MA4J8 Commutative algebra II

1 Lectures 18–22. Syzygies

Complexes and syzygies, the Koszul complex and regular sequences. Free and projective resolutions of finite modules, the Hilbert syzygies theorem and the Auslander-Buchsbaum refinement.

1.1 Introduction

Have this picture in mind: for a nice ring A and a finite A-module M,

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_{n-1} \leftarrow P_n \leftarrow 0 \tag{1.1}$$

I commonly assume

(1) (1.1) is an exact sequence of A-modules.

(1) Each P_i is a finite free A-module $P_i = b_i A = \bigoplus Ae_{ij}$.

(2) The sequence $P_0, \dots, P_n$ has length n (or $\le n$).

(3) (Sometimes) A, the modules P_i and the maps are graded.

This object appears frequently in all kinds of arguments, and is called a *finite free resolution* of M.

I write the maps in this order for 3 reasons:

- The object under study is M, and the surjective map $P_0 = b_0 A \twoheadrightarrow M$ means choosing generators of M. The argument at its most basic starts here.
- In general, whether the free resolution ends after n steps with an injective map $P_{n-1} \leftarrow P_n$ from a free module P_n is part of the problem: it fails in general, and the Hilbert syzygies theorem gives conditions under which it holds.
- If the free modules have specified bases $P_i = b_i A = \bigoplus Ae_{ij}$, each map $P_{i-1} \leftarrow P_i$ is a $(b_{i-1} \times b_i)$ matrix M_i, taking column vector $U = (u_1, \dots u_{b_i}) \in P_i$ to product $M_i U \in P_{i-1})$. Writing the maps in this order gives composition of maps as $M_1 M_2 = 0$ etc.

I spell out (1.1): $P_0 = b_0 A$ is a free A-module of rank b_0 mapping surjectively to M – it specifies b_0 generators of M. Exactness of (1.1) at P_0 means that P_1 maps surjectively to $\ker\{P_0 \to M\}$, so P_1 corresponds to writing generators for the submodule of A-linear relations between the given generators of M.

Now P_2 corresponds to the relations between the relations, that are called *syzygies*. (Greek for "yoke" – the relations are yoked together like a pair of oxen in ploughing, or are subject to linear dependence relations like stars in conjunction.) I give a discussion from scratch.

Examples Let A be an integral domain, and $x \in A$ a nonzero element. This gives the s.e.s. $0 \to A \xrightarrow{x} A \to A/(x) \to 0$ that we have seen many times. The principal ideal xA is isomorphic to A, that is, it is a free module of rank 1. This is the *only case* when an ideal is a free module!

Suppose $f, g \in A$ are coprime elements of a local integral domain, for example $x, y \in k[x,y]_{(0,0)}$. You might think that if f, g are algebraically independent, the ideal $I = (f, g)$ could be isomorphic to the direct sum $Af \oplus Ag$.

Of course this never happens. Even in this simplest case, the f and g may be algebraically independent (in the sense of eliminating different variables), but they are not A-linearly independent. In fact, the map $A \leftarrow 2A$ that takes $(1,0) \mapsto f$ and $(0,1) \mapsto g$ does $(a,b) \mapsto af + bg \in A$. This always has $(-g, f)$ in its kernel. Stupid, but true!

If A is a UFD and f, g have no common factors then $af = -bg$ if and only if

$$f = -bc \text{ and } g = ac \text{ for some } c \in A. \tag{1.2}$$

This gives the s.e.s.

$$0 \leftarrow I \leftarrow 2A \leftarrow A \leftarrow 0 \quad \text{with maps } (f, g) \text{ and } \begin{pmatrix} -g \\ f \end{pmatrix} \tag{1.3}$$

as the free resolution of the ideal I. Or we might choose to write

$$0 \leftarrow A/I \leftarrow A \leftarrow 2A \leftarrow A \leftarrow 0 \tag{1.4}$$

as the free resolution of the quotient ring A/I.

It is also common to rephrase this as the exact complex

$$A \leftarrow 2A \leftarrow A \leftarrow 0 \quad \text{or} \quad P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow 0 \tag{1.5}$$

with 0th homology $H_0(P_\bullet) = A/I$. This the *Koszul complex* of (f, g), and I elaborate on it later under weaker assumptions.

For $f, g \in k[x_1, \ldots, x_n]$ with no common factors, the variety $V(I) = V(f, g) \subset \mathbb{A}^n$ is a codimension 2 complete intersection. Its coordinate ring $k[V] = A/I$ (or its local ring $\mathcal{O}_{V,P}$) has the free resolution of length 2 given by the Koszul complex of (f, g).

These ideas are close to some of the foundations of homological algebra. I can't do all of this, but I run through some of it, especially the ideas related to the Hom functor and its derived Ext^* treated in terms of projective resolutions (usually free resolutions as above), and get some results related to duality.

Projective modules Most of what I say uses finite free modules $F = \bigoplus Ae_i$ (I also write nA as above). Projective is a mild generalisation of free, and projective modules appear everywhere in the literature. The main case of interest is finite modules over local rings (or graded rings), when projective is equivalent to free.

Definition 1.1 A module P is *projective* if every homomorphism to a quotient module M/L lifts to M. To spell that out: let $p\colon M \twoheadrightarrow N$ be a surjective map (homomorphism) and $f\colon P \to N$ any map.

Then there exists a map $g\colon P \to M$ such that f is the composite $f = pg$.

The covariant functor $\mathrm{Hom}_A(P, ?)$ is automatically left exact, and a simple restatement of the definition is that P is projective if it is an exact functor, that is, for every s.e.s. $0 \to L \to M \to N \to 0$ the sequence

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0 \qquad (1.6)$$

is exact. Think about it.

