of the inclusions*

$$\mathcal{L}(A-P)\subset\mathcal{L}(A) \quad \text{and} \quad \mathcal{L}(B-P)\subset\mathcal{L}(B) \tag{6.4}$$

can be strict.

The proofs are completely formal.

(a) Take a local parameter z_P at P and write $d_P\in\mathbb{Z}$ for the multiplicity of P in D. The condition $\operatorname{div} f + D\ge 0$ at P is equivalent to $z_P^{d_P}f$ regular at P, so $z_P^{d_P}f\in\mathcal{O}_{C,P}$. For $s\in\mathcal{L}(D)$, if $z_P^{d_P}s$ vanishes at P then $z_P^{d_P-1}s$ is also regular at P, so $f\in\mathcal{L}(D-P)$. If this holds for every $s\in\mathcal{L}(D)$ then (i) holds. Otherwise $z_P^{d_P}s$ is a unit at P for some s, in which case (ii) holds.

(a) reflects the fact that the powers of the maximal ideal of a DVR are principal $m^d=(z^d)$, with successive quotients the residue field $m^{d-1}/m^d\cong A/m$.

(b) If $\mathcal{L}(D)\ne 0$ then for any nonzero $f\in\mathcal{L}(D)$, the divisor $\operatorname{div} f+D$ is effective, and (ii) holds for any P not in its support.

(c) This follows from the basic property $v(fg)=v(f)+v(g)$ of a discrete valuation: at any $P\in C$, suppose $f_1=z_P^{d_1}\cdot u_1$ and $f_2=z_P^{d_2}\cdot u_2$, with units $u_1,u_2\in\mathcal{O}_{C,P}^\times$ and $v_P(f_i)=d_i$. Then $f_1f_2=z_P^{d_1+d_2}u_1u_2$ with u_1u_2 a unit, so that $v_P(f_1f_2)=d_1+d_2$.

(d) This holds because $\operatorname{div}(fg)=\operatorname{div} f+\operatorname{div} g=\operatorname{div} f+D_1-D_2$. Thus $\operatorname{div} f\ge -D_1$ if and only if $\operatorname{div}(fg)\ge -D_2$.

(e) This follows from (c): $f\in\mathcal{L}(A)\setminus\mathcal{L}(A-P)$ and $g\in\mathcal{L}(B)\setminus\mathcal{L}(B-P)$ would give $fg\in\mathcal{L}(A+B)$ and $v_P(fg)=a_P+b_P$, where the coefficients of A and B at P are a_P and b_P, so $fg\notin\mathcal{L}(A+B-P)$. Q.E.D.

6.1 Main Proposition (I)

A principal divisor has degree 0*:* $\deg(\operatorname{div} f)=0$ *for* $f\in k(C)^\times$. Since we interpret the divisor of f as $\operatorname{div} f=\text{zeros of } f-\text{poles of } f$, this says that any rational function has the same number of zeros and poles.

Corollary 6.2 *(1) If* $\deg D<0$ *then* $\mathcal{L}(C,D)=0$.

(2) $l(D)=\dim\mathcal{L}(C,D)\le 1+\deg D$ *for any divisor* D.

Proof If $0 \neq f \in \mathcal{L}(C, D)$ then $\operatorname{div} f + D$ is an effective divisor, so has degree ≥ 0, hence $\deg D \geq 0$. This proves 1.

(2) follows from (1) by induction on $\deg D$ and Standard Trick (a). Suppose $\deg D \geq 0$ and let $P \in C$ be any point. Then $\deg(D - P) = \deg D - 1$ so by induction $l(D - P) \leq \deg D$, and (a) gives $l(D) \leq 1 + \deg D$. □

Corollary 6.3 *If $A = \sum P_i$ is an effective divisor (repeated points allowed) then $l(D - A) \geq l(D) - \deg A$.*

This follows by repeated use of Standard Trick (a): in passing from D to $D - P_1 - \cdots - P_i$, the dimension of $\mathcal{L}(D - P_1 - \cdots - P_i)$ decreases by at most 1 at each step. Q.E.D.

