MA4L7 Algebraic curves

Example Sheet 2. Deadline Wed 5th Feb at 12:00

Exercise 2.1 (Number of forms of degree n) Write $S_n = S^n(x,y,z) = k[x,y,z]_n$ for the space of homogeneous forms of degree n in x,y,z. Find $\dim S_n$. [Hint: Guess the answer by doing $n = 0,1,2,3$. Prove it by induction on n. (A basic calculation.)]

Exercise 2.2 (Hyperplane divisor H) Let $C_a \subset \mathbb{P}^2_{\langle x,y,z\rangle}$ be a nonsingular plane curve of degree a, given by $F_a = 0$. Extend the definition of the valuation v_P from rational functions $f \in k(C)^\times$ to homogeneous forms $G \in k[x,y,z]_n$ (restricted to C) by choosing a linear form that is nonzero at P (say x), and setting $v_P(F) = v_P(G/x^n)$.

Write $H = \operatorname{div} z = \sum v_P(z)P$ for the divisor of z on C_a. This is the same thing as the intersection "with multiplicities" of C_a with the "line at infinity" $z = 0$ of [UAG, Chap. 1].

If $L \subset \mathbb{P}^2$ is any line, show that $\operatorname{div}(L)$ is a divisor of degree a linearly equivalent to H.

Exercise 2.3 (RR for a plane conic) Up to projective equivalence, any conic $C \subset \mathbb{P}^2$ is $xz = y^2$, and is isomorphic to $\mathbb{P}^1$ under the Veronese embedding $(u,v) \mapsto (u^2, uv, v^2)$. Computing $\mathcal{L}(C,D)$ should not be too onerous. Show that restricting $k[x,y,z]_n$ to C defines a vector space $\mathcal{L}(C,nH)$ of dimension $2n+1$ (of homogeneous forms restricted to C). [Hint: You could use the relation $xz - y^2$ systematically to reduce the power of y to 0 or 1 (do $y^n \mapsto xz \times y^{n-2}$, and repeat), to get $S^n(x,z) \oplus S^{n-1}(x,z)\cdot y$.]

Use the lines L and their divisors as in Ex. 2.2 to prove that any point of C is linearly equivalent to $P = (0,0,1)$. The divisor of x is $2P$. Show that $\mathcal{L}(C,2nP) = \left\{F_n/z^n\right\}$ for $F_n \in k[x,y,z]$.

Finally, $\mathcal{L}(C,(2n-1)P) = \left\{F_n/z^n\right\}$ for $F_n \in m_P \cdot k[x,y,z]_{n-1}$ where $m_P = (x,y)$.

Exercise 2.4 (RR for a plane cubic) Let $C = C_3$ be a nonsingular plane cubic, and H its hyperplane divisor of degree 3 as in Ex. 2.2. Assume known that $\mathcal{L}(C,H) = [x,y,z]$, that is, $l(C,H) = 3$.

(i) For any two point $P,Q \in C$ show that there is a third point $S \in C$ with $P + Q + S \overset{\text{lin}}{\sim} H$.

(ii) Deduce that $l(P+Q) = 2$. [Hint: Write $P + Q \overset{\text{lin}}{\sim} H - S$. The point is to figure out a reason in geometry why $\mathcal{L}(C, H-S) \subsetneq \mathcal{L}(C,H)$.]

(iii) Show that $l(C,P) = 1$ for any point $P \in C$. In other words, no-one achieves the permitted pole at P, and the vector space $\mathcal{L}(C,P)$ consists of the constants only.

(iv) Deduce that $P \overset{\text{lin}}{\sim} P'$ on a cubic curve C if and only if $P = P'$.

(v) Show that $l(C, P_1 + P_2 + P_3) = 3$ for any 3 distinct points $P_i \in C$. [Hint: Argue on the vector subspaces $\mathcal{L}(P_1+P_2), \mathcal{L}(P_1+P_32) \subset \mathcal{L}(P_1+P_2+P_3)$.]

(vi) Show that $l(D) = \deg D$ for any divisor D of degree > 0. (This is long and tedious, but maybe worth thinking about.)