A free module F is projective: take a basis $F = \bigoplus Ae_i$. Then $M \to N$ is surjective, so $f(e_i)$ is the image of some $v_i \in M$ and the map $P \to M$ taking $e_i \mapsto v_i$ is defined and does everything required. (The same argument as in Year 1 linear algebra.)

A module is projective if and only if it is a direct summand of a free module. In fact let $x_i \in P$ be a generating set; set $F = \bigoplus Ae_i$ for the free module with basis e_i enumerated by the same set as x_i, and consider the short exact sequence

$$0 \to K \to F \to P \to 0, \qquad (1.7)$$

where $F \to P$ takes $e_i \mapsto x_i$. If P is projective the lift $g\colon P \to F$ splits the s.e.s., so $F = P \oplus K$. And conversely. Please think about this if you haven't met it before.

If A, m, k is a local ring, a finite projective module P is free by an obvious application of Nakayama's lemma: in fact $V = P/mP$ is a finite dimensional k-vector space. Choose $e_i \in P$ that map to a basis of V. Nakayama's lemma implies that the e_i generate P, that is, $\bigoplus Ae_i \to P$ is surjective. Then P is a direct summand of the free module $F = \bigoplus Ae_i$. Moreover, the complementary summand is zero because the number of e_i equals the dimension of V.

Matsumura [Ma, p. 10–11] proves the same assertion not assuming P finite by an intricate transfinite induction (due to Kaplansky).

1.2 Regular sequences and the Koszul complex

I go back to the Koszul complex. Let A be a ring and I an ideal, and let M be an A-module (the case $M = A$ is often the most useful).

Definition 1.2 A sequence of elements $x_1, \dots, x_n \in I$ is a *regular sequence* for M if

(1) x_1 is a regular element for M (that is, a nonzerodivisor)

(2) x_2 is a regular element for M/x_1M, and generally, each element x_i is a regular element for $M/(x_1M + x_2M + \cdots + x_{i-1}M)$.

(3) $M/(x_1M + \cdots + x_nM) \neq 0$.

The *I-depth* of M is defined as the maximum length n of a regular sequence $x_1, \dots, x_n$ in I.

If $x \in A$ is a nonzerodivisor of A then the quotient $A/(x)$ comes in a s.e.s. $0 \to A \xrightarrow{x} A \to A/(x) \to 0$ where the first two elements are isomorphic. This corresponds to the idea of cutting an n-dimensional variety V by a hypersurface section. In geometry, this is a really obvious thing to try, but there is a hidden difficulty. The point is to make sure that this is a "clean" cut, meaning that we have the whole ideal of the section (as a geometric locus), and don't have to clean up nilpotents after the cut.

The next section discusses examples where this obvious cutting fails.

I now give a first introduction to the relations between regular sequences and the Koszul complex, restricted to length 2: if A, I are given and $x, y \in I$, the Koszul complex

$$0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow 0 \tag{1.8}$$

with $P_0 = A$, $P_1 = 2A$, $P_2 = A$, and the first map (x, y) and second map $\binom{-y}{x}$. The complex (1.8) is clearly always defined (the composite is zero).

Proposition 1.3 *(1) Assume that* (x, y) *is a regular sequence. Then*

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow 0 \tag{1.9}$$

is exact at P_1 *and* P_2. (The conditions A a UFD and f, g coprime used in the above introduction are not the real issue.)

(2) If x *is a regular element then* $H_1(K(x, y)) = 0$ *implies that* y *is regular for* A/x*, so that* (x, y) *(in that order) is a regular sequence.*

(3) Assume in addition that A, m *is local Noetherian, and* $x, y \in m$*. Then* $H_1(K(x, y)) = 0$ *implies also that every* $a \in A$ *with* $xa = 0$ *(that is, in the colon ideal* $\ker x = (0 : x)$*) is a multiple of* y*. Therefore*

$$\ker x = y(\ker x), \tag{1.10}$$

so Nakayama's lemma implies that x *is regular.*

(4) The complex $K(x, y)$ *is symmetric in* x, y *up to isomorphism, so that in the local Noetherian case,* (x_1, x_2) *a regular sequence implies that* (x_2, x_1) *is also.*

Proof (1) $P_2 \to P_1$ takes $c \in A$ to $(-yc, xc)$, and already the second factor is injective (regardless of y).

For exactness at P_1, the homology $H_1(K(x, y))$ computes the module quotient

$$\{(a, b) \mid xa + yb = 0\} \;/\; \{(-yc, xc) \text{ for } c \in A\}. \tag{1.11}$$

Let $(a, b) \in P_1$ with $xa + yb = 0 \in P_0$. This means that yb is a multiple of x. The regular sequence assumption is that y is a nonzerodivisor modulo x: however $xa + yb = 0 \in P_0$ means that multiplication by y takes the class of b in $A/(x)$ to $yb = 0 \in A/(x)$, so b was already in (x).

Now set $b = xc$. Then $xa + yb = 0$ gives $x(a + yc) = 0$. But x was a nonzerodivisor of A, so that in turn $a = -yc$. Thus the complex is exact at P_1.

(2) Conversely: if $H_1 = 0$, an element b such that $xa+yb = 0$ is $b = xc$, so that if $yb = 0 \in A/(x)$ it follows that b is already a multiple of x. Therefore y is a nonzerodivisor for $A/(x)$.