Motivation for (I) On a compact Riemann surface, we can prove Main Proposition (I) by contour integration and the Cauchy integral theorem. In fact, let f be a global meromophic function and write

$$\mathrm{d} \log f = \frac{\mathrm{d} f}{f} \quad \text{or locally} \quad \tfrac{\mathrm{d} f / \mathrm{d} z}{f} \mathrm{d} z$$

for its logarithmic derivative. This has pole of order 1 with residue $v_P(f)$ at every zero or pole of f: for where $f = z^n \cdot f_0$ with f_0 a unit, we get $\mathrm{d} \log f = \frac{n}{z} + \text{regular}$. (Check that this works in all three cases $n > 0$, $n = 0$ and $n < 0$.) The integral $\frac{1}{\times 2\pi i} \oint \mathrm{d} \log f$ around a contour thus counts the zeros and poles in the interior of the contour.

Take a countour Γ that divides the surface up into an interior containing all the zeros and poles and an exterior containining no zeros and poles. Then $\frac{1}{2\pi i} \oint \mathrm{d} \log f = \deg(\operatorname{div} f)$ if we view Γ as surrouning its interior, and $= 0$ if we view it as surrounding its exterior. Equating the two gives $\deg(\operatorname{div} f) = 0$.

On a compact Riemann surface, Cor. 6.2, (1) includes the statement that there are no global holomorphic functions other than the constants. In complex analysis, this follows from the Maximum Modulus principle: the modulus $|f|$ of a global holomorphic function f would be a continuous function, and on a compact space this would take a maximum value at some point P. But then the modulus would be constant, and hence also f is constant.

6.2 Main Proposition (II)

There exist a family D_n of divisors on C for which $\deg D_n \to \infty$ while the difference $1 + \deg D_n - l(D_n)$ remains bounded, say $1 + \deg D_n - l(D_n) \leq N$.

Corollary 6.4 *Assume this. Then the same bound* $1 + \deg D - l(D) \le N$ *holds for every divisor* D *on* C*.*

Proof The first step is to show that for every D, there is some n such that $\mathcal{L}(D_n - D) \ne 0$. In fact if $D = A - B$ with A, B effective divisors, choose n such that $l(D_n) > \deg A$. Corollary 6.3 implies that $\mathcal{L}(D_n - A) \ne 0$.

Thus replacing $D_n - D = D_n - A + B$ by a linearly equivalent divisor, we can assume it is effective, say $D_n - D \sim \Delta > 0$; I can turn that around to $D \sim D_n - \Delta$. Now Corollary 6.3 again implies that $l(D) \ge l(D_n) - \deg\Delta$, so that

$$\begin{aligned} 1 + \deg D - l(D) &\le 1 + \deg D + \deg\Delta - l(D_n) \\ &\le 1 + \deg D_n - l(D_n) \le N. \qquad \Box \end{aligned}$$

6.3 Motivation for (II)

In complex analysis, this is the hard part of Riemann–Roch, that requires partial differential equations to prove the existence of harmonic functions (satisfying the Laplace equation $\Delta f =$, with singularities at the poles interpreted in terms of boundary value problems), and then the Cauchy–Riemann equations to link harmonic functions and holomorphic functions. Riemann's own motivation for the statement (that he never proved correctly) involved the ideas of electrostatics: a pointwise electric charge must defines a harmonic potential "for physical reasons".

In algebraic geometry (II)) is straightforward. A projective curve is birational to a plane curve, say via a morphism $f\colon C \to \overline{C}_a \subset \mathbb{P}^2$, where $\overline{C}_a$ is a plane curve of degree a (usually singular) defined by $F_a(x, y, z) = 0$. Choose coordinates x, y, z so that the line $z = 0$ meets $\overline{C}$ only at nonsingular points (or even transversally), and set $H = \operatorname{div} z$ for the divisor "at infinity". It is the effective divisor defined by z/x away from the line $x = 0$ and by z/y away from $y = 0$, and it has degree a because F_a cuts out a points with multiplicity on the line $z = 0$.

Now any degree n form $G_n(x, y, z)$ defines a rational function G_n/z^n on C with poles at most nH. It is an exercise to see that this restriction provides a subspace of $\mathcal{L}(C, nH)$ of dimension $1 - \binom{a-1}{2} + an$.

6.4 Definition of genus $g(C)$ and immediate consequences

In view of Corollary 6.4, it makes sense to define

$$g(C) = \max\bigl\{1 + \deg D - l(D) \bigm| \text{for every divisor } D \text{ on } C\bigr\}.$$

It then follows formally that

$$l(D) \geq 1 - g + \deg D \quad \text{for every } D, \tag{6.5}$$

and equality holds for some D.

