MA4L7 Algebraic curves

Example sheet 5, Deadline Wed 11th Mar

Counter-examples to regular sequence

Exercise 5.1 In $\mathbb{A}^4_{\langle x,y,z,t\rangle}$ let $V = \Pi_{x,y} \cup \Pi_{z,t}$ be the union of the x,y- and z,t-coordinate planes. Verify that its ideal is

$$I_V = (z,t) \cap (x,y) = (xz, xt, yz, yt),$$

so that its coordinate ring is $k[V] = k[x,y,z,t]/(xz,xt,yz,yt)$.

Consider the hyperplane section $V \cap H$ where H is the hyperplane $y = t$. The geometric locus $V \cap H$ is the union of the x- and z- coordinate lines, so that its ideal in $k[x,y,z,t]$ is $I(V \cap H) = (y,t,xz)$.

Write $\overline{y}$ for the residue of y in the quotient ring $k[V]/(y-t)$. Prove that $\overline{y} \neq 0$, but $\overline{y}$ is annihilated by the maximal ideal (x,y,z,t) at 0. [Hint: View the quotient ring $k[V]/(y-t)$ as

$$k[x,y,z,t]/(y-t, xz, xt, yz, yt) = k[x,y,z]/(xz, xy, yz, y^2).] \qquad (5.1)$$

Note that $y - t$ is a nonzero divisor in $k[V]$, but it does not extend to a regular sequence of length 2, because the quotient $k[V]/(y-t)$ contains the element $\overline{y}$ that is killed by every element of the maximal ideal.

Taking a hyperplane section $V \cap H$ is an obvious geometric activity. In algebra, the equation of H in $k[V]$ is a nonzerodivisor, but it does not follow that it generates a radical ideal. The example is similar to the ideal $(xy, y^2) \subset k[x,y]$, a standard example in primary decomposition.

Exercise 5.2 In the example V of the preceding exercise, calculate the dimension of the quotient ring $k[V]/(x-z, y-t)$. You might expect that since V is a union of 2 planes, it is a surface of degree 2, and the answer should be 2.

Exercise 5.3 (Macaulay quartic curve $\Gamma_4 \subset \mathbb{P}^3$) The Macaulay quartic curve $\Gamma_4 \subset \mathbb{P}^3$ is the image of $\mathbb{P}^1_{\langle u,v\rangle}$ under the embedding u^4, u^3v, uv^3, v^4.

The coordinate ring of the affine cone $V \subset \mathbb{A}^4$ over Γ_4 is the subring $A \subset k[u,v]$ generated by $x = u^4, y = u^3v, z = uv^3, t = v^4$. Show that the missing element $w = u^2v^2 \in k[u,v]$ is integral over $k[V]$, and that its products xw, yww, zw, tw with the generators of the maximal ideal $m = (x,y,z,t)$ are in $k[V]$; as a slightly tricky exercise, show that its ideal I_V is

$$I_V = (xt - yz, x^2z - y^3, xz^2 - y^2t, yt^2 - z^3). \qquad (5.2)$$

Let $a \in m$ be any nonzero element; $k[V]$ is an integral domain, so a is a nonzerodivisor. Prove that any other element $b \in m$ is a zerodivisor in the module $k[V]/(a)$. [Hint: Show that aw has nonzero residue $\overline{aw} \in k[V]/(a)$.]

Exercise 5.4 In Ex. 5.3, calculate the dimension of the quotient $k[V]/(x,t)$. In other words, calculate $k[y,z]$ modulo the ideal given by the 4 equations (5.2) with x,t set equal to zero.

Compare this with what you might expect from the degree of $\Gamma_4 \subset \mathbb{P}^3$.

Exercise 5.5 Recall the example of the cuspidal cubic $k[x,y]/(y^2 - x^3) = k[t^2,t^3]$. You can describe the subring $k[t^2,t^3] \subset k[t]$ as keeping every monomial except t.

In 2 dimensions, consider the subring of $k[x,y]$ containing every monomial except y. Show that it is generated by $a = x, b = xy, c = y^2, d = y^3$ and related by the ideal of 2×2 minors

$$\bigwedge^2 \begin{pmatrix} b & d & ac & c^2 \\ a & c & b & d \end{pmatrix} = 0$$

The common ratio $b/a = d/c$ is the missing element y. Notice that the last two columns have $c \times (a,c)$ over (b,d), which inverts the first two columns and multiplies by $c = y^2$. Because of this repetition, you can check that the ideal is generated by just 4 of the 6 minors: $(ad-bc, a^2c-b^2, ac^2-bd, c^3-d^2)$.