Exercise 2.5 (Group law on a plane cubic) Let $C = C_3$ be as in Ex. 2.4 and $O \in C$ any point. Using the results of Ex. 2.4, show that for $P, Q \in C$, there is a unique point $R \in C$ with $R \overset{\text{lin}}{\sim} P + Q - O$. Show how to construct it using the popular secant-tangent construction of [UAG, Chap. 2].

Deduce that specifying the sum of any 3 collinear points to be a constant determines a group law on C.

Note Divisors on a curve form an Abelian group $\operatorname{Div} C$ by definition. The principal divisors form a subgroup group $\operatorname{PDiv} C \subset \operatorname{Div} C$ because of $\operatorname{div}(fg), \operatorname{div}(f/g) = \operatorname{div} f \pm \operatorname{div} g$. Degree is a homomorphism $\operatorname{Div} C \to \mathbb{Z}$, so divisors of degree 0 form a subgroup $\operatorname{Div}^0 C$. My Main Proposition I says that principal divisors have degree 0, so that $\operatorname{PDiv} C \subset \operatorname{Div}^0$. So $\operatorname{Div}^0 / \overset{\text{lin}}{\sim}$ is a group, basically by definition. Ex. 2.5 says that the group law on C with $O \in C$ as unit element is

$$(P,Q) \mapsto (P-O, Q-O) \mapsto (P+Q-2O) \mapsto R \overset{\text{lin}}{\sim} P+Q-O. \qquad (2.1)$$

so that the group law on C given by the secant-tangent construction is the same as $\operatorname{Div}^0 / \overset{\text{lin}}{\sim}$, apart from the little sidestep $P \mapsto P - O$ to degree 0.

The complicated proof of associativity in [UAG, Chap. 2] is covered here by the fact that $S \overset{\text{lin}}{\sim} S'$, and therefore $S = S'$

Exercise 2.6 (Degree of a principal divisor) A rational function $f \in k(C_a)$ is $f = G_m/H_m$ for $G_m, H_m \in S_m$ with $H_m \neq 0$ on C_a. Assume Bézout's theorem as stated in [UAG, Chap. 1], and determine $\deg(\operatorname{div}(G_m))$. Deduce the identity $\deg(\operatorname{div} f) = 0$. (Compare also Shafarevich, Basic Algebraic Geometry, Chap. 3 for a different treatment.)

Exercise 2.7 (RR space of mH) For $G_m \in S_m$ not vanishing on C_a, the restricted rational function $g = G_m/z^m \in k(C_a)$ is in $\mathcal{L}(C_a, mH)$. Calculate the dimension of the subspace defined by these restricted forms. [Hint: $G_m \in S_m$ restricted to C_a vanishes on C_a if and only if G_m is in the ideal of multiples of F_a. That is, multiplication by F_a gives an exact sequence

$$0 \to S_{m-a} \to S_m \to \mathcal{L}(C_a, mH). \tag{2.2}$$

The argument in the respective cases $m \geq a$ and $m < a$ is the same, but the answers should be different.]

Prove that

$$l(C_a, mH) = \dim \mathcal{L}(C_a, mH) \geq \binom{m+2}{2} - \binom{m-a+2}{2} \tag{2.3}$$

if $m \geq a$. Rewrite this as $1 - g + \deg(mH)$ for appropriate g.

Assume that (2.2) is also exact on the right (proved later in the course). Deduce the exact formula $l(C_a, mH) = 1 - g + \deg(mH)$ for $m \geq a$.

Exercise 2.8 (Canonical class $K_C = (a-3)H$) Consider (2.3) for $m = a, a-1, a-2, a-3$. You get into philosophical questions about interpreting the binomial coefficient $\binom{n}{2}$ when $n \leq 2$. Show that the formula works as stated for $m \geq a - 2$.

By considering $m = a - 3$, show that C_a has a divisor K_C so that $\deg K_C = 2g - 2$ and $l(K_C) = g > 1 - g + \deg K_C$.