We say that D is *regular* if $l(D) = 1 - g + \deg D$. Otherwise, the difference $l(D) - (1 - g + \deg D)$ is the *irregularity* of D. The full form of RR includes a formula for the irregularity of D. See 6.6 below.

I use this definition for the discussion of the following sections, as the most appropriate for the logical purpose of proving the RR theorem and using it in algebraic geometric applications. I discuss later how it relates to several other definitions in algebraic geometry, topology, analysis and different types of cohomology.

Proposition 6.5 *(1) Every divisor D of degree $\geq g$ has $\mathcal{L}(D) \neq 0$.*

(2) There exists a divisor D_{g-1} with $\deg D_{g-1} = g - 1$ and $\mathcal{L}(D_{g-1}) = 0$.

(3) Equality holds in (6.5) for every D with $\deg D \geq 2g - 1$.

Proof (a) is clear. For (b), let D be some divisor for which equality holds in (6.5). If $\deg D \geq g$ then $\mathcal{L}(D) \neq 0$, so that Standard Trick (b) applies: $l(D - P) = l(D) - 1$ for all but finitely many points of C. It follows that equality also holds in (6.5) for $D - P$, having smaller dimension $l(D - P)$. Induction on $l(D)$ takes us down to $\mathcal{L}(D) = 0$ and equality still holding in (6.5), which is (b).

Now for (c), choose D_{g-1} as in (b), and suppose that $\deg D \geq 2g - 1$. Then $\deg(D - D_{g-1}) \geq g$ and so $\mathcal{L}(D - D_{g-1}) \neq 0$ by (a), and $D - D_{g-1}$ is linearly equivalent to an effective divisor A with $\deg A = \deg D - g + 1$. As before, I can turn this around to $D_{g-1} \sim D - A$. Then Corollary 6.3 gives $l(D_{g-1}) \geq l(D) - \deg A$, so that

$$l(D) \leq 0 + \deg A = 1 - g + \deg D.$$

Together with (6.5) this proves (c). Q.E.D.

6.5 Main Proposition (III)

With g defined as above, there exists a divisor K with $\deg K = 2g - 2$ and

$$l(K) = g > 1 - g + \deg K.$$

This statement is considerably more subtle, and is the key point that will occupy us in the final stages the proof.

This $K = K_C$ is called a *canonical divisor* of C. We see shortly that it is unique up to linear equivalence, and its divisor class is called the *canonical class* of C. It is irregular, the biggest irregular divisor, and contains every irregular divisor class. The following property allows it to control the irregularity of every divisor, as I prove in the next section.

Lemma 6.6 *$\mathcal{L}(C, K_C) = \mathcal{L}(C, K_C + P)$ for every $P \in C$.*

Proof We already know this: Main Proposition (III) says that $l(K) = g$, whereas $\deg K + P = 2g - 1$, so that also $l(K + P) = g$ by Proposition 6.5, (3).

6.6 Proof of RR assuming Main Propositions (I–III)

Theorem 6.7 *Let C, $g = g(C)$ and K_C be as above. For every divisor D on C*

$$l(D) - l(K - D) = 1 - g + \deg D \tag{6.6}$$

Proof, Step 1 Equality holds in (6.6) if $\deg D \geq 2g - 1$ or $\deg D < 0$. In the first case, $K - D$ has degree < 0 so $l(K - D) = 0$ by Corollary 6.2, and $l(D) = 1 - g + \deg D$ by Proposition 6.5, (3). In the same way, if $\deg D < 0$ then $l(D) = 0$ and $l(K - D) = 1 - g + \deg(K - D) = g - 1 - \deg D$ so (6.6) also holds.

So there is nothing to prove unless $\deg D$ is in the range $[0, \dots, 2g - 2]$.

Step 2 Consider any increasing chain of divisors

$$D_{-1} < D_0 < D_1 < \cdots < D_{2g-2} < D_{2g-1}$$

with $\deg D_i = i$ for $i = -1, \dots, 2g - 1$, with each step adding one point: $D_{i+1} = D_i + P_i$. Then Standard Trick (a) gives

$$l(D_{i+1}) = l(D_i) \quad \text{or} \quad l(D_i) + 1.$$

The chain has $2g$ steps, starting at $l(D_{-1}) = 0$ and ending at $l(D_{2g-1}) = g$. Thus exactly g steps go up by 1, and g steps remain fixed.

