MA4L7 Algebraic curves. Assessed worksheet 1

Deadline: Wed 22nd Jan 12:00 noon

The course assumes the material of [UAG, Chaps. 3–4], or the equivalent sections of Christian Boehning's lecture course MA4A5. In particular, I write $X \subset \mathbb{A}^n$ for an irreducible affine variety in $\mathbb{A}^n$ (over an algebraically closed field k, with coordinates $x_1, \dots, x_n$) defined by the prime ideal I_X, and write $k[X] = k[x_1, \dots, x_n]/I_X$ for its coordinate ring. Alternatively, $A = k[X]$ is a finitely generated k-algebra that is an integral domain, and $X = \operatorname{Spec} A$.

Anyone not familiar with this, please reread Chaps. 3–4 [UAG] or equivalent treatments. The course works with curves. This leads to some simplifications: $\dim X = 1$ can be taken to mean that X is irreducible and every strict subvariety of X is a finite set. The coordinate ring $k[X]$ has prime ideal 0 (because it is an integral domain), and every other prime ideal is maximal – this is practically the definition of $\dim X = 1$. So we don't need to worry too much about the Zariski topology or the difference between Spec and Specm. A Zariski open is just the complement of a finite set, just as number theorists say "except for finitely many primes".

Exercise 1.1 (Reminders about affine varieties and NSS.) Use the NSS to establish the bijections

$$\Big\{\text{max'l ideals of } k[X]\Big\} \longleftrightarrow \Big\{\text{max'l ideals of } k[x_{1\dots n}] \text{ containing } I_X\Big\}$$
$$\longleftrightarrow \Big\{m_P = (x_i - a_i) \;\big|\; i \in [1..n]), \text{ where } P = (a_{1\dots n}) \in X\Big\}.$$

and

$$\Big\{\text{prime ideals of } k[X]\Big\} \longleftrightarrow \Big\{\text{prime ideal of } k[x_{1\dots n}] \text{ containing } I_X\Big\}$$
$$\longleftrightarrow \Big\{\; I_Y \text{ with } Y \subset X \text{ irreducible subvariety.}\Big\}$$

Exercise 1.2 (Reminder about rational versus regular functions.) X affine irreducible with affine coordinate ring $k[X]$ and function field $k(X)$. Use the NSS to prove that for $f \in k(X)$, regular at every $P \in X$ implies $f \in k[X]$. That is rational plus everywhere regular implies polynomial.

Moreover for $0 \neq g \in k[X]$, if $f \in k(X)]$ is regular at every $P \in X$ with $g(P) \neq 0$ then $f \in k[X][\frac{1}{g}]$.

Exercise 1.3 (Reminder about nonsingular points $P \in C$ curves and DVRs.) Recall the definition of DVR from lectures or one of the textbooks.

Recall the definition of tangent space $T_{X,P}$ to an affine variety, and the definition of $P \in X$ nonsingular.

Prove that $P \in X$ is a nonsingular point of a curve if and only if the local ring $\mathcal{O}_{X,P}$ is a DVR.

Exercise 1.4 (Reminder about normal rings.) Show that $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$. More generally, if A is a UFD, prove that A is integrally closed in its fraction field $K = \operatorname{Frac} A$. Deduce that a DVR is integrally closed.

Exercise 1.5 Prove the following lemma: let A be a ring, $f \in A[x]$ a monic polynomial and write (f) for the principal ideal generated by f. Then

$$A \text{ is a field} \iff A[x]/(f) \text{ is a field.}$$

Exercise 1.6 (The "baby case" of RR: rational functions on $\mathbb{P}^1$, or global meromorphic functions on the Riemann sphere $\mathbb{C} \cup \{\infty\}$.)

Let u, v be homogeneous coordinates on $\mathbb{P}^1$, and write $x = v/u$ for the affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$. Set $P = (1:0)$ and $Q = (0:1) \in \mathbb{P}^1$.