As in (3), assume $x, y \in m$. Suppose $a \in A$ is such that $xa = 0$. Then the element $(a, 0) \in P_1$ is in the kernel of $P_1 \to P_0$. Then $H_1 = 0$ gives that a is a multiple of y. This proves that $\ker x = y(\ker x)$. Since $y \in m$, we have $\ker x = m(\ker x)$. Now we are in the local Noetherian set-up, so Nakayama's lemma implies that $\ker x = 0$. Therefore x is a nonzero divisor, and (2) gives that x, y is a regular sequence.

(4) is obvious. □

Without local (3) and (4) fail: Consider for example,

$$A = k[x, y, z]/(x-1)z \quad \text{with the sequence } x_1 = x,\ x_2 = (x-1)y. \tag{1.12}$$

Then multiplication by x is injective, so x_1 is regular, and $A/(x) = k[y]$, so $(x-1)y$ acts injectively on it, and x_1, x_2 is a regular sequence. However, $(x-1)y$ kills z, so that x_2 is not regular.

The statement of the proposition applies verbatim with A replaced by an A-module M, and the sequence by

$$M \leftarrow 2M \leftarrow M \leftarrow 0. \tag{1.13}$$

For (3–4) we of course require A, m local Noetherian and M finite.

1.3 Examples of depth 0 and depth 1

Let A, m be a local ring. Then an A-module M has m-depth zero if and only if every $f \in m$ is a zero divisor of M. By basic facts on primary decomposition, this happens if and only if m is an associated prime of M, in other words, there exists a nonzero $x \in M$ with $mx = 0$.

1. Embedded point The ideal $I = (xy, y^2) \subset A = k[x, y]$ is a key case of primary decomposition. You can describe I as the functions f that satisfy two conditions

- f vanishes on the x-axis $y = 0$.
- f is singular at $(0, 0)$. Equivalently: it has multiplicity ≥ 2. It belongs to m^2 where $m = (x, y)$; it has zero derivatives $\partial f/\partial x = \partial f/\partial y = 0$.

In the quotient A/I, the element y satisfies $y^2 = 0$, so it takes the value zero everywhere, and $my = 0$, so y is in the ideal away from the origin, but $y \notin I$, so its class is not zero in A/I. It is just a little piece of nilpotent fluff hanging onto the line at 0, but it causes difficulties in different arguments.

The submodule $(y)/I \subset A/I$ is nonzero, but annihilated by m, so is isomorphic to $k = A/m$. This makes m an associated prime of A/I. Since $my = 0$, no element of m is a nonzerodivisor for A/I, so that A/I has m-depth 0.

In primary decomposition, we can write

$$I = (y) \cap (x, y)^2, \tag{1.14}$$

but equally well $I = (y) \cap (y^2, x)$ or $(y) \cap (y^2, x - ay)$. (If a curve already contains the x-axis, requiring it to be tangent to any other curve through $(0, 0)$ forces it to be singular.)

2. Transverse planes in $\mathbb{A}^4$ Start from two transverse planes

$$X = \mathbb{A}^2_{\langle x,y \rangle} \cup \mathbb{A}^2_{\langle z,t \rangle} \quad \text{with} \quad I_X = (z, t) \cap (x, y) = (xz, xt, yz, yt), \tag{1.15}$$

and cut it by a general hyperplane through the origin, say

$$H : (y + t = 0). \tag{1.16}$$

Geometrically, the hyperplane cuts the first $\mathbb{A}^2$ in the line $y = 0$, and the second $\mathbb{A}^2$ in the line $t = 0$. So H cuts X simply in the line pair $xz = 0$ in the plane $\mathbb{A}^2_{\langle x,y \rangle}$ given by $y = t = 0$ in $\mathbb{A}^4$. Obvious isn't it? Yes set-theoretically, but *not* as far as the ideals are concerned.

The ideal I_X does not have any linear entries, so cutting by $y + t = 0$ gives $\mathbb{A}^3_{\langle x,y,z \rangle}$ with $t = -y$, and the ideal of I_X restricted to $\mathbb{A}^3$ is

$$J = (xz, xy, yz, y^2). \tag{1.17}$$

The geometric picture also wants $y = t = 0$, but as in the first example, the element y is not in the restricted ideal. Instead, $y \in k[x, y, z]/J$ is a nonzero nilpotent element with

$$my = (xy, zy, y^2) = 0. \tag{1.18}$$

Thus the origin of the line pair is an embedded point of $k[x, y, z]/J$.

3. Missing monomial The polynomial ring $k[x, y]$ is polynomial functions on the plane $\mathbb{A}^2$. The condition $\partial f/\partial x(0, 0) = 0$ defines the subring $B \subset k[x, y]$ based by every monomial except x. One sees that it is generated by

$$u = x^2, \ v = x^3, \ w = y, \ z = xy. \tag{1.19}$$

The ideal of relations between u, v, w, z is

$$J = (v^2 - u^3, z^2 - uw^2, uz - vw, vz - u^2 w). \tag{1.20}$$

In Magma:

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RR<x,y,u,v,w,z> := PolynomialRing(Rationals(),6);
L := [-u+x^2,-v+x^3,-w+y,-z+xy]; I := Ideal(L); IsPrime(I);
MinimalBasis(EliminationIdeal(I,2));
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Obviously B is an integral domain, so every nonzero element is regular.

The image of $\mathbb{A}^2$ under the polynomial map to $\mathbb{A}^4$ given by $(x, y) \mapsto (u, v, w, z)$ might seem to be a perfectly nice variety $V = V(J) \subset \mathbb{A}^4$ with coordinate ring $B = k[u, v, w, z]/J$, having a little cusp at the origin a bit like the cuspidal cubic we know from primary school. However, the unquiet spirit of the departed monomial x still haunts B and V.

Write $m = (u, v, w, z)$ for the maximal ideal at the origin. Although $x \notin B$, his product with anything in m is in B. Any section of V through the origin is marked by an embedded point, a little nilpotent submodule not accounted for by the restriction of J.

