MA4JB Commutative algebra II
Example sheet 2 (first draft, to continue)

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(Should have gone earlier)
Exercises about division with remainder

Let a,b be coprime integers. You know the result that there exist x,y with a*x + b*y = 1. Find the smallest range for the values of x,y as a function of a,b.

[Hint: Consider the possible congruences mod a*b for a*x + b*y for x in [0..b-1] and y in [0..a-1]. Deduce the number of different values you hit.]

Where does the argument use division with remainder?

Let f,g in k[x] let coprime polynomials of degree d = deg f and e = deg g. Prove that the polynomials
 a*f + b*g with a of degree <= e-1 and b of degree <= d-1
are linearly independent. Deduce that they base the vector space k[x]_{<=d+e-1}.

Where does that argument use division with remainder?

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Exercises on p-adic completion

Marco's notes state ZZ_p = ZZ[[T]]/(T-p). Determine if this is true.

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Calculation in p-adic integers

The p-adic integers ZZ_p is defined as lim_{<-} ZZ/p^n.

For p = 5, what is h = 3 + 2*p + 2*p^2 + .. 2*p^n ?
Hint. First calculate the infinite sum using the geometric progression formula. (Formal power series means ignore convergence.) Multiply h by the denominator this suggests to get result. Think about how you prove it.

Show that for p = 5, t = 2 + 3*p + p^2 + 3*p^3 +..
(with coefficients [2,1] recurring) equals 1/3 in ZZ_p.

Show that the terms in the p-adic expansion of 1/3 recur with period 1 if p == 1 mod 3 and with period 2 if p == 2 mod 3.

Show that the p-adic expansion of 1/a in ZZ_p has recurrent terms with period r where p^r == 1 mod a. Compare with the familiar result for the expansion of 1/a as a decimal.

Show that a p-adic number is rational if and only if its
expansion is eventually recurrent.

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Exercise on direct proof of Nakayama's lemma.
A local ring, M a finite module. Then m*M = M implies M = 0. You
can do this by applying the determinant trick to get a*M = 0 for
some a == 1 mod m, and say that implies a is invertible. Or more
simply choose a finite generating set {m1..n} and use m*M = M to
deduce that mn is in the submodule generated by {m1,.. m_{n-1}}.

(Is there a statement J*M = M => M that works that with m replaced
by Jacobson radical = intersection of all maximal ideals?)

-> [Ma] Theorem 8.4
For an A-module M and ideal I, consider the quotient M -> M/IM
and elements ei in M -> eibar in Mbar

If eibar generate Mbar, is it true that ei generate M?

Add conditions that A is I-adic complete and M is I-adically
separated.

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Exercises on chain conditions.

-> If A is a Noetherian ring, and S in A a multiplicative
set, prove that S^-1A is again Noetherian.

-> Let A be a ring intermediate ring between ZZ and QQ. Is A
Noetherian? If possible, write down a counterexample.

-> Prove A Noetherian implies the formal power series ring
A[[x]] is again Noetherian.

-> Matsumura p.18 has a number of clever corollaries to the general
effect that A has an effective representation on a Noetherian module
M implies A is Noetherian.

-> u: M -> M a homomorphism of A-modules and consider the iteration
u^n (that is, u composed with itself n times). Prove that {ker u^n}
is an increasing chain of A-submodules and {Mn = im u^n(M) subset M}
a decreasing chain.

Now suppose M is Noetherian. Prove that both chains terminate, and
give a submodule M0 in M for which the restriction u| M0 : M0 -> M0
is an isomorphism. Does the same argument work if we assume instead
that M is Artinian?

-> Let N1, N2 be submodules of an A-module M, prove that M/N1 and

M/N2 are Noetherian then so is M/(N1 intersect N2). Same for Artinian. Does M/(N1 intersect N2) Noetherian imply anything about M/N1 or M/N2?

-> Exc using Zariski topology of Spec A
If A is a Noetherian ring then the topology of Spec A is Noetherian (has the d.c.c. for closed sets, as for affine algebraic sets in [UAG]). Deduce that Spec A is covered by finitely many maximal closed sets (irreducible components), and hence that a Noetherian ring has only finitely many minimal prime ideals.

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Noetherian versus Artinian

Prove that a Noetherian ring with Krull dim A = 0 is Artinian. [Step 1. 0-dim implies the intersection J of all maximal ideals equals the nilradical. Step 2. Use Noetherian to prove J^N = 0 for some N. Step 3. Construct a filtration of A by ideals so that each I_n/I_{n-1} is isomorphic to the residue class field k(m_i) = A/m_i.]