The vector space $k[x]_{\leq d}$ has dimension $1 + d$ (with a basis you know). View it as the space of rational fuctions with pole $\leq dQ$. This is the first case of RR space $\mathcal{L}(\mathbb{P}^1, dQ)$. The equality $l(\mathbb{P}^1, dQ) = 1 - g + d$ (with $g = 0$) holds for all $d \geq -1$, and fails by 1 when $d = -2$.

Exercise 1.7 Use $(x-a)/(x-b)$ to show that for $P_1, P_2 \in \mathbb{P}^1$ there is a rational function $f \in k(\mathbb{P}^1)$ with $\operatorname{div} f = P_1 - P_2$. More generally, if $D_1 = \sum m_i P_i$ and $D_2 = \sum n_j Q_j$ are effective divisors of the same degree $d = \sum m_i = \sum n_j$ then there exists $f \in k(\mathbb{P}^1)$ with $\operatorname{div} f = D_1 - D_2$.

Exercise 1.8 Show that a divisor $D = \sum m_i P_i$ of degree $d = \sum m_i \geq -1$ has $\mathcal{L}(\mathbb{P}^1, D)$ of dimension $l(\mathbb{P}^1, D) = 1 + \deg D$.

Exercise 1.9 Let $A \subset B_1 \subset B_2$ be integral domains. If B_1 is finite as A-module and B_2 is finite as B_1-module prove that B_2 is finite as A-module.

Given the determinant trick [UCA, 2.7], modify the argument to prove the same statement for integral extensions.

Exercise 1.10 If X is an affine algebraic variety with coordinate ring $k[X]$ and $g \in k[X]$, the subvariety $X_g = \{P \in X \mid g(P) \neq 0\}$ is also affine, and has coordinate ring $k[X_g] = k[X][\frac{1}{g}]$ (see Ex. 1.2). The X_g, called *standard open sets*, form a basis of the Zariski topology of X.

Prove that $k[X_g]$ is a finite $k[X]$-module if and only if $1/g$ is integral over $k[X]$. [Hint: You may need to reread the treatment of finite versus integral. You need to use the determinant trick for one of the implications.]

If $k[X]$ is already normal (integrally closed in $k(X)$), this happens only if g is a unit of $k[X]$, so that $X_g = X$. Thus the inclusion $X_g \subset X$ is usually not a finite morphism.

Exercise 1.11 The nodal cubic $C \subset \mathbb{A}^2$ given by $y^2 = x^2(x+1)$ has the usual parametrisation $f\colon \mathbb{A}^1 \to C \subset \mathbb{A}^2$ given by $x = t^2 - 1$, $y = t(t^2-1)$. Show that f is finite, that is, $k[\mathbb{A}^1]$ is a finite $k[C]$-module. [Hint: $k[C] \cdot 1_{k[t]}$ contains x, y; what more do you need to get $k[\mathbb{A}^1]$? You might start by finding a basis of the vector space $k[t]/k[x,y]$.]

Now replace $\mathbb{A}^1$ by the hyperbola $H : s(t-1) = 1 \subset \mathbb{A}^2_{\langle t,s\rangle}$ and consider the polynomial map $f\colon H \to C$ given by $x = t^2 - 1$, $y = t(t^2-1)$. Show that f is a bijective map. Show that it is not finite (that is, $k[H]$ is not a finite $k[C]$-module).

Exercise 1.12 The cuspidal cubic $\Gamma : y^2 = x^3$ has parametrisation $x = t^2$, $y = t^3$. Show that it is finite. On the other hand $H = \mathbb{A}^1 \setminus 0$ defined by $st = 1$ is a nonsingular curve, and $x = t^2$, $y = t^3$ maps H isomorphically to $\Gamma \setminus (0,0)$. Show that $H \to \Gamma$ is not finite. (It misses the singular point, so we don't allow it as a resolution of singularities.)