To explain: pass to the quotient ring $B/(f)$ by any nonzero $f \in m$. The product $fx \in A$ is in B, but is *not* a multiple of f in B. Therefore fx maps to a nonzero element $\xi \in B/(f)$. Now this ξ is nilpotent, and is annihilated by every element of the maximal ideal $m/(f)$: in fact for $g \in m$, the product $g\xi$ is zero in B/f, because it is the class of $gfx = f \cdot gx$ in B.

This means that although B is an integral domain, it only has m-depth 1. The quotient B/f by any $f \in m$ has a nonzero element ξ annihilated by m, so the regular element f does not extend to a regular sequence of length 2 in m.

4. Macaulay's quartic curve The rational normal curve in $\mathbb{P}^4$ is the image of $\mathbb{P}^1_{\langle u,v \rangle}$ under its 4th Veronese map $(u^4 : u^3v : u^2v^2 : uv^3 : v^4)$. However, omitting the monomial u^2v^2 also embeds $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ by the map $(u^4 : u^3v : uv^3 : v^4)$. The affine cone over this is the subring $B \subset k[u, v]$ generated by the monomials $(x, y, z, w) = (u^4, u^3v, uv^3, v^4)$ related by

$$xw - yz, x^2z - y^3, xz^2 - y^2w, yw^2 - z^3. \tag{1.21}$$

It is interesting to carry out the same arguments as in Example 3 above to verify that B also has m-depth 1.

1.4 More Koszul complexes

The Koszul complex $K(x_1, x_2, x_3)$ of length 3 is just a bit more involved: it is

$$A \leftarrow 3A \leftarrow 3A \leftarrow A \leftarrow 0 \tag{1.22}$$

with homomorphisms given by the matrices

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}, \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{1.23}$$

The 3 columns of the first syzygy matrix give the 3 identities $x_i x_j = x_j x_i$. Moreover, these 3 are linearly dependent in A^3, as expressed by the final 3×1 matrix.

The logic is as in Proposition 1.3: in any case, (1.22) is a complex. If x_1, x_2, x_3 is a regular sequence it is exact. And the converse under similar extra assumptions. This is treated more formally below.

As you know 3 dimensions is special in lots of ways. For example, you were introduced to cross product of 2 vectors in $\mathbb{R}^3$ in applied math. This gives a skew (antisymmetric) bilinear map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, which sadly is never mentioned by our algebraists because it is too advanced for 2nd year algebra and just a special case that is too elementary for 4th year courses. In algebra, the right-hand $\mathbb{R}^3$ should really be $\bigwedge^2 \mathbb{R}^3$ (I discuss this formally below). I was interested to read that in particle physics, $\mathbb{R}^3$ has polar vectors (e.g. momentum) whereas $\bigwedge^2 \mathbb{R}^3$ has axial vectors (e.g. angular momentum).

It is a well-known problem in algebra that there is no good general ordering for the $k \times k$ minors of an $n \times m$ matrix. In (1.23) I ordered the columns vectors of the first syzygy matrix as for cross product of vectors. Dimension 3 is the last time that this rational and elegant choice is available. For $n \geq 4$ this get progressively messier, and we need a better solution.

The Koszul complex for $n = 4$ is

$$0 \leftarrow A \leftarrow 4A \leftarrow 6A \leftarrow 4A \leftarrow A \leftarrow 0 \tag{1.24}$$

with maps $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}$

$$\begin{pmatrix} 0 & x_3 & -x_2 & x_4 & 0 & 0 \\ -x_3 & 0 & x_1 & 0 & x_4 & 0 \\ x_2 & -x_1 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & -x_1 & -x_2 & -x_3 \end{pmatrix}, \begin{pmatrix} x_4 & 0 & 0 & -x_1 \\ 0 & x_4 & 0 & -x_2 \\ 0 & 0 & x_4 & -x_3 \\ 0 & -x_3 & x_2 & 0 \\ x_3 & 0 & -x_1 & 0 \\ -x_2 & x_1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{1.25}$$

Note the block form $[A \mid B]$ and $[{}^tB \,\backslash\backslash\, -{}^tA]$.

Similar exercise as to why it is exact.

1.5 Exterior algebra and general Koszul complex

This is taken from Eisenbud [Ei, pp. 427–429]. The exterior algebra provides a neat formal solution to the issue of notation.

As usual A is a ring and M and N are A-modules. I assume that you have the tensor product of modules $M \otimes_A N$ on board.

The *exterior algebra* of N over A is

$$\bigwedge N = \bigoplus_{r\geq 0} \bigwedge^r N \tag{1.26}$$

where the skew (antisymmetric) product $\bigwedge^r N$ is the quotient of the r-fold tensor $N \otimes \cdots \otimes N$ by relations $n\otimes m + m\otimes n = 0$ for all $n, m \in N$. Assume also the relations $n \wedge n = 0$ to dispel any fear of ambiguity. The image of $n \otimes m$ in $\bigwedge^2 N$ in the quotient is written $n \wedge m$. In (1.26), the product of $u \in \bigwedge^a N$ and $v \in \bigwedge^b N$ is $u \wedge v \in W^{a+b}N$, satisfying $v \wedge u = (-1)^{ab} u \wedge v$. In other words, two homogeneous elements of the exterior algebra (1.26) anticommute if a and b are both odd, and commute if either is even.

