MA4L7 Algebraic curves. First lecture 2022

Nonsingular plane curve C_a of degree a

Abstract.

Nonsingular plane curves C in PP^2 form a large class of examples. I discuss the main questions: How many functions are there on C? Rather than a rigorous treatment, I discuss without proof some instructive results, that can be treated without too many prerequisites and lead to numerology that is characteristic for the Riemann--Roch theorem. I assume the material of [UAG, Chap. 1--2]. If you are not familiar with this, please read through it in your spare time.

To say the curve C = V(g) in CC^2 defined by polynomial g(x,y) = 0 is nonsingular means that either dg/dx(P) or dg/dy(P) <> 0 for every point p in C. Over CC, if dg/dx(P) <> 0 then y is a local parameter near P (in the complex topology), and the inverse function theorem expresses x as an analytic function of y.

Think of PP^2<x:y:z> as CC^2<x,y> extended by the line at infinity (z = 0). Take C = V(G) in PP^2 defined by homogeneous G = G_a of degree a. As well as nonsingularity of the affine curve C_<z<>0> in CC^2<x/z,y/z>, I impose the additional conditions on C that its intersection with the line at infinity V(z) consists of a distinct points. Write H = P1 + .. Pa. I could suppose that the points at infinity are Pi = (1, la_i, 0), so the condition is that G(x,y,0) splits as a product of the a distinct linear factors G(x,y,0) = Prod (x - la_i*y). For the affine curve C_{z<>0} = V(g) in CC^2<x,y>, the condition on g is that its homogeneous piece of degree a (which is equal to G(x,y,0)) has a distinct roots, or that C_{z<>0} has a distinct asymptotics at infinity, in the directions x = la_i*y.

The main question addressed in this course is:
  how many functions are there on a curve C?
The question is not yet meaningful. The first basic case is this: polynomials of degree <= d in CC[z] form a vector space of dimension d+1. If we take a polynomial f(z) as a function on the complex plane CC and ask for its order of growth at infinity, the condition degree <= d can be expressed as

    |f(z)| <= (const.)*|z|^d.   (*)

Or we could extend the complex plane to the Riemann sphere, with local coordinate w = 1/z at infinity. Then the same question becomes:

Q: What is the dimension of the vector space of polynomial functions on PP^1_CC with pole of order <= d at infinity?

A: d+1. We can write this as

   L(PP^1, d*P_\infty) = {1,z,..z^d},

a vector space of dimension d+1.

Recall the theorem of Cauchy and Liouville that a complex function that is holomorphic (analytic) on the whole of CC and has bounded growth at infinity |f(z)| <= (const.) is constant. This allows us to answer the same question for holomorphic functions (rather than just polynomials): holomorphic functions on CC satisfying (*) are polynomials of degree <= d, and form the same vector space of dimension d+1.

Notice the way the original question "how many functions?" has been modified. If we ask about CC, the answer is infinity, so not useful. If it is the Riemann sphere or PP^1_CC and we ask for functions that are holomorphic everywhere, Liouville's theorem says you only get the constants, which is also not interesting. On the other hand, if we take PP^1_CC and allow poles of order d at the single point at infinity, we get the very useful answer, the vector space

   L(PP^1, d*P_\infty) = CC[z]_{deg <= d} with dim = 1 + d.

Now consider the same question for the above nonsingular curve C. How many polynomial functions are there on C? If we take the affine curve C in CC^2<x,y>, we get the affine coordinate ring CC[C] = CC[x,y]/(g), which is infinite dimensional. Informed by the above discussion, we restrict to functions on C that can be expressed as polynomials of degree <= d in x,y, and possibly interpret them somehow as holomorphic functions with bounded order of growth at infinity.

How many are there? Let's do the polynomial question for the moment. Polynomials of degree <= d in CC[x,y] are in bijection with forms (homogeneous polynomials) of degree = d in CC[x,y,z]. (Think about this if you haven't seen it before: monomials x^i*y^j with i+j <= d are identified with homogeneous monomials x^i*y^j*z^k with k = d-i-j.) Moreover, they form a vector space of dimension (d+2 choose 2) based by these monomials. (Prove this as an exercise. Then memorise the result -- you won't get much further in this course without having this at your fingertips. The plane PP^2 has a 6 dimensional space of quadric forms, a 10 dimensional space of cubic forms, and so on.)

However, that only counts the polynomials in CC[x,y], whereas we

want the dimension of their image in the coordinate ring
CC[x,y]/(g), or in the homogeneous coordinate ring CC[x,y,z]/(G).
Let me describe the calculations in homogeneous terms: a form
F of degree d vanishes on the curve C_a = V(G) if and only if
F is in the ideal G, that is F = H*G with H a form of degree
d-a. Therefore the dimension of the image is
  (d+2 choose 2) if d < a (so nothing to subtract), or
  (d+2 choose 2) - (d+2-a choose 2).
This number is the dimension of the degree d homogeneous piece
of CC[x,y,z]/(G). A brief calculation shows that for d >= a it
is equal to
  1 - (a-1 choose 2) + a*d
(see below, and redo the calculation for yourself until you have
it in memory).

