RU ready? — MA426 Elliptic curves

Elliptic curves live in several different worlds of math:

- analysis and complex function theory
- number theory
- algebraic geometry.

The main reason for their importance is simple: equations of the form

$$\text{quadratic in } y = \text{cubic or quartic in } x$$

(for example, $y^2 = x^3 + 2x + 1$) provide the first cases of math problems that cannot be solved rationally: for example, the arc length of the ellipse leads to integrals such as $\int \frac{dx}{y}$ where $y^2 = 1 + x^4$, for which there is no formula in terms of "elementary" functions. Nevertheless, the resulting integrals have a rich structure. In Diophantine number theory and in algebraic geometry, problems that allow rationally parametrised solutions may be "too easy", general problems "too difficult", whereas those that translate into elliptic curves are "just right", and so tend to be the most interesting.

The course will try to give the flavour of elliptic curves in each of these contexts. While avoiding detailed technical development of the different subjects, it will outline the connections between the approaches, and provide a glimpse of the role of elliptic curves in several currently active branches of math. A vast amount of machinery on the complex analysis, algebraic geometry and number theory of elliptic curves formed the technical framework of Wiles' proof of the Taniyama–Shimura conjecture and Fermat's last theorem; the final lectures of the course will indicate these connections very briefly.

Prerequisites and teaching The course will make full use of the students' background in 2nd year algebra and complex analysis. Different part of the course will also refer to ideas of elementary number theory and Galois theory, but the course is designed so that a large part of the material will be clear to students without these formal prerequisites.

Part of the course will run closely parallel to MA475 Riemann surfaces. There will be fortnightly worksheets and an examples class to help students get good marks on the assessed component.

Books The current literature on elliptic curves is vast. This brings its own problems with it: most of the books are heading further or in a more specialised direction than the course; and even when two texts treat the same topic, the similarity may be hidden by the different notation and conventions. Look at some of the following (among many others).

M. Reid, Undergraduate algebraic geometry, CUP, Chaps. 1–2. The material is accessible, but it only covers about 1/4 of the course material. Look esp. at the pictures in App. to Chap. 2.

W. Knapp, Elliptic curves, Princeton, esp. Chaps. 3, 4, 6, 8. This is perhaps the most attractive overall text. It contains some beautiful worked examples in Diophantine number theory, and the last chapter is a sensible guide to Taniyama–Shimura. But it is long-winded in places, and contains no exercises.

J. Silverman, The arithmetic of elliptic curves, Springer. Very thorough (there is also a second volume, for the specialist), and is an invaluable source for some material, but mostly too advanced.

H. McKean and V. Moll, Elliptic curves, CUP. This is a lovely and original book, covering a wealth of material. It goes in great detail into an original choice of topics, and you have to swallow very large chunks at a time before it starts making sense.

J.-P. Serre, A course in arithmetic, Springer. This is a brilliant little book on number theory. It is the simplest reference for many elementary points, and more particularly for basics on modular forms. (Or you can get it in French, Cours d'arithmétique, Presses Univ. de France if you can't afford Springer's prices.)