A popular device with algebraists is to declare that $\bigwedge^2 N$ is the universal A-module having a skew A-bilinear map $N \times N \to \bigwedge^2 N$. As you know, this is the categorical statement that $\bigwedge^2 N$ is the solution to the UMP for skew maps $N \times N$ to an A-module. (Similarly for $\bigwedge^r N$.) Since the algebraic rules (A-bilinear and skew) are laid out in advance, it can be constructed as the A-module of linear combinations $\sum a_{ij} n_i \wedge n_j$ quotiented by those rules only.

This is just a definition; in some cases the "universal" nature of the construction may give undesired consequences – e.g., if N is not a free A-module then $N \otimes N$ or $\bigwedge^2 N$ may have torsion elements that you were not expecting.

For N an A-module and $x \in N$, the Koszul complex $K(x)$ is defined as the graded exterior product $\bigwedge N$ with differential multiplication by x:

$$K(x) : 0 \to A \to N \to \bigwedge^2 N \to \cdots \to \bigwedge^r N \to \cdots \tag{1.27}$$

Each differential $d_x \colon \bigwedge^r \to \bigwedge^{r+1}$ takes $a \mapsto x \wedge a$. The notation is very slick: the composite d_x^2 of two differentials involves multiplying by $x \wedge x = 0$, so is zero. The construction is coordinate-free, and the definition also highlights the functoriality of the construction.

1.6 Koszul complex $K(x_1, \ldots, x_n, M)$

The only case we use is the free module of rank n

$$N = nA = \bigoplus Ae_i \quad \text{with basis } e_1, \ldots, e_n, \tag{1.28}$$

and $x = \sum x_j e_j$.

Then $\bigwedge N$ is the free module of rank $2^n = \sum_i \binom{n}{i}$: the degree r component $\bigwedge^r N$ is generated by skewnomials

$$\bigwedge^r N = \bigoplus A e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_r} \quad \text{with } 1 \leq i_1 < \cdots < i_r \leq k. \tag{1.29}$$

The differential $d_x \colon \bigwedge^r N \to \bigwedge^{r+1} N$ is premultiplication by $x = \sum x_j e_j$, that is, $a \mapsto x \wedge a$. Acting on the skewnomial basis it does

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \mapsto \sum x_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_r}. \tag{1.30}$$

The formula in (1.30) politely conceals a pile of unsightly notation – this is more-or-less the formula for the $(r+1) \times (r+1)$ minors of a matrix by expanding them along the j row.

In detail, each term $x_j e_j$ of x multiplies the skewnomial. If j equals one of the subscripts i_l skewsymmetry gives zero. Otherwise, the subscript j is either $< i_1$, or fits between i_l and i_{l+1} for some l, or is $> i_r$, and that term of the skew product is then

$$= (-1)^l x_j e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge e_j \wedge e_{i_{l+1}} \wedge \cdots \wedge e_{i_r}. \tag{1.31}$$

The ± 1 is the sign of the permutation taking e_j to its rightful place after the first l of the e_i.

I defined the Koszul complex $K(x_1, \ldots, x_n, A)$ for A, but there is also a Koszul complex for an A-module M given by

$$K(x_1, \ldots, x_n, M) = K(x_1, \ldots, x_n, A) \otimes M. \tag{1.32}$$

Since each term of $K(x, A)$ is a direct sum of $\binom{n}{i}$ copies of A, each term of $K(x_1, \ldots, x_n, M)$ is a direct sum of the same number of copies of M.

1.7 The top end of $K(x_1, \ldots, x_n, M)$

The differential of $K(x_1, \ldots, x_n, M)$ is increasing, going from $\bigwedge^r M \to \bigwedge^{r+1} M$. It ends with $\bigwedge^n M \to 0$.

Proposition 1.4 *The cohomology of $K(x_1, \ldots, x_n)$ at the final term equals $A/(x_1, \ldots, x_n)$. In the same way, $K(x_1, \ldots, x_n, M)$ has top cohomology $M/(x_1, \ldots, x_n)M$.*

Proof The final term K_n of the complex $K(x_1,\dots,x_n)=\bigwedge^n N$ is the free module of rank 1 Af based by the single skewnomial $f=e_1\wedge\dots\wedge e_n$ that involves all the indices $1,\dots,n$. The penultimate term $K_{n-1}=\bigwedge^{n-1}N$ is free of rank n, based by the skewnomials f_i

$$f_i=e_1\wedge\dots\wedge\widehat{e_i}\wedge\dots\wedge e_n \quad \text{for } i=1,\dots,n \tag{1.33}$$

that omit just one index i.

Now the differential d_x applied to f_i gives $(-1)^i x_i f$. This is clear from the above description. Therefore the image of d_x is the submodule of $A=Af$ generated by $(x_1,x_2,\dots,x_n)$. The cohomology $K_n/d_x(K_n)$ is the quotient module $A/(x_1,\dots,x_n)$.

The argument for $K(x_1,\dots,x_n,M)$ is the same: the final term $K(M)_n$ is a single copy $Af\otimes M$ of M; the penultimate term $K(M)_{n-1}$ is the direct sum of n copies of M based by $Af_i\otimes M$, and the differential $d_x\colon K(M)_{n-1}\to K(M)_n$ multiplies the ith summand by x_i, with image x_iM. Thus the quotient $K(M)_n/d_x(K(M)_{n-1})$ is as stated. □

`[Ma, p. 127] uses a descending notation, where Pk has basis e{i1..ik} and the differential omits each i one at a time with the appropriate sign change. Relating the two notations is straightforward.`

1.8 Tensor product by $K(x)_\bullet$

Let $L_\bullet$ be a complex with differentials $d_L\colon L_i\to L_{i-1}$. For $x\in A$, the basic Koszul complex $K(x)_\bullet$ with entry x is $0\to A\xrightarrow{x}A\to 0$, with first term A of degree 1 mapping to A of degree 0.

