Homework sheet on the McKay correspondence

This is a preliminary version, and will be updated. Please let me know if you have suggestions for more exercises, or improved hints.

1. As discussed in 1.1, BD_{4n} has the 2-dimensional representations

$$V_i \cong \mathbb{C}^2, \quad \text{with action} \quad \alpha = \begin{pmatrix} \varepsilon^i & 0 \\ 0 & \varepsilon^{-i} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ (-1)^i & 0 \end{pmatrix}.$$

Show how to split these into eigenlines when $i = 0, n$, and work out what the relation $V_i \otimes Q = V_{i-1} \oplus V_{i+1}$ means for $i = 0, 1$ and $n-1, n$. Complete the verification that the McKay quiver of BD_{4n} is the completed Dynkin diagram $\widetilde{D}_{n+2}$.

2. Calculate the Gonzalez-Sprinberg–Verdier sheaves in these cases. In other words, write down $\mathrm{Hom}_G(V_i, \mathbb{C}[x, y])$ as a module over the ring of invariants.

3. In the McKay quiver for $G \subset \mathrm{SL}(2, \mathbb{C})$, show that $V_i \to V_j$ if and only if $V_j \to V_i$. In the McKay quiver for $G \subset \mathrm{SL}(3, \mathbb{C})$, show that every arrow $V_{i_1} \to V_{i_2}$ is contained in a triangle $V_{i_1} \to V_{i_2} \to V_{i_3} \to V_{i_1}$.

4. The binary tetrahedral group $\mathrm{BTet} \subset \mathrm{SL}(2, \mathbb{C})$ is generated by BD_8 and the matrix

$$c = \frac{1}{2} \begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \overline{\varepsilon} & \overline{\varepsilon} \\ -\varepsilon & \varepsilon \end{pmatrix}$$

where $\varepsilon = \sqrt{i} = \frac{1+i}{\sqrt{2}} = \exp \frac{2\pi i}{8}$ is a primitive 8th root of 1. Show that $c^3 = -1$

5. Consider the action of $\mathrm{BTet} \subset \mathrm{SL}(2, \mathbb{C})$ on the quadratic forms in x, y, that is, the second symmetric power $S^2 Q = \mathbb{C}^3$ of the given representation $Q = \mathbb{C}^2$. Show that the central element of BTet acts trivially, that a, b act as the Vierergruppe $V_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$, and that c acts by permuting cyclically the 3 characters $xy, \frac{1}{2}(x^2 + y^2), \frac{1}{2}(x^2 - y^2)$.

6. In Ex. 4, show that c normalises BD_8, and that together they generate a group BTet of order 24.

Calculate the invariants of BTet and show that the quotient singularity $\mathbb{C}^2/\,\mathrm{BTet}$ is isomorphic to the hypersurface $E_6 : (x^2 + y^3 + z^4 = 0) \subset \mathbb{C}^3$.

Find the irreducible representations of BTet, and show that the McKay quiver is the completed Dynkin diagram $\widetilde{E}_6$. [Hint: $\mathrm{BTet}\,/\,\mathrm{BD}_8 \cong \mathbb{Z}/3$ gives 3 one dimensional representations; the given two dimensional representation Q and its twists by these; the only place work left to do is to prove that the 3-dimensional representation S^2Q is irreducible, and to calculate $Q \otimes S^2Q$.]

7. Reduce the DHVW formula (3.1) to the Hirzebruch–Höfer and Roan form (3.2). [Hints: rearrange the sum as a double sum over all $g \in G$ and all h in the centraliser of g. See [HH] for details.]

8. The cyclic quotient singularity $\frac{1}{7}(1,2,4)$. The ring of invariants is generated by the 13 elements:

$$\begin{array}{ccccc} x^7 & x^5y & x^3y^2 & xy^3 & y^7 \\ x^3z & & xyz & y^5z & \\ x^2z^3 & & y^3z^2 & & \\ xz^5 & yz^3 & & & \\ z^7 & & & & \end{array}$$

Write out enough of the defining equations of $X \subset \mathbb{C}^{13}$ to prove that the tangent cone at the origin is a union of 3 surface scrolls $\mathbb{F}_2$ meeting fibre-to-negative section, and forming a generalised del Pezzo surface. [Hint: Do the AHilb homework sheet if you need preliminary practice. Write v_0, v_1, v_2, v_3, w_0 for the 5 monomials along the top row, continue cyclically with $w_0, w_1, \dots$ down the right side, and call $u = xyz$. The main equations of X are a whole pile of quadratic monomials equations such as $v_0v_2 = v_1^2$, $v_3w_1 = uw_0$, etc. More interesting are those that are not homogeneous quadratic: $v_2w_0 = v_3^3$, etc. Taking the tangent cone consists simply in taking the leading terms, e.g., throw away the cubic term to get $v_2w_0 = 0$, etc. If you stare at the Newton polygon for a while, the equations of the 3 scrolls will come into focus.]

The first blowup is nonsingular, and is a crepant resolution. Show that $H_2(Y, \mathbb{Z}) = \mathbb{Z}^3$ and $H_4(Y, \mathbb{Z}) = \mathbb{Z}^3$, and hence calculate $e(Y)$.

