MA4L7 Algebraic curves

Example Sheet 3. Deadline Wed 19th Feb at 12:00

Exercise 3.1 (Linear-bilinear lemma) Let V_1, V_2, W be vector spaces and $\varphi\colon V_1 \times V_2 \to W$ a bilinear map. Prove the following.

Lemma *Assume that* $\varphi(v_1, v_2) \neq 0 \in W$ *for every nonzero* $v_1 \in V_1$ *and* $v_2 \in V_2$. *Then the subspace of* W *spanned by the image of* φ *has dimension* $\geq \dim V_1 + \dim V_2 - 1$.

Hint Consider the subvariety of $V_1 \otimes V_2$ of primitive tensors $\{v_1 \otimes v_2\}$; that is, the tensors of rank 1. This is the affine cone over the Segre product $\mathbb{P}(V_1) \times \mathbb{P}(V_2) \subset \mathbb{P}(V_1 \otimes V_2)$. It has dimension $n_1 + n_2 - 1$. The assumption on φ is that the kernel of $\varphi\colon V_1 \otimes V_2 \to W$ intersects it in 0 only.

Exercise 3.2 (Clifford's theorem) A divisor D is *irregular* if $\mathcal{L}(D) \neq 0$ and $\mathcal{L}(K - D) \neq 0$, that is, we can take both D and $K - D$ to be effective.

Prove that $d \geq 2r$ for any irregular divisor D defining a g^r_d. In other words, the fastest growth of $l(D)$ among all curves C and divisors D is given by the ng^1_2 on a hyperelliptic curve discussed in Ex. 3.3. [Hint: Consider the multiplication map $\mathcal{L}(D) \times \mathcal{L}(K - D) \to \mathcal{L}(K)$, and put together the RR formula with the inequality of the lemma.]

Exercise 3.3 (RR spaces $\mathcal{L}(C, ng^1_2)$ for hyperelliptic curve C) Let C be hyperelliptic of genus $g \geq 2$ (assume $\frac{1}{2} \in k$). It has a double cover $\varphi_D\colon C \to \mathbb{P}^1$, so a divisor $|D|$ with $\deg D = 2$ and $l(D) = 2$. This is called a g^1_2. Write $t_1, t_2 \in \mathcal{L}(C, D)$ for a basis, so $x = t_1/t_2$ is a parameter on $\mathbb{P}^1$.

The field extension $k(\mathbb{P}^1) \subset k(C)$ is a quadratic extension given by $z^2 = F_{2g+2}(x)$, with Galois action $i\colon z \mapsto -z$ that acts on C interchanging the two sheets of the double cover – the *hyperelliptic involution.*

The monomials $S^n(t_1, t_2) = \{t_1^n, t_1^{n-1}t_2, \dots, t_2^n\}$ are linearly independent in $\mathcal{L}(nD)$ for each n, because x is transcendental over k. Calculate the dimension of $S^n(t_1, t_2)$ for $n = 1, \dots, g$. Using this, show that $(g - 1)D$ is irregular (that is, strict inequality in RR). Deduce that $K_C \sim (g - 1)D$. On the other hand, gD has degree $> 2g - 2$, so is regular.

Next, use RR to show that $\mathcal{L}((g+1)D)$ is strictly bigger than $S^{g+1}(t_1, t_2)$. Show that the complementary basis element s can be chose so that $z = s/t_2^{g+1}$ is anti-invariant under the hyperelliptic involution, giving the new generator with $z^2 = F_{2g+2}(x)$.

Prove that the monomials $S^n(t_1, t_2)$ and $S^{n-g-1}(t_1, t_2) \cdot z$ form a basis of $\mathcal{L}(nD)$ for every n.

Exercise 3.4 (Degree 5 divisor on a genus 2 curve) Let C be a curve of genus $g = 2$ and D a divisor of degree 5. Use the criterion involving $\mathcal{L}(D - P - Q) \subset \mathcal{L}(D$ to prove that D is very ample and φ_D maps C isomorphically to $C \subset \mathbb{P}^3$ of degree 5.

Compare the dimension of the space of quadrics of $\mathbb{P}^3$ with $\mathcal{L}(C, 2D)$, and deduce that C is contained in a quadric hypersurface $Q \subset \mathbb{P}^3$.

If Q is nonsingular (say $Q : (x_0x_3 = x_1x_2)$), it is $\mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding. In this case, prove that C is a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Deduce that $D = KC + B$ where B is a free g^1_3.

If Q is a quadric of rank 3 (that is, the ordinary quadratic cone $x_0x_2 = x_1^2$), prove that C passes through the vertex $(0, 0, 0, 1)$, and deduce that $D \overset{\text{lin}}{\sim} P + 2g^1_2$.

Exercise 3.5 Let C and D be as in Ex. 3.4. Consider $\mathcal{L}(D - K_C)$ and determine the possible base points of $|D - K_C|$. Recover the result of Ex. 3.4 without involving the geometry of the embedding $C \subset \mathbb{P}^3$.

Exercise 3.6 (Degree 4 divisor on a curve of genus 2) Suppose that $\Gamma_4 \subset \mathbb{P}^2_{\langle x,y,z\rangle}$ is a plane quartic curve with a node or cusp at $(1, 0, 0)$ and no other singularities; assume its equation is $x^2a_2 + xb_3 + c_4$, with a, b, c forms in y, z of the indicated degree. Show that projection from P defines a 2-to-1 cover from the resolution $C \to \mathbb{P}^1_{\langle y,z\rangle}$ ramified in the discriminant sextic $b^2 - 4ac$. Deduce that C is birational to a hyperelliptic curves of genus 2.