Exercise 2.9 (RR for $K_C + P$) With $C_a \in \mathbb{P}^2$ and $K_C = (a-3)H$ as above, prove that $\mathcal{L}(K_C + P) = \mathcal{L}(K_C)$ for any $P \in C$. [Hint. Pick a line L through P. The divisor $\operatorname{div} L - P$ is what you get by taking the intersection $C_a \cap L$ as in [UAG, Chap 1] consisting of a points counted with multiplicity, and decrease the multiplicity of P by 1 (usually from 1 to 0). Now consider $\mathcal{L}(C_a, (a-2)H)$ from Q.4 above, and impose the conditions of vanishing on the divisor $\operatorname{div} L - P$.]

Extended worked example: RR for $C_{a_1,a_2} \subset \mathbb{P}^1 \times \mathbb{P}^1$

Consider the product surface $\mathbb{P}^1_{\langle x,y\rangle} \times \mathbb{P}^1_{\langle z,t\rangle}$. At any point of $\mathbb{P}^1 \times \mathbb{P}^1$ at least one of (x, y) and one of (z, t) is nonzero; where $y = t = 1$ (say), the surface just reduces to $\mathbb{A}^2_{x,z}$.

Let $C = C_{a_1,a_2} \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a nonsingular curve given by a bihomogeneous form F_{a_1,a_2} (meaning a polynomial that is homogeneous of degree a_1 in (x, y) and homogeneous of degree a_2 in (z, t)).

The line $L : (x = 0)$ is of bidegree $(1, 0)$, and cuts out a divisor on C of degree a_2 that I continue to denote L. Likewise, $M : (z = 0)$ is of bidegree $(0, 1)$ and cuts out a divisor M on C of degree a_1. I want to calculate the RR space $\mathcal{L}(C, b_1L + b_2M)$. The argument is instructive and is parallel to Ex. 2.7, 2.8.

Bihomogeneous forms G of bidegree (b_1, b_2) form a vector space with basis the monomials

$$x^i y^{b_1-i} z^j t^{b_2-j} \quad \text{for } 0 \le i \le b_1 \text{ and } 0 \le j \le b_2, \tag{2.4}$$

so has dimension $(b_1 + 1)(b_2 + 1)$. (It is naturally the tensor product $S^{b_1}(x, y) \otimes S^{b_2}(z, t)$.)

For any such G, the expression $G/x^{b_1}z^{b_2}$ defines a rational function on $\mathbb{P}^1 \times \mathbb{P}^1$, and restricting it to C clearly gives either zero if G is divisible by F_{a_1,a_2}, or a rational function $\rho(G) \in \mathcal{L}(C, b_1L + b_2M)$. This provides an exact sequence analogous to (2.2), and proves that

$$\begin{aligned} l(C, b_1L + b_2M) &\ge (b_1 + 1)(b_2 + 1) - (b_1 - a_1 + 1)(b_2 - a_2 + 1) \\ &= a_1b_2 + a_2b_1 - (a_1 - 1)(a_2 - 1) + 1. \end{aligned} \tag{2.5}$$

On the other hand, the divisor $b_1L + b_2M$ on C has degree $a_1b_2 + a_2b_1$, so (2.5) can be rewritten

$$l(C, b_1L + b_2M) \ge 1 - g + \deg(b_1L + b_2M), \tag{2.6}$$

where $g = (a_1 - 1)(a_2 - 1)$.

Arguing as in Ex. 2.8, and assuming that the restriction map is surjective, we get equality in (2.6) provided $b_1 \ge a_1 - 2$, $b_2 > a2 - 2$ or $b_1 > a_1 - 2$, $b_2 \ge a2 - 2$ (one of the inequalities must be strict).

The limiting case $b_1 = a_1 - 2$, $b_2 = a_2 - 2$ gives the canonical class $K_{C_{a_1,a_2}} = (a_1 - 2)L + (a_2 - 2)M$. The following point is where irregularity slips into the argument (leading to strict inequality in the RR formula): the last term $(b_1 - a_1 + 1)(b_2 - a_2 + 1)$ in (2.5) is there in the regular case $b_1 \ge a_1$, $b_2 \ge a_2$ as the dimension of the kernel of the restriction map. However, if $b_1 < a_1$ or $b_2 < a_2$ this kernel is zero, so it is a mistake to subtract off the negative quantity $(b_1 - a_1 + 1)(b_2 - a_2 + 1)$.

Please compare this with your solution to Ex. 2.8.