Exercise 1.13 A popular exercise in algebraic number theory. Let d be an integer, not a perfect square. Determine the ring of integers of the number field $\mathbb{Q}[\sqrt{d}]$. [Hint: Write $A = \mathbb{Z}[\sqrt{d}]$ inside its field of fractions $\mathbb{Q}[\sqrt{d}]$. The question is to determine (for $a, b \in \mathbb{Q}$) when $a + b\sqrt{d}$ is the root of a monic equation over $\mathbb{Z}$. The solution involves first getting rid of any square factor in d, then dividing into cases according to $d \equiv 1, 2$ or 3 modulo 4.]

Exercise 1.14 Let A be a UFD with field of fractions $K = \operatorname{Frac} A$, and assume $1/2 \in A$. For square-free $a \in A$, consider the quadratic field $K(\alpha)/K$ where $\alpha = \sqrt{a}$. Show that $A[\alpha] \subset K(\alpha)$ is integrally closed. [Hint: find the minimal polynomial of $c + d\alpha$ and show $d \in A$.]

Exercise 1.15 Let k be an algebraically closed field of characteristic $\neq 2$, and write $A = k[x]$. For $d(x) \in k[x]$ a polynomial, determine the normalisation of $A[\sqrt{d}]$. [Hint: As above, the question is when $a + b\sqrt{d}$ in $k(x)[\sqrt{d}]$ is the root of a monic polynomial. The main issue is to get rid of any square factor in d.]

Exercise 1.16 Let A be a UFD with $K = \operatorname{Frac} A$, and assume $1/3 \in A$. Let $a, b \in A$ be square-free coprime elements. Consider the cubic extension field $L = K(\sqrt[3]{a^2b})$ generated by y with minimal polynomial $y^3 = a^2b$. Prove that y and $z = y^2/a$ are integral over A, and find relations expressing the quadratic expressions y^2, yz, z^2 as lower degree quantities.

Prove that these 3 relations generate the ideal of all relations holding between y, z is generated by. [Hint: $y^3 = a^2b$ must be a linear combination of them.]

Exercise 1.17 Continuing the preceding exercise. It is given that $X = e+cy+dz \in L$ has minimal polynomial $(X-e)^3 - 3abcd(X-e) - ab(ac^3+bd^3)$, deduce that $A[y, z]$ is the integral closure of A in L.

Compared to the quadratic case, these computations are remarkable tricky, even assuming the cubic extension is cyclic.

Exercise 1.18 If $a = (x-1)(x-2)$ and $b = x(x+1)$, determine the normalisation of the affine plane curve $y^3 = ab^2$.

Exercise 1.19 Following on from the cuspidal cubic $y^2 = x^3$, determine the normalisation of $k[x, y]/(y^2 - x^5)$. Same question for $k[x, y]/(y^3 - x^7)$.

Exercise 1.20 More generally, if a, b are coprime, find the normalisation of $x^a = y^b$. [Hint: If you want to write $x = t^a$ and $y = t^b$, you are on the right track. However, for this to be a normalisation, you still have to establish that $t \in \operatorname{Frac}(A)$ where $A = k[x, y]/(x^a - y^b)$. In other words, express t in terms of x and y.]

Exercise 1.21 Let $K \subset L$ be a finite field extension. Recall from Galois theory that any $y \in L$ has a *minimal polynomial*, an irreducible polynomial

$$p(T) = T^d + c_{d-1}T^{d-1} + \cdots + c_1T + c_0 \in K[T]$$

such that $p(y) = 0$, so that $K[y] = K[T]/(p(T))$; it follows that $K[y] = K(y)$ is a field, since $(p(T))$ is a maximal ideal. We say that L/K is a *primitive extension* with generator y if $L = K(y)$.

Consider the multiplication map $\mu_y \colon L \to L$ consisting of multiplication by y. If L/K is a primitive extension, write out the matrix of μ_y in the basis $1, y, \dots, y^{d-1}$, and deduce that its trace is $\operatorname{Tr}_{L/K} \mu_y = -c_{d-1}$.

In general, prove that the trace of μ_y equals $-c_{d-1} \times [L : K(y)]$. [Hint: let b_j for $j = 1, \dots, [L : K(y)]$ be any basis of $L/K(y)$, and calculate the trace of μ_y in the basis $y^i b_j$ of L/K.