Recall that K_C is the final irregular divisor. Prove that for any curve C of genus ≥ 1 and any $P, Q \in C$, we have $l(K + P + Q) - l(K) = 1$, so the morphism φ_D corresponding to $D = K + P + Q$ cannot distinguish the two points P, Q. In other words, $\varphi_D(P) = \varphi_D(Q)$.

Now suppose that $g = 2$, and let D be any divisor of degree 4. Show that $l(D - K_C) > 0$, so that D is linearly equivalent to $K + P + Q$. Prove that $\varphi_D \colon C \to \mathbb{P}^2$ either maps C to a quartic curve $\Gamma_4 \subset \mathbb{P}^2$ with a node at $\varphi(P) = \varphi(Q)$ (resp., cusp if $P = Q$), or is a double cover of a plane conic (in the case $D - K_C = g^1_2$, that is, $D = 2g^1_2$).

Exercise 3.7 (Past exam question) (i) Let $A \subset K$ be a subring of a field. If $y \in K$ is integral over A, prove that the subring $A[y] \subset K$ is a finite A-module (finitely generated as A-module).

Generalise the statement to the subring $A[y_1, \ldots, y_n] \subset K$ generated by finitely many integral elements. [The proof is not required.]

(ii) Suppose now that $A \subset B \subset K$ with the subring B a finite A-module. Prove that any $b \in B$ is integral over A. [Hint: Choose generators of B over A, and write out the A-linear map of multiplication by b as a matrix with entries in A. Argue on the determinant of b times the identity minus this matrix.]

(iii) Deduce from (i–ii) that the sum and products of elements of K that are integral over A are again integral over A, hence that the integral closure of A in K is a subring.

(iv) Calculate the integral closure of the ring $A = k[x, y]/(y^3 - x^8)$ in its field of fractions.

Explain briefly how taking normalisation (integral closure) provides the nonsingular model of the curve $C \subset \mathbb{A}^2$ given by $y^3 = x^8$.

Status: (i–iii) is bookwork; (iii) looks obvious, but depends on (ii), which is tricky to do directly. (iv) is unseen, but similar to material on the example sheets.

Exercise 3.8 (Past exam question) (i) Give the definition of discrete valuation ring. Explain how it relates to the notion of nonsingular point of a curve C. Show how to define the divisor $\operatorname{div} f$ of a function $f \in k(C)^\times$, and explain what it means in terms of zeros and poles of f at points of C.

(ii) Let $C_a \subset \mathbb{P}^2$ be a nonsingular curve of degree a in $\mathbb{P}^2$ with homogeneous coordinates x_1, x_2, x_3. Give the definition of multiplicity of intersection $\operatorname{mult}_P(C_a, L)$ of C_a with a line $L \subset \mathbb{P}^2$ at $P \in C$, and relate it to the divisor of $L/x_i \in k(C)^\times$ (for appropriate choice of x_i).

Write $\operatorname{div} L$ for the divisor on C corresponding to $\sum \operatorname{mult}_P(C, L)$. Prove that the divisors $\operatorname{div} L$ for different L are all linearly equivalent.

From now on let $C = C_3$ be a nonsingular plane cubic curve. You may assume as given that $g(C) = 1$, and that every line $L \subset \mathbb{P}^2$ meets C in 3 points counted with multiplicity.

(iii) Use RR to prove that any divisor D of degree 1 is linearly equivalent to P for a unique point $P \in C$. For $P_1, P_2, P_3 \in C$, give a geometric construction for the point Q that is linearly equivalent to $P_1 + P_2 - P_3$.

(iv) Write A for the group of divisors of degree 0 modulo linear equivalence. For $O \in C$ a marked point, show that the map $P \mapsto [P-O] \in A$ defines a bijective map $C \to A$. Prove that C has a group law with O as unit element such that the sum $\sum \operatorname{div} L$ of the 3 points of $L \cap C$ is constant.

Status: Bookwork. (iii) relates to the geometric construction of the group law.

Exercise 3.9 (Past exam question) (i) Let C be a nonsingular projective curve. Define a divisor on C, linear equivalence of divisors, and the Riemann–Roch space $\mathcal{L}(C, D)$. Prove that if D_1 and D_2 are linearly equivalent then $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$.

(ii) Prove that for a given divisor D and any $P \in C$,

$$l(D - P) = \text{either } l(D) - 1 \quad \text{or} \quad l(D).$$

If $l(D) \neq 0$, prove that the second possibility occurs for at most a finite number of points of C.

(iii) Give the full statement of the Riemann–Roch theorem, assuming the definition of the canonical divisor class K_C, and use it to prove the following assertions:

(a) $\deg K_C = 2g - 2$ and $l(K_C) = g$.

(b) For any integer n with $0 \leq n \leq g$, there exist n points $P_1, \ldots, P_n$ such that $l(P_1 + \cdots + P_n) = 1$.

(c) For any integer m with $g - 2 \leq m \leq 2g - 2$, there exists a divisor D on C with $\deg D = m$ for which $l(D) = m - g$.

Status: (i–ii) and (iii,a) is bookwork. (iii.b–c) are unseen, but follow from the methods of argument around the proof of RR.