Normal characterises DVR

Proposition 0.1 *Let A be a Noetherian local integral domain having Krull dimension* 1.

Then A is normal if and only if A is a DVR.

Note To say that A is a local integral domain just means that 0 is a prime ideal and there is a single maximal ideal m. Adding 1-dimensional to that means that A has $0 \subsetneq m$ as its only prime ideals.

Normal is the condition that A is integrally closed in its field of fractions $K = \operatorname{Frac} A$.

Step 1: Lemma *Let A be a 1-dimensional Noetherian integral domain. Then $m \neq 0$, and for any nonzero $x \in m$, there exists an element $y \in A \setminus (x)$ for which*

$$m = \{z \in A \mid zy \in (x)\}. \tag{0.1}$$

Consider the cyclic module $A/(x)$, and for $y \in A \setminus (x)$, write $\overline{y} \in A/(x)$ for its image under the quotient map. Then $\overline{y} \in A/(x)$ has annihilator ideal $\operatorname{Ann} \overline{y}$. In terms of $y \in A$ this means that $y \notin (x)$, and $\operatorname{Ann} \overline{y}$ is the ideal of A given by

$$\operatorname{Ann} \overline{y} = ((x) : y) = \{z \in A \mid zy \in (x)\}. \tag{0.2}$$

By the Noetherian assumption, the set Σ of all ideals of this form has a maximal element.

Claim 0.2 *If $y \in A \setminus x$ has the property that $\operatorname{Ann} \overline{y}$ is maximal among all the ideals of the form $\operatorname{Ann} \overline{y}$ then it is prime. Since I currently assume that A is a 1-dimensional local domain, and $\operatorname{Ann} \overline{y}$ contains at least (x), the only possibility is $\operatorname{Ann} \overline{y} = m$.*

Proof Work with $y \in A$. Suppose $z_1, z_2 \notin \operatorname{Ann} \overline{y}$. Then $z_2 y \notin (x)$, so that $\operatorname{Ann}(z_2 \overline{y})$ contains $\operatorname{Ann} \overline{y}$, and the maximal assumption implies they are equal. Then

$$z_1 \notin \operatorname{Ann} \overline{y} \implies z_1 z_2 y \notin (x), \tag{0.3}$$

and also the product $z_1 z_2 \notin (x)$. Therefore the ideal $\operatorname{Ann} \overline{y}$ is prime.

Note The submodule $A \cdot \overline{y} \subset A/(x)$ is then isomorphic to the residue field A/m. This means that m is an *associated prime* of $A/(x)$. Thus the result is part of primary decomposition: quite generally any finite module M has a Jordan-Hölder sequence with successive quotients of the form A/P with $P \in \operatorname{Spec} A$ an associated prime of M.

Step 1 did not use integrally closed.

Step 2 If x, y are as in Step 1, then the element $y/x \in \operatorname{Frac} A$ is not in A, but $m(y/x) \subset A$. Therefore, there is a dichotomy: either

(I) $(y/x)m \subset m$. Standard use of the determinant trick (see for example [UCA 2.6]) gives that y/x is integral over A. Then since A is normal, this implies that $y/x \in A$, which contradicts $y \notin (x)$.

(II) $(y/x)m = A$. Then there is $t \in m$ such that $(y/x)t = 1$, that is, $t = x/y$. Then $m = (t)$ is principal: in fact for $z \in m$,

$$(y/x)z \in A, \quad \text{so that} \quad z = t((y/x)z) \in (t). \quad \square \tag{0.4}$$