The ring $A = k[a,b,c,d] \subset k[x,y]$ is the coordinate ring of the image of $\mathbb{A}^2 \to \mathbb{A}^4$ given by $(x,y) \mapsto (a,b,c,d)$.

Prove that the quotient $A/(a)$ of A by the principal ideal (a) contains a nonzero element whose product with any of a,b,c,d is zero. Thus the regular element $a \in A$ does not extend to a regular sequence of length 2.

Exercise 5.6 (Regular sequences and free module) Let S be a local ring and $x \in m$ an element of its maximal ideal. Let M be a S-module that is finite as S-module. Suppose that x is a nonzero divisor of M and set $\overline{M} = M/xM$, giving the exact sequence

$$0 \to M \xrightarrow{x} M \to \overline{M} \to 0.$$

Prove that M is free over S if and only if $\overline{M}$ is free over $\overline{S} = S/(x)$.

If S is a graded ring, $s \in S$ a homogeneous element of positive degree, and M a graded module, use the same argument to prove that M is a free graded module over S if and only if $\overline{M}$ is a free graded module over $\overline{S} = S/(x)$.

Exercise 5.7 ($R(C,D)$ for $D = g^1_2$ on a hyperelliptic curve) Let C be the hyperelliptic curve treated in Ex. 3.3. The equation $z^2 = f_{2g+2}$ corresponds to the quadratic extension field $k(C)/k(x)$ of $k[x]$, with Galois group $\mathbb{Z}/2$ generated by the hyperelliptic involution, and z a -1-eigenform. The calculation of Ex. 3.3 shows that $R(C, g^1_2) = k[t_1, t_2, z]$ with t_1, t_2 coordinates on $\mathbb{P}^1$ giving a basis of $\mathcal{L}(C, g^1_2)$. The new element z in degree $g+1$ has $\operatorname{div} z = Q_1 + \cdots + Q_{2g+2}$ where $Q_i \in C$ are the branch points over the roots of f.

Thus $R(C,D)$ is a free module of rank 2 over the polynomial ring $S = k[t-1, t_2] = R(\mathbb{P}^1, P_\infty)$, with generators $1, z$ of degrees 0 and $g+1$. That is:

$$R(C,D) = S \cdot 1 \oplus S \cdot z = S \oplus S[-(g+1)].$$

Exercise 5.8 (K_C for hyperelliptic curve) It follows that the dual module

$$\mathcal{K}_C = \operatorname{Hom}(R(C,D), S(-2)) \cong S(-2) \oplus S(g-1) \cong R(C,D)[-(g-1)].$$

Verify that $\deg K_C = 2g-2$ and $l(K_C) = g$, and $l(nD) - l(K_C - nD) = 1 - g + 2n$ for all n.

Exercise 5.9 (Plane curve $C_a \subset \mathbb{P}^2$) Let $C_a : (F_a = 0) \subset \mathbb{P}^2_{\langle x,y,z\rangle}$ be a nonsingular plane curve, and assume that $(0,0,1) \notin C_a$, so the defining equation F_a is monic in z. Write $H = \operatorname{div} z$ for the hyperplane divisor. The projection of C to $\mathbb{P}^1_{\langle x,y\rangle}$ is a morphism of degree a, and correponds to the inclusion $S = k[x,y] \subset R(C,H)$. Show that $R(C,H)$ viewed as a module over S is free with generators $1, z, \ldots, z^{a-1}$, so that $R(C,H) = \bigoplus_{i=0}^{a-1} S(-i)$.

Use this to calculate $l(nH)$ and verify that for $n \geq a-2$ it can be written in the form $1 - g + na$ with $g = \sum_{i=1}^{a-1}(i-1) = \binom{a-2}{2}$.

Determine the dual module $\mathcal{K} = \operatorname{Hom}(R(C,H), S(-2))$ and prove that it is isomorphic to $R(C,H)(a-3)$.