This numerology is what I was aiming for. The nonsingular plane
curve C_a of degree a has genus g = (a-1 choose 2), and for
d >= a, the dimension of forms of degree d is
  1 - g + a*d.
I can replace a form F_d(x,y,z) of degree d by the rational
function F_d/z^d, a polynomial F_d(x/z,y/z,1) of degree <= d in
the affine coordinates x/z, y/z. As a rational function in CC(C)
it has poles of order <= d at every point of the line at infinity
so poles of order <= d*H where H = P1 + .. Pa. Thus
  l(C, d*H) = dim L(C, d*H) >= 1 - g + a*d, and equality for d >= a.

The calculation:
(d+2)(d+1)/2 - (d-a+2)(d-a+1)/2
  = 1/2*(d^2 + 3*d + 2 - (d^2-2*a*d+a^2 + 3*(d-a) + 2))
[then cancel d^2 and 3*d and 2]
  = 1/2*(2*a*d - a^2 + 3*a) = -1/2*(a^2-3*a) + a*d
  = 1 - (a-1 choose 2) + a*d.

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This is just simple numerology. The deg d homogeneous piece
(CC[x,y,z]/(G))_d is a sensible space of forms (homogeneous
polynomials) on C with dimension given by the RR formula.

It is a fact (to be proved later) that the Riemann-Roch space of
rational functions of (elements of the function field CC(C)) with
pole of order <= d*H is actually given by the above space of
homogeneous forms on C, so what I calculated is L(C, d*H).

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Some cultural conclusions:

Remember the formula: g = (a-1 choose 2), and

L(C, D) >= 1 - g + deg D, with equality if deg D >> 0.

Remember the binomial coefficients: forms of degree d on PP^n
is a vector space of dimension (d+n choose n).

There is more interesting numerology in the formulas: I calculated
the dimension of (forms of degree d restricted to C_a) to be

  ((d+2) choose 2) - ((d-a+2) choose 2) when d >= a
  = 1 - g + a*d

or ((d+2) choose 2) if d < a (because there is nothing to subtract).

However, the formula 1 - g + a*d actually still holds for d = a-1
and d = a-2, because the polynomial formula for the binomial
coefficient
  (n choose 2) = 1/2*n*(n-1)
gives 0 when n = 1 or n = 0. However, the polynomial formula is off
by 1 when d = a-3. If fact the formula gives (-1 choose 2) = 1.

Please do not to dismiss this as a trivial glitch. The hard point
in the proof of RR is the existence of a divisor KC of degree
2*g-2, but L(C,KC) of dimension g > 1-g + 2*g-2 (strict
inequality), and (a-3)*H provides exactly this for the plane
curve. It has degree (a-3)*a and L(C,(a-3)*H) = (a-1 choose 2).

You can follow this further down: define the "irregularity" of d*H as
the difference in the strict inequality
  irreg(d*H) = dim L(C, d*H) - (1 - g + a*d).
Then you will notice that for d <= a-3, the irregularity of d*H
equals L(C, (a-3-d)*H). This looks forward to the eventual full form
of the RR formula:
  l(C,D) - l(C, KC-D) = 1 - g + deg D,
in which the irregularity is itself treated as a different RR space
irreg(D) = l(C, KC-D) for the divisor KC-D.

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g = (a-1 choose 2) is the number appearing in the RR formula
L(C, D) >= 1 - g + deg D. The statement of RR is in algebraic
geometry. However, over CC it is also the genus of the Riemann
surface corresponding to C. (I treat this as an exercise in
topology, and not a logical component of the course.)

Exc. Over CC, the curve C_a in PP^2_a is a 1-dimensional compact
complex manifold, an oriented surface of genus g, in the topological
sense of a sphere with g handles. The best way to see this is to
calculated the Euler number, which comes 2 - 2*g. The easiest case

for calculations is
  G = z^a - Prod (x - s_i*y).
Then the continuous map C -> PP^1<x,y> is a branched cover: the equation has a single root over each point P_i, and a points over the complement S^2 minus a points. Euler number is simply additive, so
  e(C) = a + a*(2-a) = 3*a - a^2 = 2 - 2*g
  where g = 1/2*(a-1)*(a-2) = (a-1 choose 2)

Exc. An alternative calculation is to view C_a as a neighbour of the singular curve obtained as a union of a lines. This has a connected components, and (a choose 2) nodes. Smooth away each node produces a S^1 vanishing cycle.
  disjoint union of a copies of PP^1 has Euler number 2*a
  identifying (a choose 2) pairs of points to nodes reduces
  that by (a choose 2), then growing an S^1 over each node
  - (a choose 2)
  2*a - 2*(a choose 2) = 2*a - a*(a-1) = 2 - 2*g
  where g = (a^2 - 3*a + 2)/2 = (a-1 choose 2).

Exc. Polynomials in k[x] of degree <= d form a vector space of dimension d+1. This is true for d >= 0, but also for d = -1. It is false for d = -2.

Exc. Polynomials in k[x,y] of degree <= d form a vector space of dimension (d+2 choose 2) = 1/2*(d+2)*(d+1). This is true for d >= 0 (and I ask you to remember it, please). For what values
  d = -1, d = -2, d = -3, etc.
does it fail, and when it fails, by how much?

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