Step 3 The same thing applies to the chain $K_C - D_i$ for i decreasing from $2g-1$ down to -1. This also has $2g$ steps, starting from degree -1 and going up to degree $2g-1$; each step from $i+1$ down to i is $K - D_i - P_i < K - D_i$. As in Step 3, exactly g steps go up by 1, and g steps remain fixed.

However, since $D_{i+1} = D_i + P_i$, if both the inclusions $\mathcal{L}(D_i) \subset \mathcal{L}(D_{i+1})$ and $\mathcal{L}(K - D_{i+1}) \subset \mathcal{L}(K - D_i)$ were strict, Standard Trick (e) would imply that $\mathcal{L}(K) \subset \mathcal{L}(K + P_i)$ is strict, which Lemma 6.6 forbids. Therefore for each i, *exactly one* of the two inclusions is strict.

Step 4 For every D with $0 \le \deg D \le 2g-1$ and every $P \in C$, the pair $D - P < D$ is contained in a chain as in Step 2 (in many ways). In fact, I can just subtract off one by one any $\deg D$ points from $D - P$ to get down to degree -1, and add any $2g - 1 - \deg D$ points to get up to degree $2g-1$.

It follows from Step 3 that *exactly one* of the inclusions $\mathcal{L}(D-P) \subset \mathcal{L}(D)$ and $\mathcal{L}(K-D) \subset \mathcal{L}(K-D+P)$ is strict. That is,

$$\begin{aligned} &\textit{either} \quad l(D) - l(D-P) = 1 \quad \text{and} \quad l(K-D+P) - l(K-D) = 0, \\ &\textit{or} \qquad\ l(D) - l(D-P) = 0 \quad \text{and} \quad l(K-D+P) - l(K-D) = 1. \end{aligned} \tag{6.7}$$

Step 5 Theorem 6.7 now follows by induction, starting from $\deg D = -1$. In fact, if (6.6) holds for $D - P$, it follows for D by (6.7).

6.7 Motivation for (III)

On a compact Riemann surface S, the canonical class corresponds to the space of holomorphic 1-forms Ω^1_S. A holomorphic 1-form is locally of the form $s = g(z)\mathrm{d}z$ with g a holomorphic function. We also have meromorphic 1-forms obtained by allowing $g(z)$ to be a meromorphic function, and $K = \operatorname{div} s$ is formed from the zeros and poles of the $g(z)$.

Now Lemma 6.6 corresponds to the statement that a meromorphic 1-form s on a Riemann surface cannot have a simple pole at P as its only pole. In fact, the integral of s on a contour around P gives $\oint s =$ residue of S of P, but the same contour can be viewed as bounding the exterior of S on which s is holomorphic. Therefore the residue would be zero, so that s does not actually have a pole.

In analysis, a meromorphic function f with pole of order d at P has the local form

$$f(z) = \frac{a_d}{z^d} + \frac{a_{d-1}}{z^{d-1}} + \cdots + \frac{a_1}{z} + \text{regular}$$

with the *principal part* having d free parameters $a_1, \ldots, a_d$. Corollary 6.2 corresponds to the idea that allowing poles on an effective divisor D allows principal parts depending on a vector space of dimension $\deg D = \sum d_P$.

Corresponding to the g-dimensional RR space $\mathcal{L}(C, K_C)$ of Main Proposition (III), an analytic definition of the genus $g(S)$ is as the dimension of the space of global holomorphic 1-forms. Now given any global holomorphic 1-form s, contour integration provides a linear relation on the possible principal parts of f. Indeed, if we take a contour going around all the poles of f then $\frac{1}{2\pi i \times} \oint fs$ equals the sum of the residues of fs. Viewing the contour as going around its exterior, we see the integral is zero. In other words, the g holomorphic 1-form of S provide g linear conditions on the possible principal parts, which explains the right-hand side $1 - g + \deg D$ of the RR formula. The irregularity of D covers the possibility that these conditions are not linearly independent, and this also explains the formula $l(K - D)$ for the irregularity of D.