Write $L(x)_\bullet$ for the tensor product $L_\bullet\otimes K(x)_\bullet$ with the 2-term Koszul complex. Since $K(x)$ consists of 2 terms of degree 1 and degree 0, with differential $x\colon A\to A$ decreasing degrees by 1, the tensor product is the following extension of $L_\bullet$ by $L[1]_\bullet$:

$$\begin{array}{rccccccc} L[1]_\bullet: & \cdots\to & L_p & \to & L_{p-1} & \to & L_{p-2} & \to\cdots \\ & & & \searrow & & \searrow & & \\ L_\bullet: & \cdots\to & L_{p+1} & \to & L_p & \to & L_{p-1} & \to\cdots \end{array} \tag{1.34}$$

The top line $L[1]_\bullet$ is the complex obtained by shifting the degree of $L_\bullet$ up by 1: it has L_p in degree $p+1$, that is $L[1]_{p+1}=L_p$, so that the three columns in (1.34) have terms of the same homological degree, respectively $p+1,p,p-1$.

The tensor product $L(x)_\bullet$ is the direct sum of top and bottom rows, with the differential

$$d_\otimes(\xi,\eta) = (d_L\xi + (-1)^p x\eta, d_L\eta) \quad \text{for } \xi \in L_p \text{ and } \eta \in L[x]_p = L_{p-1}. \tag{1.35}$$

Each parallelogram of (1.34) has sloping arrows given by multiplication by x, and the alternating $\pm$ ensure that these anticommute. The condition $d^2 = 0$ for $L(x)_\bullet$ to be a complex follows. (This is the usual argument for tensor product of complexes).

Proposition 1.5 *The tensor product complex $L(x)_\bullet$ fits in a short exact sequence of complexes*

$$0 \to L_\bullet \to L(x)_\bullet \to L[1]_\bullet \to 0. \tag{1.36}$$

The resulting long exact homology sequence does

$$\begin{array}{cccccc} \cdots \to & H_p(L_\bullet) & \to & H_p(L(x)_\bullet) & \to & H_{p-1}(L[1]_\bullet) \to \\ \xrightarrow{(-1)^{p-1}x} & H_{p-1}(L_\bullet) & \to & \cdots & & \end{array} \tag{1.37}$$

Moreover, multiplication by x acts by zero on the homology of the tensor product complex. That is, $x \cdot H_p(L(x)) = 0$.

Proof The lower row of (1.34) has no arrows going out of it, so $L_\bullet$ is a subcomplex of $L(x)_\bullet$, with quotient the top row $L[1]_\bullet$, establishing the s.e.s.

For the boundary map, an element of $H_{p-1}(L[1]_\bullet)$ is represented by a cycle $\eta \in L_p$ with $d_L(\eta) = 0$. It is the image of $(0,\eta) \in L(x)_{p+1}$ that has differential $((-1)^{p-1}x\eta, 0)$. This is the assertion of (1.37).

For the final statement, an element of $H_p(L(x))$ is represented by a cycle $(\xi,\eta) \in L_p \oplus L_{p-1}$ with differential $(d\xi + (-1)^{p-1}x\eta, d\eta) = (0,0)$, which I spell out as

$$x\eta = (-1)^p d\xi \quad \text{and} \quad d\eta = 0. \tag{1.38}$$

From this, we calculate the boundary of $(0, (-1)^p\xi)$ in $L(x)_\bullet$ to be $x(\xi,\eta)$. This proves that x times our cycle is a boundary. $\square$

Theorem 1.6 (Ma, Th 16.5) *(1) If $x_1, \dots, x_n$ is a regular sequence for M then the Koszul complex $K(x_1,\dots,x_n,M)$ has $H_0 = M/(x_{1\dots n})$ and $H_p = 0$ for $p > 0$.*

(2) If (A,m) is local and $x_1,\dots,x_n \in m$ then a stronger form of the converse holds. Namely, $M \neq 0$ and $H_1(K(x_1,\dots,x_n,M)) = 0$ implies that $x_1,\dots,x_n$ is a regular sequence for M.

In the particular case $M = A$, it follows that $K(x_1,\dots,x_n,A) = 0$ is a finite free resolution of the quotient $A/(x_1,\dots,x_n)$, as in the introduction.

Both parts are proved by induction on n, applying Proposition 1.5 with

$$L = K(x_1, \ldots, x_{n-1}, M) \quad \text{and} \quad L(x_n) = K(x_1, \ldots, x_n, M). \tag{1.39}$$

Proof of (1) We assume $x_1, \ldots, x_n$ is a regular sequence, so we can assume by induction that $K(x_1, \ldots, x_{n-1}, M)$ is exact except at H_0, where $H_0(K(x_{1,\ldots,n-1}, M)) = M/((x_1, \ldots, x_{n-1})M)$. Everything we need now comes from Proposition 1.5.

For $p \geq 2$, the homology H_p of the extended complex $L(x_n)$ is sandwiched between two groups that are zero by induction. For $p = 1$ the end of the long exact sequence (1.37) includes

$$\begin{aligned} 0 = H_1(L) \to H_1(K(x_1, \ldots, x_n, M)) \to H_0(L) \\ \xrightarrow{\pm x_n} H_0(L) \to H_0(K(x_1, \ldots, x_n)) \to 0. \end{aligned} \tag{1.40}$$

Since x_n is regular for $M/((x_1, \ldots, x_{n-1})M)$, this implies $K(x_1, \ldots, x_n, M)$ is exact at H_1 and has $H_0 = M/((x_1, \ldots, x_n)M)$, which proves (1).