Exercise 5.10 (Trigonal curve) Let D be a g^1_3 on a curve C, that is, $\deg D = 3$, $l(D) = 2$ and $\mathcal{L}(D) = [s_1, s_2]$ with disjoint effective divisors $D_i = D + \operatorname{div} s_i$. Viewed as a module over $S = k[s_1, s_2]$, the graded ring $R(C,D)$ is the free module generated by elements $1, z, w$, with $z \in \mathcal{L}(aD)$ and $w \in \mathcal{L}(bD)$ of degree $0 < a \leq b$, so that $R(C,D) = S \oplus S(-a) \oplus S(-b)$.

This implies that $l(nD) = (n+1)^+ + (n-a+1)^+ + (n-b+1)^+$. Spell out precisely what this means for n is the different intervals between $[-1, 0, a-1, a, b-1, b]$. Show that $g = (a-1) + (b-1)$, and that $l(nD) = 1 - g + 3n$ for $n \geq b-1$.

Carry out the same calculations for the dual module $\mathcal{K} = \mathrm{Hom}(R(C,D), S(-2))$ and verify the RR formula for all n.

Exercise 5.11 (Calculating with $R(C,D) = \bigoplus S(-a_i)$) Assume as given that $R(C,D) = \bigoplus_{i=1}^{d} S(-a_i)$ with $0 = a_1 < a_2 \le \cdots \le a_d$, and give a formula for $l(C,nD)$. For $n \ge a_d - 1$, express the answer in the form $l(C,nD) = 1 - g + nd$ for the appropriate value $g = \sum_{i=2}^{d}(a_i - 1)$. (cf. Proposition 9.4, (C) in the notes of my 2019 course).

Exercise 5.12 (RR calculations for $\mathcal{K} = \mathrm{Hom}_S(R(C,D), S(-2))$) With the above assumption $R(C,D) = \bigoplus S(-a_i)$, the dual module

$$\mathcal{K} = \mathrm{Hom}_S(R(C,D), S(-2)) \cong \bigoplus S(a_i - 2)$$

Calculate $l(K)$ and $l(K - nD)$ for $n \ge 0$, and verify the RR formula for nD. (cf. Proposition 9.6 in my 2019 notes).

Exercise 5.13 (Plane curve with ordinary multiple points) Let $\Gamma \subset \mathbb{P}^2$ be a plane curve of degree a having an ordinary multiple point of multiplicity m at Q. The normalisation (resolution of singularties) $C \to \Gamma$ has m points $P_1, \dots, P_m$ over Q, corresponding to the m tangent branches of Γ at Q.

Treat $C \to \Gamma$ as local or affine. (This means shrink Γ to an affine neighbourhood of Q, and take the inverse image of that in C, which contains all of $P_1, \dots, P_m$. Or just treat Γ as the local ring $\mathcal{O}_{\Gamma,Q}$ contained in the semilocal ring $\bigcap \mathcal{O}_{C,P_i} \subset k(C)$.)

The Brill–Noether method developed in Fulton's book asserts that forms on $\mathbb{P}^2$ of degree $n \ge a-3$ vanishing $m-1$ times at Q (that is, in the conductor ideal $\mathcal{C} = m_Q^{m-1}$) map surjectively to the RR space $\mathcal{L}(C, K_C + (n-a+3)H)$. Here H is the hyperplane section divisor, and K_C is the divisor $(a-3)H - (m-1)\sum P_i$.

Calculate the degree of all the divisors involved, and verify that the RR theorem hold for them. (That is, equality if $n \ge a - 2$, the value of g if $n = a - 3$, and the difference $l(D) - l(K - D)$ when $n < a - 2$.)

The point of Fulton's book is that every curve C is birational to a plane curve Γ with ordinary multiple points Q_i of order m_i, and the adjoint curves of degree $n + a - 3$ give an exact description of the RR spaces of $K_C + aH$. This is the Brill–Noether method of proof of RR. At the same time, it provides a vast catalogue of examples of constructions of curves.

Exercise 5.14 (Conductor ideal (Harder)) As in Ex. 5.13, let $\Gamma \subset \mathbb{P}^2$ be a plane curve of degree a having an ordinary multiple point of multiplicity m at Q.