9. Same exercise for $\frac{1}{13}(1,3,9)$.

10. Write down the junior and seniors for $\frac{1}{7}(1,2,4)$ and $\frac{1}{13}(1,3,9)$.

11. Show that the age actually depends on the choice of ε: taking ε^{-1} interchanges juniors and seniors in the examples just worked out.

12. Check age $-1 =$ discrepancy in some numerical examples.

13. Verify the formulas of Vafa and Zaslow for B_2 and B_4.

14. The group BD_{4n} has $n+3$ conjugacy classes. Calculate E_g and F_g for each, and calculate the conjugacy classes $\mapsto$ exceptional curves direction of McKay.

15. For each curve E_i in the resolution of D_{n+2}, write down a transverse coordinate, and a parametrised little loop γ around E_i. Chase it back up to a conjugacy class of BD_{4n}.

16. Let $X = \frac{1}{3}(1,1,2,2)$. This is a terminal 4-fold singularity. Resolve X by blowing up, and check that $h_{\mathrm{string}}(X) = 1 + \mathbb{L} + \mathbb{L}^2$. This agrees with the McKay correspondence, but a priori there is no particular reason to get an integral motive.

17. Let X be the germ of hypersurface singularity $x^2+y^2+z^2+t^2=0$. Resolving X by a blowup, prove that $h_{\mathrm{string}}(X) = 1+\mathbb{L}$. [Hint: a blowup gives $Q = \mathbb{P}^1 \times \mathbb{P}^1$ as exceptional divisor with discrepancy 1.]

Now do the same with $x^2+y^2+z^2+t^4=0$. In fact $x^2+y^2+z^2+t^{2n}$ has a resolution with $\mathbb{P}^1$ as fibre, so that $h_{\mathrm{string}}(X) = 1+\mathbb{L}$ however you calculate it.

18. Let X be the germ of hypersurface singularity $x^2+y^2+z^2+t^3=0$. Show that $h_{\mathrm{string}}(X) = \mathbb{L} + \frac{1+\mathbb{L}^2}{[\mathbb{P}^4]}$ and $e_{\mathrm{string}}(X) = 1\frac{2}{5}$. [Hint: A blowup resolves X, introducing the quadric cone as exceptional divisor with discrepancy 1. To get a normal crossing divisor you have to blow up again, giving $K_Y = 4E_1 + E_2$, where $E_1 = \mathbb{P}^2$ and $E_2 = \mathbb{F}_2$.] I have no idea what (if anything) the answers mean.

19. Same question for $x^2+y^2+z^2+t^5=0$ and $x^2+y^2+z^2+t^{2n+1}=0$.

20. A cylinder set in $J_\infty Y$ is a set of the form $\pi_k^{-1}(B_k)$ for a constructible set $B_k \subset J_k Y$. Show that its measure given in (4.3) is well defined (independent of k) and finitely additive.

21. Let $P \in S$ be a nonsingular point of an algebraic surface, with local coordinates (x, y), and consider the blowup $\sigma\colon S_1 \to S$, which has exceptional curve E with discrepancy 1. Write $j_\sigma\colon J_\infty S_1 \to J_\infty S$ for the induced map on infinite jet space, and $j_{\sigma,k}\colon J_k S_1 \to J_k S$ for its kth order truncation. For any k and $t \geq 0$, prove that $j_{\sigma,k}$ maps the strata of arcs in S_1 with $\gamma \cdot E = t$ as a $\mathbb{A}^t$-bundle. [Hint: An arc $\gamma \in J_\infty S$ is $(x(z), y(z))$ for formal power series $x(z), y(z) \in k[\![z]\!]$. If $y(z)$ has smaller order of vanishing then the corresponding arc in S_1 is $(x(z)/y(z), y(z))$. The point is just that if $y(z)$ has order t then multiplication by $y(z)$ as a map from $k[z]/(z^k) \to k[z]/(z^t)$ has t-dimensional kernel.]

22. Do the exercise of Example 4.2 of the paper. [Hint: the *formule clef* says that $[Y'] = [Y \setminus C] + [C \times ¶^{c-1}]$, and the exceptional divisor E has discrepancy $c - 1$. If $C \cap D = \emptyset$, we are done. Otherwise, ...]

23. The main point of the proof of birational invariance: given $\varphi\colon Y' \to Y$, consider the set $F^{-1}_{D'-D}(t)$ of arcs $\gamma \in J_k Y'$ such that $(\operatorname{Jac}\varphi) \cdot \gamma = t$. Prove that it has a stratification such that $j_\varphi\colon J_k Y' \to J_k Y$ maps each stratum as a $\mathbb{C}^t$ bundle. [Hint: An induction on t and k. Lifting a finite jet in $J_k Y$ to $J_k Y'$ given the $(k-1)$th order lifting, you write down linear conditions in terms of the Jacobian matrix of φ; each zero gives you an extra dimension of freedom. In other words, the order of zeros of the Jacobian determinant on γ is the number of linear algebra conditions lost of passing from $J_k Y' \to J_k Y$.]

24. In Step I of the proof of Theorem 44, I explained that g is well defined using the intuition of the fundamental group. Give a correct argument in terms of arcs and Galois theory. [Hint:]

25. Expand Step II of the proof of Theorem 44. [Hint:]

26. Understand the shift in notation explained at the end of the proof of Theorem 4.4 of the paper.

27. A cluster of degree 2 is a tangent vector. An arc of degree k is not quite the same thing as a cluster of degree k.