Proof of (2) Since the $x_i \in m$ and $M \neq 0$, Nakayama's lemma gives $M/((x_1, \ldots, x_n)M) \neq 0$ and, of course, also $M/((x_1, \ldots, x_i)M) \neq 0$ for $i < n$).

We assume $H_1(K(x_1, \ldots, x_n, M)) = 0$. The first aim is to show that $H_1(K(x_1, \ldots, x_{n-1}, M)) = 0$, which will allow us to assume by induction that $x_1, \ldots, x_{n-1}$ is a regular sequence. In fact, the terms immediately before it $H_1(K(x_1, \ldots, x_n, M)) = 0$ in the long exact sequence (1.37) are $H_1(L) \xrightarrow{x_n} H_1(L)$. Thus $H_1(L) = x_n H_1(L)$, so Nakayama's lemma implies that $H_1(L) = 0$.

Now we know that $x_1, \ldots, x_{n-1}$ is a regular sequence, and the same exact sequence continues with

$$0 \to H_0(L) \xrightarrow{\pm x_n} H_0(L) \to H_0(K(x_1, \ldots, x_n, M)) \to 0. \tag{1.41}$$

Therefore x_n is a nonzerodivisor for $H_0(L) = H_0(K(x_1, \ldots, x_{n-1}), M) = M/()(x_1, \ldots, x_{n-1})M)$. $\square$

Appendix: Tensor product of complexes $(L., d_L) \otimes (M., d_M)$

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Make the double complex  Li tensor_A Mj for all i, j
with two differentials
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    d_L x 1_M  decreasing i
    1_L x d_M  decreasing j

The corresponding single complex is
    sum_{i+j=k} (Li tensor Mj), with the differential
           dk = sum dL_i + (-1)^{jk} dM_j.
Here the (-1)^jk has the effect of introducing one minus sign
in each square, so that instead of commuting, the arrows now
anticommute, making dk.d_{k-1} = 0 to make the sum a complex.
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1.9 Hilbert syzygies theorem

I discuss the Hilbert syzygies theorem in more-or-less the original form. Let $S = k[x_1, \dots, x_n]$ be a graded polynomial ring over a field k, and write $m = (x_1, , \dots, x_n)$ for the graded maximal ideal.

Theorem 1.7 (Syzygies theorem (1890)) *Let M be a finite graded S-module.*

Then there exist a finite free resolution of the form (1.1)

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_{n-1} \leftarrow P_n \leftarrow 0 \tag{1.42}$$

Overall shape of the proof Work by induction on n. There are two ideas: First, if one of the generators x_i is a nonzerodivisor for M, we can assume the result for $N = M/x_iM$ by induction, and lift the finite free resolution of N to one for M, using simple diagram chasing. The condition that x_i is a nonzerodivisor is used here to ensure that the snake lemma gives exact sequences as usual.

Next, if all the x_i annihilate something in M (for example if $m \in \operatorname{Ass} M$), choose generators $m_1, \dots, m_{b_0} \in M$, and write $p\colon P_0 = b_0 S \twoheadrightarrow M$ for the standard surjective map. Now switch attention to $\ker p$. This is a submodule of the free S-module P_0, so it is torsion-free: every nonzero element is a nonzerodivisor, so the first idea certainly applies to this.

Roughly speaking, the first step assumes depth $M > 0$ and decreases the dimension by passing to the hyperplane section $x_i = 0$. The second step increases the depth if necessary, thus making the first step applicable. I treat this first in a naive way, as if we were still in the 1890s, but we can soup up the result by turning on some more recent technology, as I sketch later.

Theorem 1.8 (Hilbert syzygies + Auslander–Buchsbaum) *Let S, m be a regular local ring of dimension n, and M a finite graded S-module of* m-depth $\geq d$. *Then M has a finite free resolution of length $\leq n - d$.* Proof omitted. The modern form is in [Ma] and [Ei].

The induction starts at $n = 0$, with the statement that a finite dimensonal vector space has a basis. The first step in detail. Suppose $n \geq 1$. Assume that x_n is a nonzerodivisor for M.

Consider the standard short exact sequence

$$0 \to M \xrightarrow{x_n} M \xrightarrow{\pi} N \to 0. \tag{1.43}$$

Now N is a finite module over $\overline{S} = k[x_1, \dots, x_{n-1}]$, so by induction, it has a finite free resolution by graded free $\overline{S}$-modules:

$$0 \leftarrow N \leftarrow Q_0 \leftarrow Q_1 \leftarrow \cdots \leftarrow Q_{n-1} \leftarrow Q_{n-1} \leftarrow 0 \tag{1.44}$$

Each Q_i is a finite free graded module. Set[1] $Q_i = \bigoplus_{j=1}^{b_i} \overline{S}(-a_{ij})$. Write $p\colon Q_0 \twoheadrightarrow N$ – its image is generated by the images $n_j \in N_{d_{n_j}}$ of the basis elements of Q_0.

Lemma 1.9 (Hyperplane section principle) *(1) Suppose given homogeneous generators $m_i \in N$ and homogeneous elements $m_j \in M$ such that $m_j \mapsto n_j$. Then the m_i generate M.*

(2) Now assume that x_n is M-regular (a nonzerodivisor of M). Write $P_0 = b_0 S$ for the free S-module corresponding to the generators m_j, and $K_0 = \ker\{P_0 \twoheadrightarrow M\}$ giving the s.e.s.

$$0 \to K_0 \to P_0 \to M \to 0. \tag{1.45}$$

Write $Q_0 = b_0\overline{S} \to N$ for the same construction for the generators n_i of N of $\overline{S}$, and $L_0 = \ker\{Q_0 \to N\}$. Then $K_0 \to L_0$ is surjective.