The conductor of the normalisation

$$\mathcal{C} = [k[\Gamma] : k[C]] = \mathrm{Ann}(k[C]/k[\Gamma]) = \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_\Gamma)$$

is the ideal of functions f in $k[\Gamma]$ so that $f \cdot k[C] \subset k[\Gamma]$. By considering the quotient rings $\mathcal{O}_{\Gamma,Q}/m_Q^N$ and $\bigoplus \mathcal{O}_{C,P_i}/m_{P_i}^N$ for any $N \geq m$, prove that $\mathcal{C} = m_Q^{m-1}$.

[Hint: This is all finite dimensional linear algebra. The quotient ring $\mathcal{O}_{\Gamma,Q}/m_Q^N$ is isomorphic to the polynomial ring $k[x_1, x_2]/(x_1, x_2)^N$, so Taylor series in 2 variables up to degree N. Similarly each $\mathcal{O}_{C,P_i}/m_{P_i}^N$ is Taylor series in 1 variable up to degree N.]

Exercise 5.15 (Past exam question) Part 1. The proof of RR used in the course was based on three main propositions. The first two of these are:

(I) A principal divisor has degree zero: $\deg(\operatorname{div} f) = 0$ for all $f \in k(C)^\times$.

(II) There exists a sequence of divisors D_n of degree tending to $+\infty$ such that the difference $\deg D_n + 1 - l(D_n)$ is bounded.

Use (I) and (II) [together with the fact that $l(D-P) = l(D)$ or $l(D) - 1$ for every D and P] to prove the following results:

(i) The maximum $g = \max_D\{\deg D + 1 - l(D)\}$ taken over all divisors D is well defined, so that the Riemann–Roch inequality $l(D) \geq 1 - g + \deg D$ is satisfied for every divisor D.

(ii) With g as in (i), every divisor D of degree $\geq g$ has $l(D) > 0$, so is linearly equivalent to an effective divisor.

(iii) There exists a divisor D of degree $g - 1$ for which $l(D) = 0$, so that the RR inequality is equality.

(iv) $l(D) = 1 - g + \deg D$ holds for every divisor D of degree $\geq 2g - 1$.

Part 2. Suppose that $g(C) = 2$ and $\deg D = 4$. Prove that $l(D - K_C) \neq 0$, and deduce that φ_D is not an embedding. Show that φ_D is either a generically 2-to-1 map of C to a plain conic, or maps C birational to a quartic curve $\overline{C}$ with a node or cusp as its only singularity. Explain which divisors D correspond to each case. [You may use the criteria on embeddings, and standard properties of the canonical map of a genus 2 curve.]

Status: Part 1 is bookwork. The whole proof of RR is too long for an exam question, but it is fair to state parts of it as given, and ask for the proof of the next part. Part 2 is part of a past exam questions (that is basically too hard), and was discussed on an example sheet.

Exercise 5.16 (Past exam question) (i) Let C be a nonsingular projective curve and A, B divisors on C. Prove that multiplication in $k(C)$ defines a k-bilinear map

$$\mathcal{L}(C, A) \times \mathcal{L}(C, B) \to \mathcal{L}(C, A + B).$$

(That is, $f \in \mathcal{L}(C, A)$ and $g \in \mathcal{L}(C, B)$ implies $fg \in \mathcal{L}(C, A + B)$.)

(ii) Let C be a nonsingular projective curve and D a divisor of degree d with $\mathcal{L}(C, D)$ of dimension 2, with basis s_1, s_2. Say what it means for the linear system $|D|$ to be a free g^1_d.

(iii) Let $|D|$ be a free g^1_d as in (ii), and A any divisor. Determine the intersection

$$s_1 \cdot \mathcal{L}(C, A) \cap s_2 \cdot \mathcal{L}(C, A) \subset \mathcal{L}(C, A + D).$$

Deduce that the subspace $s_1 \cdot \mathcal{L}(C, A) + s_2 \cdot \mathcal{L}(C, A)$ in $\mathcal{L}(C, A + D)$ has dimension equal to $2l(A) - l(A - D)$.

(iv) Now assume in addition that $\deg A - \deg D \geq 2g - 1$. Prove that the image of the multiplication map of (i) spans $\mathcal{L}(A + D)$.

Status: (ii–iii) is part of the Castelnuovo free pencil trick, that was lectured. (iv) is unseen, but not difficult.