(3) Under the same assumption, a finite free resolution $Q_\bullet \twoheadrightarrow N$ can be lifted to a resolution $P_\bullet \twoheadrightarrow M$ of the same shape (the same Betti numbers and graded pieces of the same degree).

(1) If the rings were local, I could just say that the m_j generate M modulo mM, so the result follows by Nakayama's lemma. Finite graded modules

[1] The notation $S(-a)$ means the module S graded in degree $-a$. The only point of this is to keep track of the grading – my resolution complexes have morphisms with entries $S(-a) \to S(-b)$ given by polynomials of degree $a - b \geq 0$, so I can view the morphisms as having degree 0.

offer a different (and much older) trick: induction on the degree of homogeneous elements. In fact, for $c \in M$, write $\pi(c) \in N$ as the combination $\pi(c) = \sum \alpha_j n_j$. Then $c - \sum \alpha_j m_j$ is in $\ker \pi$, so is divisible by x_n. That is, $c - \sum \alpha_j m_j = x_n c'$ with $\deg c' = \deg c - 1$. Now by induction on the degree we can assume that $c' \in \sum A m_j$, which proves the lemma.

(2) We have seen that we can get generators of M (giving P_0 with the surjective map $P_0 \twoheadrightarrow M$) by lifting generators of N. We want to deal with the kernel K_0 of $P_0 \twoheadrightarrow M$ in the same way, in terms of the kernel L_0 of $Q_0 \twoheadrightarrow N$. (2) asserts that the surjective map $P_0 \to Q_0$ induces a surjective map $K_0 \to L_0$. I prove this using the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & K_0 & \to & P_0 & \to & M & \to & 0 \\ & & \downarrow x & & \downarrow x & & \downarrow x & & \\ 0 & \to & K_0 & \to & P_0 & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_0 & \to & Q_0 & \to & N & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{1.46}$$

In (1.46) the horizontal rows $0 \to K_0 \to P_0 \to M \to 0$ are exact, and the maps $P_0 \twoheadrightarrow Q_0$ and $M \twoheadrightarrow N$ are surjective by construction.

It is here that the assumption that x is M-regular is needed: the top right vertical map $M \xrightarrow{x} M$ is injective. Then the snake lemma (the long exact sequence $0 \to \ker \to \ker \to \ker \xrightarrow{\delta} \operatorname{coker} \to \operatorname{coker} \to \operatorname{coker} \to 0$) then implies that in the bottom row, $L_0 = \ker\{Q_0 \to N\}$ coincides with the cokernel of $K_0 \xrightarrow{x} K_0$. Therefore $K_0 \twoheadrightarrow L_0$, as required.

(3) follows by applying (1) and (2) repeatedly.

Proof of Theorem 1.7 If some x_i is M-regular, the Lemma allows us to decrease the dimension of S. If we can't do that, choose generators of M and the corresponding surjection $P_0 \twoheadrightarrow M$ from a free module P_0. The kernel $K_0 = \ker\{P_0 \twoheadrightarrow M\}$ is a submodule of a free module, so is torsion free. In this case, every nonzero element of S is M-regular, and in particular x_n. Then we can decrease n by passing to the quotient by x_n. The initial step of passing from M to K_0 added 1 to the length of the resolution chain, but the next step cuts the dimension down by 1, so by induction, we get a free graded resolution of length $\leq n$. □

Notice that the worst score for the length of a free resolution is given by $M = S/m = k$, with length n given by the Koszul complex $K(x_1, \dots, x_n)$. We set $P_0 = S$, and the kernel $K_0 = \ker S \to k$ is the maximal ideal m itself. This is torsion free, but has depth only 1 for the reason described in Section 1.3: in this case $K_0/x_n K_0$ as an $\overline{S}$ module is isomorphic to the quotient field k as the module $k[x_0, \dots, x_{n-1}]/(x_0, \dots, x_{n-1})$.

1.10 Regular local ring

Theorem 1.10 *Let A, m, k be a Noetherian local ring, and $n = \dim A$. Then A is* regular *if*

- *(i) The associated graded ring $\operatorname{Gr} A = \bigoplus_k m^{k-1}/m^k$ is isomorphic to the polynomial ring $k[t_1, \dots, t_n]$.*
- *(ii) m/m^2 has dimension n as a k-vector space.*
- *(iii) The maximal ideal m is generated by n elements.*

(i–iii) also imply:
The maximal ideal m is generated by a regular sequence.

This is easy: (i) implies (ii) is obvious. (ii) implies (iii) follows as usual from Nakayama's lemma: if $x_1, \dots, x_n \in m$ generate m/m^2 then they also generate m. For (iii), if $x_1, \dots, x_n$ generate m then polynomials of degree d base m^d/m^{d+1}. A linear dependence between them would imply that $\dim A < n$ (by the Hilbert series characterisation of dimension), so that $\operatorname{Gr} A$ is the symmetric k-algebra on $x_1, \dots, x_n$.

If x_1 maps to t_1 is (i), then x_1 is a nonzerodivisor of A. Applying this to $A/(x_1)$ and using induction gives that $x_1, \dots, x_n$ is a regular sequence. □

Remark 1.11 The simple-minded statement and proof I gave of Theorem 1.7 extends readily to the case of A a regular local ring of dimension n. As in the above proof, we can always pass to $K_0 = \ker\{P_0 \to M\}$ that is torsion-free (because it is a submodule of a free module). Then any element $x \in m \setminus m^2$ can be used in place of x_n in the argument of Lemma 1.9 decreasing the dimension by 1.

However, this is not quite enough to prove the Auslander–Buchsbaum form of the theorem in general. This needs some characterisations of depth in terms of homological algebra and some more work. See [Ma] and [Ei].